应用数学和力学, 第 30 卷 第 5 期
2009 年 5 月 15 日出版

Applied Mathematics and Mechanics
Vol. 30, No. 5, May. 15, 2009

文章编号: 1000-0887(2009)05-0547-09

正交异性双材料 II 型界面裂纹问题研究*

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(郭兴明推荐)

摘要: 探讨正交异性双材料 II 型界面裂纹问题, 给出了它的力学模型. 将控制方程化为广义重调和方程, 借助复变函数方法推出了含两个应力奇异指数的应力函数. 基于边界条件得到了两个八元非齐次线性方程组. 求解该方程组, 在双材料工程参数满足适当的条件下确定了两个实应力奇异指数. 根据极限的唯一性定理推出了应力强度因子的公式和裂纹尖端应力场的理论解. 作为特例, 当两种正交异性材料相同时, 可以推出正交异性单材料 II 型断裂的已有结果.

关 键 词: II 型界面裂纹; 应力强度因子; 双材料; 正交异性

中图分类号: O346.1; O174.5 文献标识码: A

DOI: 10.3879/j.issn.1000-0887.2009.05.005

引 言

本文讨论正交异性双材料 II 型界面裂纹尖端附近的力学性态. 为了不遗漏影响界面断裂的因素, 我们假设在应力函数的表示式中包含两个应力奇异指数 $\lambda_m (m = 1, 2)$. 为了讨论的全面性, 假设 λ_m 有 3 种情形[1-2]: (i) $\lambda_m = -1/2 + i\varepsilon_m$; (ii) $\lambda_m = -1/2 + c + i\varepsilon_m$; (iii) $\lambda_m = -1/2 + \varepsilon_m$; 其中 ε_m 或 c, ε_m 是实数. 情形(i)见文献[3-11], 情形(ii)见文献[12], 因为 $\lambda_m (m = 1, 2)$ 是复数, 所以裂纹尖端附近的应力有振荡奇异性. 情形(iii)见文献[11, 13-14], 因为 $\lambda_m (m = 1, 2)$ 是实数, 所以裂纹尖端附近的应力没有振荡奇异性. 限于篇幅, 本文仅给出情形(iii)的推导过程, 但与文献[13]的求解方法不同, 与文献[11, 14]的力学模型不同.

1 力学模型

设 $y > 0$ 部分为第 1 种正交异性材料 $j = 1$, 其弹性常数为 $E_{11}, E_{12}, \nu_{11}, \mu_1$; $y < 0$ 部分为第 2 种正交异性材料 $j = 2$, 其弹性常数为 $E_{21}, E_{22}, \nu_{21}, \mu_2$; $y = 0, |x| < a$ 部分为裂纹面; $y = 0, |x| > a$ 部分为双材料粘接界面. 研究正交异性双材料 II 型界面裂纹尖端附近的力学性态归结为求解下列偏微分方程的边值问题:

1) 应力函数 $U_j(x, y)\ (j = 1, 2)$ 满足控制方程[10,15]:

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* 收稿日期: 2008-10-18; 修订日期: 2009-03-26
基金项目: 山西省自然科学基金资助项目(2007011008)
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$$(b_{22})_j \frac{\partial^4 U_j}{\partial x^4} + [2(b_{12})_j + (b_{66})_j] \frac{\partial^4 U_j}{\partial x^2 \partial y^2} + (b_{11})_j \frac{\partial^4 U_j}{\partial y^4} = 0 \qquad (j = 1, 2); \tag{1}$$

2) Ⅱ型界面裂纹的边界条件同文献[4, 13]. 在裂纹面上受剪应力 τ 作用, 且拉应力为0:

$$y = 0,\ |x| < a:\ (\sigma_y)_1 = (\sigma_y)_2 = 0,\ (\tau_{xy})_1 = (\tau_{xy})_2 = -\tau; \tag{2}$$

在粘结界面上应力和位移连续:

$$y = 0,\ |x| > a:$$

$$(\sigma_y)_1 = (\sigma_y)_2,\ (\tau_{xy})_1 = (\tau_{xy})_2,\ (u)_1 = (u)_2,\ (v)_1 = (v)_2; \tag{3}$$

在无穷远处应力自由:

$$\sqrt{x^2 + y^2} \to +\infty:\ (\sigma_y)_1 = (\sigma_y)_2 = 0,\ (\tau_{xy})_1 = (\tau_{xy})_2 = 0. \tag{4}$$

其极坐标形式为[3, 16]

$$\theta = \pm\pi:\ (\sigma_\theta)_1 = (\sigma_\theta)_2 = 0,\ (\tau_{r\theta})_1 = (\tau_{r\theta})_2 = -\tau, \tag{5}$$

$$\theta = 0:\ (\sigma_\theta)_1 = (\sigma_\theta)_2,\ (\tau_{r\theta})_1 = (\tau_{r\theta})_2,\ (u_r)_1 = (u_r)_2,\ (u_\theta)_1 = (u_\theta)_2, \tag{6}$$

$$r \to +\infty:\ (\sigma_\theta)_1 = (\sigma_\theta)_2 = 0,\ (\tau_{r\theta})_1 = (\tau_{r\theta})_2 = 0. \tag{7}$$

