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一角点支撑对边两边固支正交各向异性矩形薄板振动的辛叠加方法*

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摘要： 运用辛叠加方法研究了一角点支撑对边两边固支的正交各向异性矩形薄板的振动问题.首先由边界条件出发, 将原振动问题分解为两个对边简支的子振动问题, 再根据 Hamilton 体系的分离变量法分别得到两个子振动问题的级数展开解,然后利用叠加方法得到原振动问题的辛叠加解.为了在具体计算中确定所得辛叠加的级数展开项,对该解计算正交各向异性矩形薄板的情形进行了收敛性分析.应用所得辛叠加解分别计算了一角点支撑对边两边固支的各向同性和正交各向异性矩形薄板的振动频率, 进而给出了正交各向异性方形薄板的前 8 阶振动频率所对应的模态.

关　键　词： 正交各向异性矩形薄板；　Hamilton 体系；　辛叠加方法；　振动

中图分类号： O302　　**文献标志码：** A　　DOI：10.21656/1000-0887.450001

A Symplectic Superposition Method for Vibration of the Orthotropic Rectangular Thin Plate Point-Supported at a Corner and Clamped at its Opposite Edges

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Abstract: The symplectic superposition method was used to study the vibration problem of the orthotropic rectangular thin plate point-supported at a corner and clamped at its opposite edges. Firstly, based on the boundary conditions, the original vibration problem was decomposed into 2 subproblems with 2 opposite edges simply supported. Next, the series expansion solutions to the 2 sub-vibration problems were obtained based on the separation variable method in the Hamiltonian system. Then the symplectic superposition solution to the original vibration problem was obtained with the superposition method. To determine the terms of the series expansion of the obtained symplectic superposition solution in specific calculations, the convergence analysis of the solution for calculating orthotropic rectangular thin plates was performed. The symplectic superposition solution was also used to calculate the vibration frequencies of the isotropic and orthotropic rectangular thin plate point-supported at a corner and clamped at its opposite edges, respectively, and to give the modes corresponding to the 1st 8 vibration frequencies of an orthotropic square thin plate.

* **收稿日期：** 2024-01-02；**修订日期：** 2024-01-31

基金项目： 国家自然科学基金(12362001;11862019);内蒙古自然科学基金(2023MS01008)

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引用格式： 叶雨农, 额布日力吐. 一角点支撑对边两边固支正交各向异性矩形薄板振动的辛叠加方法[J]. 应用数学和力学, 2024, **45**(7): 898-906.

Key words：orthotropic rectangular thin plate；Hamiltonian system；symplectic superposition method；vibration

0　引　　言

正交各向异性板的振动问题多年来一直被学者们所研究,其中数值方法占主导地位.在 20 世纪 60 年代,Leissa[1]详细介绍了几种经典的数值方法,之后又发展出了有限元法[2]、广义微分求积法[3]和等几何分析方法[4]等较有效的数值方法.比起数值方法,解析方法可以更直观地揭示问题的本质和隐含关系.因此还有许多学者在研究相关解析方法,如变量分离法[5]、Fourier 级数法[6]、辛弹性力学方法[7]和辛叠加方法[8]等,其中辛叠加方法是辛弹性力学方法的一种推广和延伸,该方法拓宽了经典辛弹性力学方法求解力学问题解析解的范围,从而解决了许多板、梁和壳的振动以及屈曲问题,如侧裂矩形薄板的振动问题[9]、双功能梯度纳米梁系统的振动问题[10]、非 Lévy 型正交各向异性开口圆柱壳的屈曲问题[11]和 non-Lévy-type 型厚板的屈曲问题[12]等.

角点支撑的矩形薄板在机械和土木工程中有广泛的应用,如太阳能电池板、飞机和航天部件等[13-15].2014 年,文献[14]用辛叠加方法研究了一角点支撑各向同性矩形薄板的弯曲问题,之后在 2015 年,文献[16]应用辛叠加方法研究了一角点支撑各向同性矩形薄板在不同边界条件下的振动问题.从数学方程的角度来看,正交各向异性矩形薄板包含了各向同性矩形薄板的情形,在实际应用中也更为广泛.近年来,学者们应用辛叠加方法已经解决了部分正交各向异性矩形薄板的振动问题,如四边固支、四边自由和一边固支三边自由[17]等的振动问题.但是到目前为止,还未见有研究角点支撑的正交各向异性矩形薄板的振动问题.

基于上述内容,本文应用辛叠加方法研究了一角点支撑对边两边固支的正交各向异性矩形薄板的振动问题.首先将一角点支撑对边两边固支的正交各向异性矩形薄板的振动问题分解为两个对边简支的子振动问题.然后利用经典的辛弹性力学方法分别给出这两个子振动问题的级数展开解,再利用叠加法得到原振动问题的辛叠加解.之后在本文的算例中,我们应用所得的辛叠加解分别计算了一角点支撑对边两边固支的各向同性矩形薄板和正交各向异性矩形薄板的振动频率,计算各向同性矩形薄板所得的振动频率与文献[16]的计算结果高度一致,表明了所得辛叠加解的正确性.此外,本文还在算例中对所得辛叠加解计算正交各向异性矩形薄板的振动频率进行了收敛性分析,计算了正交各向异性矩形薄板在不同长宽比值下的振动频率,最后给出了正交各向异性方形板的模态.

1　Hamilton 系统

一角点支撑对边两边固支正交各向异性矩形薄板的振动方程为

$$D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}+\rho h\frac{\partial^2 w}{\partial t^2}=0, \tag{1}$$

其定义域为 $\{(x,y,t)\mid 0\leqslant x\leqslant a,\ 0\leqslant y\leqslant b,\ 0\leqslant t\}$,其中 $D_1=\dfrac{E_1h^3}{12(1-\nu_{12}\nu_{21})}$,$D_2=\dfrac{E_2h^3}{12(1-\nu_{12}\nu_{21})}$,$D_3=(D_{12}+2D_{66})$,$D_{12}=\nu_{12}D_2=\nu_{21}D_1$,$D_{66}=\dfrac{G_{12}h^3}{12}$,$\dfrac{\nu_{12}}{E_1}=\dfrac{\nu_{21}}{E_2}$.这里 $E_1,E_2,\nu_{12},\nu_{21}$ 分别为独立的弹性参数,h,ρ 和 w 分别为板的厚度、体密度和挠度.

由于弹性线性系统的法向振动是简谐振动,因此薄板法向振动的位移可以假设为

$$w(x,y,t)=W(x,y)\mathrm{e}^{\mathrm{i}\omega t}, \tag{2}$$

其中 $W(x,y)$ 为板的振型函数,ω 为角频率,i 为虚数单位.

将式(2)代入到式(1),消去 $\mathrm{e}^{\mathrm{i}\omega t}$ 可得

$$D_1\frac{\partial^4 W}{\partial x^4}+2D_3\frac{\partial^4 W}{\partial x^2\partial y^2}+D_2\frac{\partial^4 W}{\partial y^4}-\rho h\omega^2W=0. \tag{3}$$

一角点支撑对边两边固支正交各向异性矩形薄板的边界条件为

$$\begin{cases} W(0,0)=0,\ M_x|_{x=0}=V_x|_{x=0}=0,\ W|_{x=a}=\left.\dfrac{\partial W}{\partial x}\right|_{x=a}=0, \\ M_y|_{y=0}=V_y|_{y=0}=0,\ W|_{y=b}=\left.\dfrac{\partial W}{\partial y}\right|_{y=b}=0. \end{cases} \tag{4}$$

板内弯矩 M_x,M_y 以及等效剪力 V_x,V_y 可分别表示为

$$\begin{cases} M_x=-\left(D_1\dfrac{\partial^2 W}{\partial x^2}+D_{12}\dfrac{\partial^2 W}{\partial y^2}\right),\ M_y=-\left(D_2\dfrac{\partial^2 W}{\partial y^2}+D_{12}\dfrac{\partial^2 W}{\partial x^2}\right), \\ V_x=-\left[D_1\dfrac{\partial^3 W}{\partial x^3}+(D_{12}+4D_{66})\dfrac{\partial^3 W}{\partial x\partial y^2}\right], \\ V_y=-\left[D_2\dfrac{\partial^3 W}{\partial y^3}+(D_{12}+4D_{66})\dfrac{\partial^3 W}{\partial x^2\partial y}\right]. \end{cases} \tag{5}$$

假设 $\theta=\partial W/\partial y$, 并结合关系式(5),方程(3)可转化为如下 Hamilton 系统:

$$\frac{\partial \boldsymbol{U}(x,y)}{\partial y}=\boldsymbol{H}\boldsymbol{U}(x,y), \tag{6}$$

其中 $\boldsymbol{H}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{D_{12}}{D_2}\dfrac{\partial^2}{\partial x^2} & 0 & 0 & -\dfrac{1}{D_2} \\ \rho h\omega^2-\left(D_1-\dfrac{D_{12}^{\ 2}}{D_2}\right)\dfrac{\partial^4}{\partial x^4} & 0 & 0 & \dfrac{D_{12}}{D_2}\dfrac{\partial^2}{\partial x^2} \\ 0 & 4D_{66}\dfrac{\partial^2}{\partial x^2} & -1 & 0 \end{bmatrix}$, $\boldsymbol{U}(x,y)=\begin{bmatrix} W \\ \theta \\ -V_y \\ M_y \end{bmatrix}$.