2 应力函数

控制方程(1)的特征方程为[15, 17]

$$(b_{11})_j s_j^4 + [2(b_{12})_j + (b_{66})_j] s_j^2 + (b_{22})_j = 0 \qquad (j = 1, 2), \tag{8}$$

其判别式

$$\Delta_j = \left[\frac{2(b_{12})_j + (b_{66})_j}{(b_{11})_j}\right]^2 - 4\frac{(b_{22})_j}{(b_{11})_j} \qquad (j = 1, 2). \tag{9}$$

当 $\Delta_1 > 0$, $\Delta_2 > 0$ 时, 有特征根如下:

$$s_{jk} = \mathrm{i}\beta_{jk},\ s_{j(k+2)} = -\mathrm{i}\beta_{jk} \qquad (j, k = 1, 2) \tag{10}$$

且

$$\beta_{j1}^2 + \beta_{j2}^2 = \frac{2(b_{12})_j + (b_{66})_j}{(b_{11})_j},\ \beta_{j1}^2\beta_{j2}^2 = \frac{(b_{22})_j}{(b_{11})_j} \qquad (j = 1, 2). \tag{11}$$

由式(11), 控制方程(1)可改写为

$$\left\{\begin{aligned} &\left(\frac{\partial^2}{\partial x^2} + \frac{1}{\beta_{j1}^2}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{1}{\beta_{j2}^2}\frac{\partial^2}{\partial y^2}\right) U_j = 0, &(12\mathrm{a})\\ &\left(\frac{\partial^2}{\partial x^2} + \frac{1}{\beta_{j2}^2}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{1}{\beta_{j1}^2}\frac{\partial^2}{\partial y^2}\right) U_j = 0 \qquad (j = 1, 2). &(12\mathrm{b}) \end{aligned}\right.$$

由式(10), 设

$$z_{jk} = x + \mathrm{i}\beta_{jk} y = x_{jk} + \mathrm{i}y_{jk} \qquad (j, k = 1, 2), \tag{13}$$

则式(12)即式(1)可化为

$$\nabla_{j1}^2 \nabla_{j2}^2 U_j = \nabla_{j2}^2 \nabla_{j1}^2 U_j = 0 \qquad (j = 1, 2). \tag{14}$$

基于重调和方程形如 $\nabla^2\nabla^2 U = 0$, 故将式(14)称为广义重调和方程. 由方程(14)和复数表示式(13)易知, 复变量 z_{jk} $(j, k = 1, 2)$ 的解析函数的实部或虚部是控制方程(1)的解[18]. 考虑到 $k, m = 1, 2$, 由此可选取应力函数为含两个实应力奇异指数 λ_1, λ_2 的下列级数:

$$U_j(x, y) = \sum_{m=1}^{2}\sum_{k=1}^{2} \mathrm{Re}[(A_{jk,\lambda_m} - \mathrm{i}B_{jk,\lambda_m}) U_{jk,\lambda_m}(z_{jk})], \tag{15a}$$

$$U_{jk,\lambda_m}(z_{jk}) = \frac{\tau}{(\lambda_m + 2)(\lambda_m + 1)}(z_{jk} - a)^{\lambda_m + 2} \qquad (z_{jk} \neq a; j, k, m = 1, 2). \tag{15b}$$

其极坐标形式为

$$U_j(r,\theta)=\sum_{m=1}^{2}\sum_{k=1}^{2}\mathrm{Re}[(A_{jk,\lambda_m}-\mathrm{i}B_{jk,\lambda_m})U_{jk,\lambda_m}(r,\theta)], \tag{16a}$$

$$U_{jk,\lambda_m}(r,\theta)=\frac{\tau}{(\lambda_m+2)(\lambda_m+1)}r^{\lambda_m+2}(\cos\theta+\mathrm{i}\beta_{jk}\sin\theta)^{\lambda_m+2}$$
$$(r\neq 0;j,k,m=1,2), \tag{16b}$$

其中

$$z_{jk}-a=x-a+\mathrm{i}\beta_{jk}y=r(\cos\theta+\mathrm{i}\beta_{jk}\sin\theta)=r\rho_{jk}\mathrm{e}^{\mathrm{i}\varphi_{jk}}\quad(j,k=1,2). \tag{17}$$

将式(15)、(16)代入应力和应力函数之间的关系式[10,15-16],注意到式(17),得到

$$(\sigma_x)_j=\sum_{m=1}^{2}\sum_{k=1}^{2}\mathrm{Re}[(A_{jk,\lambda_m}-\mathrm{i}B_{jk,\lambda_m})(-\beta_{jk}^2)U_{jk,\lambda_m}(z_{jk})], \tag{18a}$$

$$(\sigma_y)_j=\sum_{m=1}^{2}\sum_{k=1}^{2}\mathrm{Re}[(A_{jk,\lambda_m}-\mathrm{i}B_{jk,\lambda_m})U_{jk,\lambda_m}(z_{jk})], \tag{18b}$$

$$(\tau_{xy})_j=-\sum_{m=1}^{2}\sum_{k=1}^{2}\mathrm{Re}[(A_{jk,\lambda_m}-\mathrm{i}B_{jk,\lambda_m})(\mathrm{i}\beta_{jk})U_{jk,\lambda_m}(z_{jk})], \tag{18c}$$

$$U_{jk,\lambda_m}(z_{jk})=\tau(z_{jk}-a)^{\lambda_m}\quad(j,k,m=1,2); \tag{18d}$$