通过计算可以验证上式中 4×4 的算子矩阵 $\boldsymbol{H}$ 满足 $\langle \boldsymbol{x},\boldsymbol{H}\boldsymbol{y}\rangle=\langle \boldsymbol{y},\boldsymbol{H}\boldsymbol{x}\rangle$,这里 $\langle \boldsymbol{x},\boldsymbol{H}\boldsymbol{y}\rangle=\int_0^b \boldsymbol{x}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{H}\boldsymbol{y}\mathrm{d}y$,$\boldsymbol{x}$ 和 $\boldsymbol{y}$ 为简支条件下的全状态向量,即简支条件下满足 Hamilton 系统(6)的向量; $\boldsymbol{J}=\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_2 \\ -\boldsymbol{I}_2 & \boldsymbol{0} \end{bmatrix}$ 是辛矩阵,其中 $\boldsymbol{I}_2$ 是 2 阶单位矩阵.因此 $\boldsymbol{H}$ 是 Hamilton 算子矩阵,从而式(6)为薄板方程(3)的 Hamilton 系统.

2 本征值和本征函数系

Hamilton 系统式(6)可以通过分离变量法求解,为此令

$$\boldsymbol{U}(x,y)=\boldsymbol{X}(x)Y(y). \tag{7}$$

将式(7)代入式(6),可得到

$$\frac{\boldsymbol{H}\boldsymbol{X}(x)}{\boldsymbol{X}(x)}=\frac{Y'(y)}{Y(y)}=\mu, \tag{8}$$

由式(8)可得

$$\boldsymbol{H}\boldsymbol{X}(x)=\mu\boldsymbol{X}(x), \tag{9}$$

$$\frac{\mathrm{d}Y(y)}{\mathrm{d}y}=\mu Y(y), \tag{10}$$

其中 μ 为本征值,$\boldsymbol{X}(x)$ 为相应的本征函数,记 $\boldsymbol{X}(x)=[X_1(x),X_2(x),X_3(x),X_4(x)]^{\mathrm{T}}$.式(9)可改写为

$$(\boldsymbol{H}-\mu\boldsymbol{I})\boldsymbol{X}(x)=\boldsymbol{0}, \tag{11}$$

其中 $\boldsymbol{I}$ 为 4 阶单位矩阵.

在 x 方向上,对边简支条件为

$$W|_{x=0}=M_x|_{x=0}=0,\ W|_{x=a}=M_x|_{x=a}=0. \tag{12}$$

由方程(11)和边界条件(12)可求得对边简支边界条件下的 4 组本征值 μ_{ni}($n=1,2,3,\cdots;i=1,2,3,4$):

$$\mu_{n1}=-\sqrt{\frac{\alpha_n^2 D_3-\sqrt{Q_1}}{D_2}},\ \mu_{n2}=-\mu_{n1},\ \mu_{n3}=-\sqrt{\frac{\alpha_n^2 D_3+\sqrt{Q_1}}{D_2}},\ \mu_{n4}=-\mu_{n3},\tag{13}$$

其中 $\alpha_n=\dfrac{n\pi}{a}, Q_1=-D_2(-\rho h\omega^2+\alpha_n^4 D_1)+\alpha_n^4 D_3^2$.

计算可得相应的本征函数为

$$\boldsymbol{X}_{ni}(x)=[1,\ \mu_{ni},\ \mu[\mu^2 D_2-(D_{12}+4D_{66})\alpha_{n^2}],\ -\mu^2 D_2+D_{12}\alpha_n^2]^{\mathrm{T}}\sin(\alpha_n x),\tag{14}$$

其中 $n=1,2,3,\cdots;\ i=1,2,3,4$.

3　子振动问题的解

通过符号计算可以证明本征函数系(14)的辛正交性和完备性[18].再根据辛本征函数展开法[19],我们可以假设在对边简支条件(12)下 Hamilton 系统(6)的通解为

$$\boldsymbol{U}(x,y)=\sum_{n=1}^{\infty}[Y_{n1}(y)\boldsymbol{X}_{n1}(x)+Y_{n2}(y)\boldsymbol{X}_{n2}(x)+Y_{n3}(y)\boldsymbol{X}_{n3}(x)+Y_{n4}(y)\boldsymbol{X}_{n4}(x)].\tag{15}$$

将式(15)代入式(6),计算可得

$$Y_{ni}(y)=C_{ni}\mathrm{e}^{\mu_{ni}y},\tag{16}$$

这里 $C_{ni}(i=1,2,3,4;\ n=1,2,3,\cdots)$ 为待定常数.

将式(16)代回通解(15),并取第一分量,可得对边简支条件(12)下正交各向异性矩形薄板振动问题的通解:

$$W(x,y)=\sum_{n=1}^{\infty}[C_{n1}\mathrm{e}^{\mu_{n1}y}\boldsymbol{X}_{n1}(x)+C_{n2}\mathrm{e}^{\mu_{n2}y}\boldsymbol{X}_{n2}(x)+C_{n3}\mathrm{e}^{\mu_{n3}y}\boldsymbol{X}_{n3}(x)+C_{n4}\mathrm{e}^{\mu_{n4}y}\boldsymbol{X}_{n4}(x)].\tag{17}$$

4　辛叠加解析解

如图 1 所示,一角点支撑对边两边固支正交各向异性矩形薄板的振动问题(图 1(a))可以分解为如下两个子问题[16]:

(Ⅰ) 在 $x=0$ 和 $x=a$ 边简支,在 $y=0$ 和 $y=b$ 边满足如下边界条件(图 1(b))

$$\begin{cases}M_y|_{y=0}=0,\ W|_{y=0}=\displaystyle\sum_{n=1}^{\infty}E_n\sin(\alpha_n x),\\ W|_{y=b}=0,\ M_y|_{y=b}=\displaystyle\sum_{n=1}^{\infty}F_n\sin(\alpha_n x);\end{cases}\tag{18}$$

(Ⅱ) 在 $y=0$ 和 $y=b$ 边简支,在 $x=0$ 和 $x=a$ 边满足如下边界条件(图 1(c))

$$\begin{cases}M_x|_{x=0}=0,\ W|_{x=0}=\displaystyle\sum_{n=1}^{\infty}G_n\sin(\beta_n y),\\ W|_{x=a}=0,\ M_x|_{x=a}=\displaystyle\sum_{n=1}^{\infty}H_n\sin(\beta_n y),\end{cases}\tag{19}$$

其中 $\beta_n=n\pi/b,\ n=1,2,3,\cdots$.

将通解(17)代入子问题(Ⅰ)的边界条件(18),可得到子问题(Ⅰ)的基本解:

$$\begin{aligned}W_1(x,y)=\sum_{n=1}^{\infty}\frac{1}{D_2(\mu_{n1}^2-\mu_{n3}^2)}\sin(\alpha_n x)\{F_n[-\operatorname{csch}(b\mu_{n1})\sinh(y\mu_{n1})+\\ \operatorname{csch}(b\mu_{n3})\sinh(y\mu_{n3})]+E_n[\operatorname{csch}(b\mu_{n3})\sinh(b\mu_{n3}-y\mu_{n3})(-D_{12}\alpha_n^2+D_2\mu_{n1}^2)+\\ \operatorname{csch}(b\mu_{n1})\sinh(b\mu_{n1}-y\mu_{n1})(D_{12}\alpha_n^2-D_2\mu_{n3}^2)]\}.\end{aligned}\tag{20}$$

与求解子问题(Ⅰ)类似,可求得子问题(Ⅱ)的基本解:

$$W_2(x,y)=\sum_{n=1}^{\infty}\frac{1}{D_1(\xi_{n1}^2-\xi_{n3}^2)}\sin(\beta_n y)\{H_n[-\operatorname{csch}(a\xi_{n1})\sinh(x\xi_{n1})+$$

$$\operatorname{csch}(a\xi_{n3})\sinh(x\xi_{n3})] + G_n[\operatorname{csch}(a\xi_{n3})\sinh(a\xi_{n3} - x\xi_{n3})(-D_{12}\beta_n^2 + D_1\xi_{n1}^2) + \operatorname{csch}(a\xi_{n1})\sinh(a\xi_{n1} - x\xi_{n1})(D_{12}\beta_n^2 - D_1\xi_{n3}^2)]\}, \tag{21}$$

其中

$$\xi_{n1} = -\sqrt{\frac{\beta_n^2 D_3 - \sqrt{Q_2}}{D_1}},\ \xi_{n2} = -\xi_{n1},\ \xi_{n3} = -\sqrt{\frac{\beta_n^2 D_3 + \sqrt{Q_2}}{D_1}},\ \xi_{n4} = -\xi_{n3},$$

这里 $Q_2 = -D_1(-\rho h\omega^2 + \beta_n^4 D_2) + \beta_n^4 D_3^2$.