$$(\sigma_r)_j=\sum_{m=1}^{2}\sum_{k=1}^{2}\mathrm{Re}[(A_{jk,\lambda_m}-\mathrm{i}B_{jk,\lambda_m})(-\sin\theta+\mathrm{i}\beta_{jk}\cos\theta)^2U_{jk,\lambda_m}(r,\theta)], \tag{19a}$$

$$(\sigma_\theta)_j=\sum_{m=1}^{2}\sum_{k=1}^{2}\mathrm{Re}[(A_{jk,\lambda_m}-\mathrm{i}B_{jk,\lambda_m})(\cos\theta+\mathrm{i}\beta_{jk}\sin\theta)^2U_{jk,\lambda_m}(r,\theta)], \tag{19b}$$

$$(\tau_{r\theta})_j=-\sum_{m=1}^{2}\sum_{k=1}^{2}\mathrm{Re}[(A_{jk,\lambda_m}-\mathrm{i}B_{jk,\lambda_m})(-\sin\theta+\mathrm{i}\beta_{jk}\cos\theta)\times$$
$$(\cos\theta+\mathrm{i}\beta_{jk}\sin\theta)U_{jk,\lambda_m}(r,\theta)], \tag{19c}$$

$$U_{jk,\lambda_m}(r,\theta)=\tau r^{\lambda_m}(\cos\theta+\mathrm{i}\beta_{jk}\sin\theta)^{\lambda_m}\quad(j,k,m=1,2). \tag{19d}$$

由式(18)、(19),将两个 $\lambda_m(m=1,2)$ 称为应力奇异指数,有时也称为特征数.

3 应力奇异指数

将式(19)代入边界条件(5)、(6),注意到当 $\lambda_1\neq\lambda_2$ 时,r^{λ_1} 与 r^{λ_2} 线性无关,得到关于系数 B_{11,λ_m}, B_{12,λ_m}, A_{11,λ_m}, A_{12,λ_m}, B_{21,λ_m}, B_{22,λ_m}, A_{21,λ_m}, A_{22,λ_m} $(m=1,2)$ 的两个八元非齐次线性方程组,对它们的增广矩阵 $(\overline{A}_{\lambda_m})=(A_{\lambda_m},b_{\mathrm{II}})$ 通过适当的顺序消元,可以求出系数矩阵 (A_{λ_m}) 的行列式为

$$|A_{\lambda_m}|=r^{8\lambda_m}(\beta_{12}-\beta_{11})^2(\beta_{22}-\beta_{21})^2\left[\left(\frac{1}{\lambda_m}e_{12}+f_{12}\right)f_{12}+\right.$$
$$\left.g_{12}h_{12}\cot^2(\lambda_m\pi)\right]\sin^4(\lambda_m\pi), \tag{20}$$

其中

$$\begin{cases}e_{12}=\left(\dfrac{2(\nu_{11}+1)}{E_{11}}-\dfrac{1}{\mu_1}\right)-\left(\dfrac{2(\nu_{21}+1)}{E_{21}}-\dfrac{1}{\mu_2}\right),\\ f_{12}=\dfrac{\beta_{11}\beta_{12}-\nu_{11}}{E_{11}}-\dfrac{\beta_{21}\beta_{22}-\nu_{21}}{E_{21}},\end{cases} \tag{21a}$$

$$\begin{cases} g_{12} = \dfrac{\beta_{11}+\beta_{12}}{E_{11}} + \dfrac{\beta_{21}+\beta_{22}}{E_{21}}, \\ h_{12} = \beta_{11}\beta_{12}\dfrac{\beta_{11}+\beta_{12}}{E_{11}} + \beta_{21}\beta_{22}\dfrac{\beta_{21}+\beta_{22}}{E_{21}} \end{cases} \tag{21b}$$

称为双材料工程参数. 同时, 根据非齐次线性方程组有解的充分必要条件可以证明: 当 $|A_{\lambda_m}| = 0$ 且秩(A_{λ_m}) = 秩(A_{λ_m}) = 7 时, 每个非齐次线性方程组有解, 且有无穷多解(含一个自由未知量). 下面求解方程:

$$|A_{\lambda_m}| = r^{8\lambda_m}(\beta_{12}-\beta_{11})^2(\beta_{22}-\beta_{21})^2\left[\left(\frac{1}{\lambda_m}e_{12}+f_{12}\right)f_{12}+ g_{12}h_{12}\cot^2(\lambda_m\pi)\right]\sin^4(\lambda_m\pi) = 0. \tag{22}$$

若 $\sin(\lambda_m\pi) = 0$, 则 $\lambda_m = n\ (m = 1, 2;\ n = 0, \pm 1, \pm 2, \cdots)$, 与双材料工程参数 $e_{12}, f_{12}, g_{12}, h_{12}$ 无关, 应舍去.

若

$$\left(\frac{1}{\lambda_m}e_{12}+f_{12}\right)f_{12}+ g_{12}h_{12}\cot^2(\lambda_m\pi) = 0 \qquad (m = 1, 2), \tag{23}$$

其中含因子 $\cot(\lambda_m\pi)$, 可选取应力奇异指数为

$$(\text{i})\quad \lambda_m = -n-1/2+\mathrm{i}\varepsilon_m; \tag{24a}$$

$$(\text{ii})\quad \lambda_m = -n-1/2+c+\mathrm{i}\varepsilon_m; \tag{24b}$$

$$(\text{iii})\quad \lambda_m = -n-1/2+\varepsilon_m \qquad (m = 1, 2;\ n = 0, 1, 2, \cdots); \tag{24c}$$

其中 ε_m 或 c, ε_m 是实双材料弹性常数, 而 n 所取的值由边界条件(7) 确定.