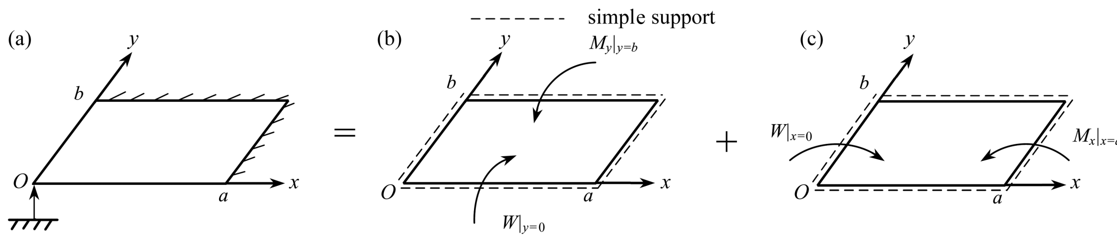

图 1 一角点支撑对边两边固支矩形薄板辛叠加解结构图

Fig. 1 Schematic of the symplectic superposition solution of the rectangular thin plate point-supported at a corner and clamped at its opposite edges

为了满足一角点支撑对边两边固支的边界条件,在边界 $y = 0$ 处两个子问题的等效剪力和弯矩之和应为零,即 $V_y|_{y=0} = 0$ 和 $M_y|_{y=0} = 0$, 计算可得

$$\frac{1}{\mu_{i1}^2 - \mu_{i3}^2}\{F_i[\operatorname{csch}(b\mu_{i1})\mu_{i1}^3 - \operatorname{csch}(b\mu_{i3})\mu_{i3}^3] + E_i[\coth(b\mu_{i3})(-D_{12}\alpha_i^2 + D_2\mu_{i1}^2)\mu_{i3}^3 + \coth(b\mu_{i1})\mu_{i1}^3(D_{12}\alpha_i^2 - D_2\mu_{i3}^2)] + (D_{12} + 4D_{66})/D_2\cdot\alpha_i^2[-F_i\operatorname{csch}(b\mu_{i1})\mu_{i1} + F_i\operatorname{csch}(b\mu_{i3})\mu_{i3} + E_i\coth(b\mu_{i3})(D_{12}\alpha_i^2 - D_2\mu_{i1}^2)\mu_{i3} + E_i\coth(b\mu_{i1})\mu_{i1}(-D_{12}\alpha_i^2 + D_2\mu_{i3}^2)]\} + \sum_{n=1}^{\infty}\frac{1}{aD_1(\alpha_i^2 + \xi_{n1}^2)(\alpha_i^2 + \xi_{n3}^2)}\{2\alpha_i\beta_n[-\cos(a\alpha_i)H_n(D_{12} + 4D_{66})\alpha_i^2 - \cos(a\alpha_i)H_nD_2\beta_n^2 - G_n\beta_n^2D_{12}^2\alpha_i^2 - G_n\beta_n^2D_{12}(4D_{66}\alpha_i^2 + D_2\beta_n^2) + G_n\beta_n^2D_1D_2(\alpha_i^2 + \xi_{n1}^2) + G_nD_1D_2\beta_n^2\xi_{n3}^2 - G_nD_1(D_{12} + 4D_{66})\xi_{n1}^2\xi_{n3}^2]\} = 0, \qquad i = 1,2,3,\cdots. \tag{22}$$

在边界 $y = b$ 处两个子问题的旋度之和应为零,即 $\partial W/\partial y|_{y=b} = 0$, 经计算可得

$$\frac{1}{D_2(\mu_{i1}^2 - \mu_{i3}^2)}\{F_i[-\coth(b\mu_{i1})\mu_{i1} + \coth(b\mu_{i3})\mu_{i3}] + E_i[\operatorname{csch}(b\mu_{i3})(D_{12}\alpha_i^2 - D_2\mu_{i1}^2)\mu_{i3} + \operatorname{csch}(b\mu_{i1})\mu_{i1}(-D_{12}\alpha_i^2 + D_2\mu_{i3}^2)]\} + \sum_{n=1}^{\infty}\frac{1}{aD_1(\alpha_i^2 + \xi_{n1}^2)(\alpha_i^2 + \xi_{n3}^2)}\{2\cos(b\beta_n)\alpha_i\beta_n[-\cos(a\alpha_i)H_n - G_nD_{12}\beta_n^2 + G_nD_1(\alpha_i^2 + \xi_{n1}^2 + \xi_{n3}^2)]\} = 0, \qquad i = 1,2,3,\cdots. \tag{23}$$

在边界 $x = 0$ 处两个子问题的等效剪力和弯矩之和应为零,即 $V_x|_{x=0} = 0$ 和 $M_x|_{x=0} = 0$, 经计算可得

$$\frac{1}{\xi_{i1}^2 - \xi_{i3}^2}\{H_i[\operatorname{csch}(a\xi_{i1})\xi_{i1}^3 - \operatorname{csch}(a\xi_{i3})\xi_{i3}^3] + G_i[\coth(a\xi_{i3})(-D_{12}\beta_i^2 + D_1\xi_{i1}^2)\xi_{i3}^3 + \coth(a\xi_{i1})\xi_{i1}^3(D_{12}\beta_i^2 - D_1\xi_{i3}^2)] + \frac{1}{D_1}(D_{12} + 4D_{66})\beta_i^2[-H_i\operatorname{csch}(a\xi_{i1})\xi_{i1} + H_i\operatorname{csch}(a\xi_{i3})\xi_{i3} + G_i\coth(a\xi_{i3})(D_{12}\beta_i^2 - D_1\xi_{i1}^2)\xi_{i3} + G_i\coth(a\xi_{i1})\xi_{i1}(-D_{12}\beta_i^2 + D_1\xi_{i3}^2)]\} + \sum_{n=1}^{\infty} -\frac{1}{bD_2(\beta_i^2 + \mu_{n1}^2)(\beta_i^2 + \mu_{n3}^2)}\{2\alpha_n\beta_i[\cos(b\beta_i)F_nD_1\alpha_n^2 + \cos(b\beta_i)F_n(D_{12} + 4D_{66})\beta_i^2 + E_n(D_{12} + 4D_{66})(D_{12}\alpha_n^2\beta_i^2 + D_2\mu_{n1}^2\mu_{n3}^2) + E_nD_1D_{12}\alpha_n^4 - E_nD_1D_2\alpha_n^2(\beta_i^2 + \mu_{n1}^2 + \mu_{n3}^2)]\} = 0, \qquad i = 1,2,3,\cdots. \tag{24}$$

在边界 $x = a$ 处两个子问题的旋度之和为零,即 $\partial W/\partial x|_{x=a} = 0$, 经计算可得

$$\frac{1}{D_1(\xi_{i1}^2 - \xi_{i3}^2)}\{H_i[-\coth(a\xi_{i1})\xi_{i1} + \coth(a\xi_{i3})\xi_{i3}] + G_i[\operatorname{csch}(a\xi_{i3})(D_{12}\beta_i^2 - D_1\xi_{i1}^2)\xi_{i3} + \operatorname{csch}(a\xi_{i1})\xi_{i1}(-D_{12}\beta_i^2 + D_1\xi_{i3}^2)]\} + \sum_{n=1}^{\infty} - \frac{1}{bD_2(\beta_i^2 + \mu_{n1}^2)(\beta_i^2 + \mu_{n3}^2)}[2\cos(a\alpha_n)\alpha_n\beta_i\cos(b\beta_i)F_n + 2\cos(a\alpha_n)\alpha_n\beta_i E_n D_{12}\alpha_n^2 - 2\cos(a\alpha_n)\alpha_n\beta_i E_n D_2(\beta_i^2 + \mu_{n1}^2 + \mu_{n3}^2)] = 0, \qquad i = 1,2,3,\cdots. \tag{25}$$

式(22)—(25)是关于 $E_m, F_m, G_n, H_n (m = 1,2,3,\cdots; n = 1,2,3,\cdots)$ 的齐次无穷方程组,为了使其有非平凡解,系数矩阵的行列式应该为零.显然齐次方程组(22)—(25)的系数矩阵的行列式是无穷阶的,实际计算中需要对无穷阶做截断处理,随着阶数的增加,可以获得更加精确的频率.为了计算方便,我们取 m 和 n 的值相等,并使方程个数与未知量个数相等,进而得到关于频率 ω 的方程.

在确定频率之后,将所得频率代回方程组(22)—(25),可以计算出相应的非平凡系数 E_m, F_m, G_n, H_n,将这些系数代入子问题的基本解(20)、(21),叠加后就可以得到辛叠加解:

$$W(x,y) = W_1(x,y) + W_2(x,y). \tag{26}$$

5 算　　例

例 1　计算一角点支撑对边两边固支的各向同性矩形薄板的振动频率, 方程(1)对应的参数分别取 $\nu_{12} = \nu_{21} = \nu = 0.3$, $D_1 = D_2 = D_3 = D$, $D_{12} = \nu D$, $D_{66} = D(1-\nu)/2$.我们应用辛叠加解(26)的前 120 项的和(即 n 取到 30)计算了不同长宽比情况下的前 8 阶频率,并与文献[16]对比, 具体数值结果见表 1(数值结果保留 5 位有效数字).