下面仅讨论情形(iii). 在裂纹尖端附近 $(z_{jk}\to a;\ r\to 0)$, 式(24c) 中 $n = 0$, 有

$$\lambda_m = -\frac{1}{2}+\varepsilon_m \qquad (m = 1, 2). \tag{25}$$

将式(25) 代入式(23), 得到

$$\left(-\frac{2}{1-2\varepsilon_m}e_{12}+f_{12}\right)f_{12}+ g_{12}h_{12}\tan^2(\varepsilon_m\pi) = 0. \tag{26}$$

将函数 $1/(1-2\varepsilon_m)$ 和 $\tan(\varepsilon_m\pi)$ 的幂级数展开式代入式(26), 略去 ε_m 的 3 次及 3 次以上的充分小量, 得到

$$(g_{12}h_{12}\pi^2 - 8e_{12}f_{12})\varepsilon_m^2 - 4e_{12}f_{12}\varepsilon_m - (2e_{12}-f_{12})f_{12} = 0. \tag{27}$$

解此一元二次方程, 当判别式

$$\Delta = 4f_{12}[(2e_{12}-f_{12})g_{12}h_{12}\pi^2 - 4(3e_{12}-2f_{12})e_{12}f_{12}] > 0 \tag{28}$$

时得到两个实根:

$$\varepsilon_m = \frac{2e_{12}f_{12} + (-1)^{m-1}\sqrt{f_{12}[(2e_{12}-f_{12})g_{12}h_{12}\pi^2 - 4(3e_{12}-4f_{12})e_{12}f_{12}]}}{g_{12}h_{12}\pi^2 - 8e_{12}f_{12}} \qquad (m = 1, 2). \tag{29}$$

对于正交异性双材料 II 型界面裂纹问题, 只有当双材料工程参数满足条件(28) 时, 可由式(29) 求出两个实双材料弹性常数 ε_1, ε_2. 将 ε_1, ε_2 代入式(25) 得到两个实应力奇异指数 λ_1, λ_2.

对原非齐次线性方程组继续求解, 将顺序消元后得到的 7 个方程的同解线性方程组进行逆序回代. 考虑到系数由边界条件(5)、(6) 确定, 即由 $\theta = \pi$, $\theta = -\pi$, $\theta = 0$, r 取任何值时的

约束条件确定,加之求出系数后 r^{λ_m} 作为因子并入应力函数里面,所以得到16个系数的求解公式如下:

$$B_{1k,\varepsilon_m}=(-1)^k\frac{\beta_{22}-\beta_{21}}{\beta_{12}-\beta_{11}}[\beta_{1k^*}g_{12}\tan(\varepsilon_m\pi)+f_{12}\cot(\varepsilon_m\pi)]a_{22,\varepsilon_m},\tag{30a}$$

$$A_{1k,\varepsilon_m}=(-1)^k\frac{\beta_{22}-\beta_{21}}{\beta_{12}-\beta_{11}}(\beta_{1k^*}g_{12}-f_{12})a_{22,\varepsilon_m}+(-1)^k\frac{1}{2(\beta_{12}-\beta_{11})\cos(\varepsilon_m\pi)},\tag{30b}$$

$$B_{2k,\varepsilon_m}=(-1)^{k-1}[\beta_{2k^*}g_{12}\tan(\varepsilon_m\pi)-f_{12}\cot(\varepsilon_m\pi)]a_{22,\varepsilon_m},\tag{30c}$$

$$A_{2k,\varepsilon_m}=(-1)^k(\beta_{2k^*}g_{12}+f_{12})a_{22,\varepsilon_m}+(-1)^{k-1}\frac{1}{2(\beta_{22}-\beta_{21})\cos(\varepsilon_m\pi)}\quad(k,m=1,2),\tag{30d}$$

其中 a_{22,ε_m} 是自由未知量; $k=1$ 时, $k^*=2$; $k=2$ 时, $k^*=1$.

若两种正交异性材料 $j=1, j=2$ 相同,则式(21)化为

$$e_{12}=0,\ f_{12}=0,\ g_{12}=2\frac{\beta_1+\beta_2}{E_1},\ h_{12}=2\beta_1\beta_2\frac{\beta_1+\beta_2}{E_1}.\tag{31}$$

将式(31)依次代入式(29)和(25),得到

$$\varepsilon_1=\varepsilon_2=0,\tag{32}$$

$$\lambda_1=\lambda_2=-\frac{1}{2},\tag{33}$$

其中式(33)与正交异性单材料Ⅱ型断裂的应力奇异指数 $\lambda=-1/2$ 是吻合的.

4 应力强度因子

将式(25)代入式(18)中,则式(18)化为

$$(\sigma_x)_j=\sum_{m=1}^{2}(\sigma_x)_{j,\varepsilon_m}=\sum_{m=1}^{2}\sum_{k=1}^{2}\mathrm{Re}[(A_{jk,\varepsilon_m}-\mathrm{i}B_{jk,\varepsilon_m})(-\beta_{jk}^2)U_{jk,\varepsilon_m}(z_{jk})],\tag{34a}$$

$$(\sigma_y)_j=\sum_{m=1}^{2}(\sigma_y)_{j,\varepsilon_m}=\sum_{m=1}^{2}\sum_{k=1}^{2}\mathrm{Re}[(A_{jk,\varepsilon_m}-\mathrm{i}B_{jk,\varepsilon_m})U_{jk,\varepsilon_m}(z_{jk})],\tag{34b}$$

$$(\tau_{xy})_j=\sum_{m=1}^{2}(\tau_{xy})_{j,\varepsilon_m}=-\sum_{m=1}^{2}\sum_{k=1}^{2}\mathrm{Re}[(A_{jk,\varepsilon_m}-\mathrm{i}B_{jk,\varepsilon_m})(\mathrm{i}\beta_{jk})U_{jk,\varepsilon_m}(z_{jk})],\tag{34c}$$

$$U_{jk,\varepsilon_m}(z_{jk})=\frac{\tau}{(z_{jk}-a)^{1/2-\varepsilon_m}}\quad(j,k,m=1,2).\tag{34d}$$

考虑到式(34b, c, d),引入应力强度因子如下:

$$(K_{\mathrm{II}})_j=-\sum_{m=1}^{2}\sum_{k=1}^{2}\lim_{z_{jk}\to a}\mathrm{Re}[(2\pi|z_{jk}-a|)^{1/2-\varepsilon_m}(A_{jk,\varepsilon_m}-\mathrm{i}B_{jk,\varepsilon_m})(\mathrm{i}\beta_{jk})U_{jk,\varepsilon_m}(z_{jk})],\tag{35a}$$

$$(K_{\mathrm{I}})_j=\sum_{m=1}^{2}\sum_{k=1}^{2}\lim_{z_{jk}\to a}\mathrm{Re}[(2\pi|z_{jk}-a|)^{1/2-\varepsilon_m}(A_{jk,\varepsilon_m}-\mathrm{i}B_{jk,\varepsilon_m})U_{jk,\varepsilon_m}(z_{jk})],\tag{35b}$$

$$(K)_j=(K_{\mathrm{I}})_j-\mathrm{i}(K_{\mathrm{II}})_j\quad(j=1,2).\tag{35c}$$

兼顾到两个裂纹尖端 $z_{jk}=a$ 和 $z_{jk}=-a$ 处的应力奇异性,可设

$$U_{jk,\varepsilon_m}(z_{jk})=\frac{\tau z_{jk}^{1-2\varepsilon_m}}{(z_{jk}-a)^{1/2-\varepsilon_m}(z_{jk}+a)^{1/2-\varepsilon_m}}\quad(j,k,m=1,2).\tag{36}$$

在裂纹尖端 $z_{jk}=a$ 附近，有

$$U_{jk,\varepsilon_m}(z_{jk})=\left(\frac{a}{2}\right)^{1/2-\varepsilon_m}\frac{\tau}{(z_{jk}-a)^{1/2-\varepsilon_m}}\qquad(z_{jk}\to a;j,k,m=1,2).\tag{37}$$

将式(37)与式(34d)对照，式(37)多一个常数因子. 考虑到系数公式(30)中有一个自由未知量 a_{22,ε_m}，所以在应力表示式(34a,b,c)中仍可选取系数与式(30)相同. 根据极限的唯一性定理，当 $z_{jk}\to a^-$ 和 $z_{jk}\to a^+$ 时，取得相同的极限 $(K_{\text{II}})_j$，由式(35a)有

$$\sum_{m=1}^{2}\sum_{k=1}^{2}\lim_{z_{jk}\to a^-}\mathrm{Re}\left\{(2\pi|z_{jk}-a|)^{1/2-\varepsilon_m}[-(A_{jk,\varepsilon_m}-\mathrm{i}B_{jk,\varepsilon_m})(\mathrm{i}\beta_{jk})U_{jk,\varepsilon_m}(z_{jk})]\right\}=$$
$$\sum_{m=1}^{2}\sum_{k=1}^{2}\lim_{z_{jk}\to a^+}\mathrm{Re}\left\{(2\pi|z_{jk}-a|)^{1/2-\varepsilon_m}[-(A_{jk,\varepsilon_m}-\mathrm{i}B_{jk,\varepsilon_m})(\mathrm{i}\beta_{jk})U_{jk,\varepsilon_m}(z_{jk})]\right\}$$
$$(j=1,2).\tag{38}$$

注意到 $(-1)^{1/2}=\mathrm{i}$，$(-1)^{\varepsilon_m}=[\mathrm{e}^{\ln(-1)}]^{\varepsilon_m}=\cos(\varepsilon_m\pi)+\mathrm{i}(-1)^{j-1}\sin(\varepsilon_m\pi)$，将式(36)、(17)代入式(38)、(35a,b)，得到

$$(\beta_{j1}A_{j1,\varepsilon_m}+\beta_{j2}A_{j2,\varepsilon_m})\cos(\varepsilon_m\pi)-(\beta_{j1}B_{j1,\varepsilon_m}+\beta_{j2}B_{j2,\varepsilon_m})\times$$
$$[1+(-1)^j\sin(\varepsilon_m\pi)]=0\qquad(j,m=1,2)\tag{39}$$