表 1　一角点支撑对边两边固支各向同性矩形薄板的频率参数 $\omega a^2\sqrt{\rho h/D}$

Table 1　Values of frequency parameter $\omega a^2\sqrt{\rho h/D}$ of the isotropic rectangular thin plates point-supported at a corner and clamped at its opposite edges

b/a	reference	mode							
		1st	2nd	3rd	4th	5th	6th	7th	8th
1.0	present	15.165	23.899	39.386	54.070	62.705	77.319	85.662	105.29
	ref. [16]	15.165	23.902	39.388	54.083	62.705	77.321	85.695	105.29
1.5	present	9.197 8	18.191	26.293	31.679	44.141	53.626	59.007	68.345
	ref. [16]	9.197 0	18.195	26.294	31.683	44.142	53.637	59.021	68.346
2	present	6.556 0	14.037	20.013	25.902	30.630	36.834	46.124	53.062
	ref. [16]	6.555 1	14.035	20.026	25.897	30.637	36.829	46.133	53.048
2.5	present	5.355 7	10.465	16.927	21.726	24.924	30.121	33.694	40.362
	ref. [16]	5.355 0	10.461	16.933	21.735	24.931	30.105	33.711	40.342
3	present	4.735 6	8.287 4	13.647	18.634	23.142	24.580	28.504	33.019
	ref. [16]	4.734 9	8.283 6	13.642	18.657	23.142	24.583	28.512	32.994
3.5	present	4.379 8	6.954 0	11.056	16.006	19.980	23.211	25.428	27.318
	ref. [16]	4.379 3	6.950 5	11.048	16.009	20.015	23.206	25.430	27.317
4	present	4.158 7	6.096 0	9.260 3	13.431	17.661	21.293	23.040	25.384
	ref. [16]	4.158 2	6.092 9	9.252 7	13.422	17.687	21.319	23.047	25.369
4.5	present	4.012 5	5.517 2	8.009 7	11.399	15.373	18.887	22.411	23.053
	ref. [16]	4.012 0	5.514 4	8.002 6	11.388	15.371	18.939	22.415	23.058
5	present	3.910 9	5.110 6	7.115 0	9.882 7	13.296	16.893	19.972	22.572
	ref. [16]	3.910 4	5.108 1	7.108 4	9.871 1	13.284	16.914	20.033	22.568

以下我们对应用辛叠加解(26)计算一角点支撑对边两边固支的正交各向异性矩形薄板的振动频率进行收敛性分析，方程(1)对应的参数分别取 $D_3 = 2D_1, D_2 = 2D_1$，$\nu_{21} = 0.3$.我们应用辛叠加解(26)研究了长宽比分别为 1 和 2 情况下的前 8 阶频率的收敛性，具体计算结果如表 2 所示(数值结果保留 5 位有效数字).

表 2 一角点支撑对边两边固支正交各向异性矩形薄板的收敛性分析 ($\omega a^2 \sqrt{\rho h/D_1}$)

Table 2 Convergence analysis of the orthotropic rectangular thin plates point-supported at a corner and clamped at its opposite edges ($\omega a^2 \sqrt{\rho h/D_1}$)

b/a	number of series terms	mode							
		1st	2nd	3rd	4th	5th	6th	7th	8th
1	50	18.929	31.932	50.967	65.823	81.800	99.774	114.65	124.19
	55	18.929	31.932	50.967	65.824	81.800	99.775	114.65	124.19
	60	18.929	31.933	50.967	65.824	81.800	99.775	114.65	124.19
	65	18.928	31.933	50.967	65.825	81.800	99.776	114.65	124.19
	70	18.928	31.933	50.967	65.825	81.800	99.776	114.65	124.19
	75	18.928	31.933	50.967	65.825	81.800	99.776	114.65	124.19
	80	18.928	31.933	50.967	65.825	81.800	99.777	114.65	124.19
	85	18.928	31.933	50.967	65.825	81.800	99.777	114.65	124.19
2	50	8.869 7	17.901	24.683	31.580	41.419	49.502	58.218	66.804
	55	8.869 6	17.901	24.683	31.580	41.419	49.502	58.220	66.804
	60	8.869 6	17.901	24.683	31.580	41.419	49.502	58.221	66.805
	65	8.869 5	17.901	24.684	31.580	41.419	49.502	58.222	66.805
	70	8.869 5	17.901	24.684	31.580	41.420	49.501	58.222	66.805
	75	8.869 4	17.901	24.684	31.580	41.420	49.501	58.223	66.805
	80	8.869 4	17.901	24.684	31.580	41.420	49.501	58.223	66.805
	85	8.869 4	17.901	24.684	31.580	41.420	49.501	58.223	66.805