和

$$(K_{\text{II}})_j=-\sum_{m=1}^{2}\tau(\pi a)^{1/2-\varepsilon_m}(\beta_{j1}B_{j1,\varepsilon_m}+\beta_{j2}B_{j2,\varepsilon_m}),\tag{40a}$$

$$(K_{\text{I}})_j=\sum_{m=1}^{2}\tau(\pi a)^{1/2-\varepsilon_m}(A_{j1,\varepsilon_m}+A_{j2,\varepsilon_m})\qquad(j=1,2).\tag{40b}$$

将式(30a,b)和(30c,d)代入式(39)，求出

$$a_{22,\varepsilon_m}^{(j)}=(-1)^{j-1}\frac{\tan(\varepsilon_m\pi)}{2(\beta_{22}-\beta_{21})f_{12}}\qquad(j,m=1,2).\tag{41}$$

将式(41)代入式(30a,b)、(30c,d)，推出16个系数公式如下:

$$B_{1k,\varepsilon_m}=(-1)^k\frac{\beta_{1k^*}g_{12}\tan^2(\varepsilon_m\pi)+f_{12}}{2(\beta_{12}-\beta_{11})f_{12}},\tag{42a}$$

$$A_{1k,\varepsilon_m}=(-1)^k\frac{\beta_{1k^*}g_{12}\sin(\varepsilon_m\pi)+f_{12}[1-\sin(\varepsilon_m\pi)]}{2(\beta_{12}-\beta_{11})f_{12}\cos(\varepsilon_m\pi)},\tag{42b}$$

$$B_{2k,\varepsilon_m}=(-1)^k\frac{\beta_{2k^*}g_{12}\tan^2(\varepsilon_m\pi)-f_{12}}{2(\beta_{22}-\beta_{21})f_{12}},\tag{42c}$$

$$A_{2k,\varepsilon_m}=(-1)^{k-1}\frac{\beta_{2k^*}g_{12}\sin(\varepsilon_m\pi)+f_{12}[1+\sin(\varepsilon_m\pi)]}{2(\beta_{22}-\beta_{21})f_{12}\cos(\varepsilon_m\pi)}\qquad(k,m=1,2).\tag{42d}$$

将式(42a,b)、(42c,d)代入式(40)，得到两种正交异性材料 $j=1,j=2$ 的应力强度因子公式如下:

$$(K_{\text{II}})_j=(-1)^j\sum_{m=1}^{2}\frac{\tau(\pi a)^{1/2-\varepsilon_m}}{2}=\sum_{m=1}^{2}(K_{\text{II}})_{j,\varepsilon_m},\tag{43a}$$

$$(K_{\text{I}})_j=(-1)^j\sum_{m=1}^{2}\frac{\tau(\pi a)^{1/2-\varepsilon_m}g_{12}\tan(\varepsilon_m\pi)}{2f_{12}}=\sum_{m=1}^{2}(K_{\text{I}})_{j,\varepsilon_m}\qquad(j=1,2).\tag{43b}$$

若两种正交异性材料 $j=1,j=2$ 相同，双材料变成了单材料. 考虑到两个应力奇异指数 λ_1，λ_2 和每种材料 $j=1$ 或 $j=2$ 的断裂性态对单材料的协同影响，正交异性单材料的应力强度因子确定如下:

$$K_{\text{II}}=\frac{[(K_{\text{II}})_2-(K_{\text{II}})_1]_{\varepsilon_m=0}}{2}=\frac{\sum_{m=1}^{2}\lim_{\varepsilon_m\to0}[(K_{\text{II}})_{2,\varepsilon_m}-(K_{\text{II}})_{1,\varepsilon_m}]}{2}=\tau(\pi a)^{1/2},\tag{44a}$$

$$K_{\text{I}} = \frac{[(K_{\text{I}})_2 + (K_{\text{I}})_1]_{\varepsilon_m = 0}}{2} = \frac{\sum_{m=1}^{2} \lim_{\varepsilon_m \to 0} [(K_{\text{I}})_{2,\varepsilon_m} + (K_{\text{I}})_{1,\varepsilon_m}]}{2} = 0, \tag{44b}$$

这与正交异性单材料Ⅱ型裂纹的应力强度因子相同. 在(44a)中用了减法, 这是基于对条件(2)中的载荷条件 $y = 0, |x| < a: (\tau_{xy})_1 = (\tau_{xy})_2 = -\tau$ 的考虑.

5 裂纹尖端应力场

由式(36)易知, 在裂纹尖端 $z_{jk} = a$ 附近, 有

$$\lim_{z_{jk} \to a} \left\{ [2\pi(z_{jk} - a)]^{1/2 - \varepsilon_m} U_{jk,\varepsilon_m}(z_{jk}) \right\} = \tau(\pi a)^{1/2 - \varepsilon_m} \quad (j, k, m = 1, 2). \tag{45}$$

由此看到

$$U_{jk,\varepsilon_m}(z_{jk}) = \frac{\tau(\pi a)^{1/2-\varepsilon_m}}{(2\pi)^{1/2-\varepsilon_m}} \frac{1}{(z_{jk} - a)^{1/2-\varepsilon_m}}, \tag{46a}$$

$$\frac{1}{(z_{jk} - a)^{1/2-\varepsilon_m}} = \operatorname{Re}\frac{1}{(z_{jk} - a)^{1/2-\varepsilon_m}} + \mathrm{i}\operatorname{Im}\frac{1}{(z_{jk} - a)^{1/2-\varepsilon_m}} \quad (z_{jk} \to a; j, k, m = 1, 2). \tag{46b}$$

在式(46)、(43)中令 $j = 1, j = 2$ 后将式(46)、(43)、(42a, b)或(42c, d)代入式(34a, b, c), 推出两种正交异性材料 $j = 1$ 或 $j = 2$ 的Ⅱ型界面裂纹尖端附近 $(z_{jk} \to a)$ 的应力表示式为

$$\begin{aligned}(\sigma_x)_j = {} & \sum_{m=1}^{2} \frac{(K_{\text{II}})_{j,\varepsilon_m}}{(2\pi)^{1/2-\varepsilon_m}} \frac{1}{\beta_{j2} - \beta_{j1}} \left\{ \left[\frac{1}{\cos(\varepsilon_m \pi)} + (-1)^j \tan(\varepsilon_m \pi) \right] \times \right. \\ & \left. \operatorname{Re}\left[\frac{\beta_{j2}^2}{(z_{j2} - a)^{1/2-\varepsilon_m}} - \frac{\beta_{j1}^2}{(z_{j1} - a)^{1/2-\varepsilon_m}} \right] + \operatorname{Im}\left[\frac{\beta_{j2}^2}{(z_{j2} - a)^{1/2-\varepsilon_m}} - \frac{\beta_{j1}^2}{(z_{j1} - a)^{1/2-\varepsilon_m}} \right] \right\} + \\ & \sum_{m=1}^{2} \frac{(K_{\text{I}})_{j,\varepsilon_m}}{(2\pi)^{1/2-\varepsilon_m}} \frac{\beta_{j1}\beta_{j2}}{\beta_{j2} - \beta_{j1}} \left\{ \operatorname{Re}\left[\frac{\beta_{j2}}{(z_{j2} - a)^{1/2-\varepsilon_m}} - \frac{\beta_{j1}}{(z_{j1} - a)^{1/2-\varepsilon_m}} \right] + \right. \\ & \left. (-1)^{j-1} [\tan(\varepsilon_m \pi)] \operatorname{Im}\left[\frac{\beta_{j2}}{(z_{j2} - a)^{1/2-\varepsilon_m}} - \frac{\beta_{j1}}{(z_{j1} - a)^{1/2-\varepsilon_m}} \right] \right\},\end{aligned} \tag{47a}$$

$$\begin{aligned}(\sigma_y)_j = {} & \sum_{m=1}^{2} \frac{(K_{\text{II}})_{j,\varepsilon_m}}{(2\pi)^{1/2-\varepsilon_m}} \frac{1}{\beta_{j2} - \beta_{j1}} \left\{ \left[\frac{1}{\cos(\varepsilon_m \pi)} + (-1)^j \tan(\varepsilon_m \pi) \right] \times \right. \\ & \left. \operatorname{Re}\left[\frac{1}{(z_{j1} - a)^{1/2-\varepsilon_m}} - \frac{1}{(z_{j2} - a)^{1/2-\varepsilon_m}} \right] + \operatorname{Im}\left[\frac{1}{(z_{j1} - a)^{1/2-\varepsilon_m}} - \frac{1}{(z_{j2} - a)^{1/2-\varepsilon_m}} \right] \right\} + \\ & \sum_{m=1}^{2} \frac{(K_{\text{I}})_{j,\varepsilon_m}}{(2\pi)^{1/2-\varepsilon_m}} \frac{1}{\beta_{j2} - \beta_{j1}} \left\{ \operatorname{Re}\left[\frac{\beta_{j2}}{(z_{j1} - a)^{1/2-\varepsilon_m}} - \frac{\beta_{j1}}{(z_{j2} - a)^{1/2-\varepsilon_m}} \right] + \right. \\ & \left. (-1)^{j-1} [\tan(\varepsilon_m \pi)] \operatorname{Im}\left[\frac{\beta_{j2}}{(z_{j1} - a)^{1/2-\varepsilon_m}} - \frac{\beta_{j1}}{(z_{j2} - a)^{1/2-\varepsilon_m}} \right] \right\},\end{aligned} \tag{47b}$$

$$\begin{aligned}(\tau_{xy})_j = {} & \sum_{m=1}^{2} \frac{(K_{\text{II}})_{j,\varepsilon_m}}{(2\pi)^{1/2-\varepsilon_m}} \frac{1}{\beta_{j2} - \beta_{j1}} \left\{ \operatorname{Re}\left[\frac{\beta_{j2}}{(z_{j2} - a)^{1/2-\varepsilon_m}} - \frac{\beta_{j1}}{(z_{j1} - a)^{1/2-\varepsilon_m}} \right] - \right. \\ & \left. \left[\frac{1}{\cos(\varepsilon_m \pi)} + (-1)^j \tan(\varepsilon_m \pi) \right] \operatorname{Im}\left[\frac{\beta_{j2}}{(z_{j2} - a)^{1/2-\varepsilon_m}} - \frac{\beta_{j1}}{(z_{j1} - a)^{1/2-\varepsilon_m}} \right] \right\} + \\ & \sum_{m=1}^{2} \frac{(K_{\text{I}})_{j,\varepsilon_m}}{(2\pi)^{1/2-\varepsilon_m}} \frac{\beta_{j1}\beta_{j2}}{\beta_{j2} - \beta_{j1}} \left\{ (-1)^j [\tan(\varepsilon_m \pi)] \operatorname{Re}\left[\frac{1}{(z_{j1} - a)^{1/2-\varepsilon_m}} - \frac{1}{(z_{j2} - a)^{1/2-\varepsilon_m}} \right] + \right. \\ & \left. \operatorname{Im}\left[\frac{1}{(z_{j1} - a)^{1/2-\varepsilon_m}} - \frac{1}{(z_{j2} - a)^{1/2-\varepsilon_m}} \right] \right\} \quad (j = 1, 2).\end{aligned} \tag{47c}$$