例 2 计算一角点支撑对边两边固支的正交各向异性矩形薄板的振动频率，方程(1)对应的参数分别取 $D_3 = 2D_1, D_2 = 2D_1$，$\nu_{21} = 0.3$.应用辛叠加解(26)的前 300 项的和(即 n 取到 75)计算了不同长宽比情况下的前 8 阶频率，具体数值结果见表 3(数值结果保留 5 位有效数字).此外，将方形板 ($b/a = 1$) 前 8 阶频率所对应的模态以三维立体图的形式画出，如图 2 所示.

表 3 一角点支撑对边两边固支正交各向异性矩形薄板的频率参数 $\omega a^2 \sqrt{\rho h/D_1}$

Table 3 Values of frequency parameter $\omega a^2 \sqrt{\rho h/D_1}$ of the orthotropic rectangular thin plates point-supported at a corner and clamped at its opposite edges

b/a	reference	mode							
		1st	2nd	3rd	4th	5th	6th	7th	8th
1	present	18.928	31.933	50.967	65.825	81.800	99.776	114.65	124.19
1.5	present	12.483	21.692	33.851	42.947	55.604	67.520	75.576	81.878
2	present	8.869 4	17.901	24.684	31.580	41.420	49.501	58.223	66.805
2.5	present	7.011 4	14.240	20.654	26.545	31.025	38.707	46.258	53.475
3	present	5.968 4	11.404	17.931	22.735	26.481	32.009	36.404	43.731
3.5	present	5.329 5	9.499 4	15.130	20.180	24.293	26.740	31.518	36.249
4	present	4.910 4	8.204 4	12.804	17.934	21.977	24.366	28.082	30.721
4.5	present	4.620 8	7.291 0	11.059	15.659	19.907	23.216	24.836	27.953
5	present	4.412 4	6.623 4	9.757 3	13.696	17.948	21.468	23.445	25.887

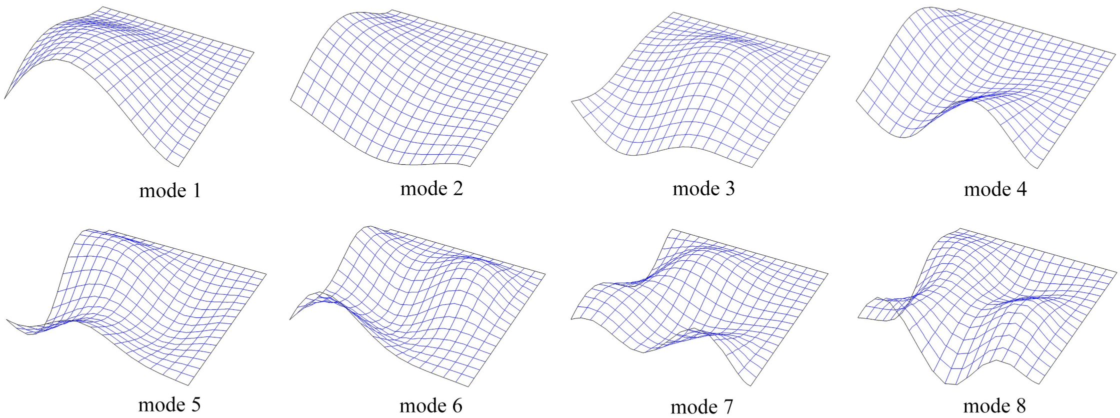

图 2　一角点支撑对边两边固支矩形薄板的前 8 阶模态

Fig. 2　The 1st 8 modal shapes of the orthotropic rectangular thin plates point-supported at a corner and clamped at its opposite edges

6　结　　论

本文应用辛叠加方法给出了一角点支撑对边两边固支正交各向异性矩形薄板振动问题的辛叠加解，并对该解计算正交各向异性矩形薄板的情形进行了收敛性分析.我们应用所得辛叠加解分别计算了各向同性矩形薄板和正交各向异性矩形薄板在不同长宽比值（b/a）下的振动频率，还以三维立体图形的形式给出了正交各向异性方形薄板的前 8 阶振动频率所对应的模态.

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