对材料 $j = 2$ 上任一点 $M_2(x, y)$ 或 $M_2(r, \theta)$, 必在材料 $j = 1$ 上有一对称点 $M_1(x, -y)$

或 $M_1(r,-\theta)$. 由式(17), 对称两点 M_2, M_1 之间, 有

$$z_{jk}-a=r\rho_{jk}e^{i\varphi_{jk}},\quad \rho_{jk}=\sqrt{\cos^2\theta+\beta_{jk}^2\sin^2\theta},\quad \varphi_{jk}=(-1)^j\arctan(\beta_{jk}\tan\theta)\quad (j,k=1,2). \tag{48}$$

当两个正交异性材料 $j=1$, $j=2$ 相同时, 双材料变成了单材料, 这时 $\beta_{2k}=\beta_{1k}=\beta_k(k=1,2)$. 由式(48)可以看出

$$\rho_{jk}=\rho_k=\sqrt{\cos^2\theta+\beta_k\sin^2\theta},\quad \varphi_{jk}=(-1)^j\varphi_k=(-1)^j\arctan(\beta_k\tan\theta)\quad (j,k=1,2). \tag{49}$$

由式(17)、(46b)、(48)、(49), 易知

$$\begin{cases}\mathrm{Re}\dfrac{1}{(z_{jk}-a)^{1/2-\varepsilon_m}}=\mathrm{Re}\dfrac{1}{(z_k-a)^{1/2-\varepsilon_m}},\\ \mathrm{Im}\dfrac{1}{(z_{jk}-a)^{1/2-\varepsilon_m}}=(-1)^j\mathrm{Im}\dfrac{1}{(z_k-a)^{1/2-\varepsilon_m}}\end{cases}\quad (j,k,m=1,2). \tag{50}$$

鉴于每种材料 $j=1$ 或 $j=2$ 的应力 $(\sigma_x)_j$, $(\sigma_y)_j$, $(\tau_{xy})_j$ 和两个应力奇异指数 λ_1, λ_2 对单材料的应力 σ_x, σ_y, τ_{xy} 的协同影响, 仿照式(44)的考虑, 由式(32)、(50)、(47)、(43), 可以推出与文献[15, 17]相同的正交异性单材料Ⅱ型裂纹尖端附近($z_k\to a$)的应力表示式.

6 结　　论

本文对正交异性双材料Ⅱ型界面裂纹附近的力学性态进行理论探讨. 利用边界条件, 通过求解非齐次线性方程组, 在双材料工程参数满足适当的条件下求出了两个实应力奇异指数 $\lambda_m=-1/2+\varepsilon_m(m=1,2)$. 通过计算一元二次方程的根, 得到了两个实双材料弹性常数 $\varepsilon_m(m=1,2)$. 借助有关公式, 推出了Ⅱ型界面裂纹尖端附近的应力强度因子和应力场. 两个常数 ε_1, ε_2 不仅影响着每个应力强度因子, 还影响着每种材料 $j=1$ 或 $j=2$ 的裂纹尖端附近的应力. 同时应力强度因子和Ⅱ型界面裂纹尖端附近的应力都有混合型断裂特征, 但因为应力奇异指数是实数, 所以没有应力振荡奇异性. 从两个应力奇异指数 λ_1, λ_2 和每种材料 $j=1$ 或 $j=2$ 的力学性态对单材料的断裂存在着协同影响的观点出发, 求出了正交异性单材料Ⅱ型裂纹尖端附近的应力奇异指数, 应力强度因子和应力场.

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Researches on Interface Crack Problems for Mode Ⅱ of Double Dissimilar Orthotropic Composite Materials

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Abstract: The fracture problems near interface crack tip for mode Ⅱ of double dissimilar orthotropic composite materials are studied. The mechanical models of interface crack for mode Ⅱ were given. By translating the governing equations into generalized bi- harmonic equations, the stress functions containing two stress singularity exponents were derived with the help of a complex function method. Based on the boundary conditions, a system of non- homogeneous linear equations was found. Two real stress singularity exponents were determined under appropriate conditions of bi- material engineering parameters through solving this system. According to the theorem of limit uniqueness, both the formulae of stress intensity factors and theoretical solutions of stress field near interface crack tip were deduced. When the two orthotropic materials are the same, the stress singularity exponents, stress intensity factors and stresses for mode Ⅱ crack of orthotropic single material were obtained.

Key words: interface crack for mode Ⅱ; stress intensity factor; double materials; orthotropic