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# 非均匀磁场下 Maxwell 磁纳米流体的拉伸流动与磁扩散分析*

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**摘要：** 磁性纳米颗粒可以提升聚合物的导电性和导热性等性能,被广泛应用于机械、生物医学、能源存储等领域.当外界施加非均匀磁场时,感应磁场在高 Reynolds 数的情况下不可忽略.为探究磁性纳米颗粒对层流边界层内黏弹性流体非稳态拉伸流动与磁扩散的影响,将时间分布阶 Maxwell 本构方程与动量方程耦合,建立了二维不可压缩 Maxwell 磁纳米流体的速度与磁扩散偏微分方程组.采用有限差分法进行数值分析,通过控制磁性纳米颗粒种类、体积分数和磁参数大小,分析了流体的速度和感应磁场在边界层内的分布.研究发现:在熔融聚合物中添加 $Fe_2O_3$ 纳米颗粒后,流体的速度、感应磁场最大,速度和磁边界层的厚度最厚;Maxwell 纳米流体的松弛时间参数增大,速度与磁扩散均减小;另外,随着磁参数增大,流体的速度边界层厚度减小,磁边界层厚度增大;$Fe_3O_4$ 纳米颗粒的体积分数越大,流体流动越快,感应磁场越小.因此,非均匀磁场下在聚合物中添加磁性纳米颗粒的研究,为改善材料的性能给予了可参考的数据.

**关　键　词：** Maxwell 流体；　磁性纳米颗粒；　感应磁场；　数值差分格式
**中图分类号：** O357　　　**文献标志码：** A　　　DOI：10.21656/1000-0887.440164

# Stretching Flow and Magnetic Diffusion Analysis of Maxwell Magnetic Nanofluids in Non-Uniform Magnetic Fields

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**Abstract**: Magnetic nanoparticles can enhance the electrical and thermal conductivity of polymers, which are widely used in fields such as machinery, biomedicine, and energy storage. When a non-uniform magnetic field is imposed externally, the induced magnetic field cannot be ignored in the case of high Reynolds numbers. To

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explore the effects of magnetic nanoparticles on the unsteady flow and magnetic diffusion of viscoelastic fluid over the stretching sheet within the laminar boundary layer, the time distributed-order Maxwell constitutive equation was coupled with the momentum equation to establish partial differential equations for the velocity and magnetic diffusion of a 2D incompressible Maxwell magnetic nanofluid. Numerical analysis was performed with the finite difference method, and the velocity and the induced magnetic field distribution of the fluid within the boundary layer were analyzed by control of the magnetic nanoparticle type, the volume fraction and the magnetic parameter magnitude. The results show that, the velocity and induced magnetic field of the fluid are the largest in the case of $Fe_2O_3$ nanoparticles added to molten polymers, besides, the velocity and magnetic boundary layer thickness is the largest. With the increase of the Maxwell nanofluid relaxation time parameter, both the velocity and the magnetic diffusion will decrease. In addition, the velocity boundary layer thickness and the magnetic boundary layer thickness of the fluid decrease with the magnetic parameter. The larger the volume fraction of $Fe_3O_4$ nanoparticles is, the faster the fluid flow and the smaller the induced magnetic field will be. Therefore, the study of the addition of magnetic nanoparticles to polymers in non-uniform magnetic fields gives referential data for improving material properties.

**Key words**: Maxwell fluid; magnetic nanoparticle; induced magnetic field; numerical difference scheme

## 0　引　　言

聚合物是经聚合反应生成的高分子化合物,而熔融的聚合物在力的作用下可以流动,是一类同时具有黏性和弹性的黏弹性流体[1].在聚合物中加入纳米颗粒可有效改善聚合物的性能,提高其韧性和传热速率,在工业和生物医学等领域被广泛应用,增加了能源的利用率[2-3].磁性纳米颗粒具有磁性特征且具有纳米颗粒的独特效应,常见的磁性纳米颗粒有金属铁、钴、镍、金属氧化物 $Fe_2O_3$ 和 $Fe_3O_4$ 等,在熔融聚合物中加入微量的磁性纳米颗粒,可以使材料的各种性能如导电性、导热性、阻隔性等提高.Zainal 等[4]应用边界层理论研究了包含电磁流体动力学在拉伸板上复合纳米流体的非稳态常驻点流动.Sheikholeslami 等[5]研究了非均匀磁场对 $Fe_3O_4$-水基纳米流体强制对流换热的影响.

对磁性纳米流体施加外磁场,可以在非接触的条件下对其流动产生影响.这种非接触操控方式在实际操作中更便于实现,且操控范围广,不受 pH 值、离子强度、表面电荷和温度的影响[6].在多数实际情况下,外加磁场是可变的,近年来,很多研究探讨了非均匀磁场对纳米流体流动的影响.Sheikholeslami 等[7]研究了变磁场下磁场力对纳米流体传热的影响.Shaker 等[8]研究了非均匀磁场对磁性纳米流体在开腔通道内混合对流换热的影响,发现腔加热壁的上角附近产生了涡流.磁流体力学(MHD)是结合经典流体力学和电动力学的方法研究导电流体和磁场相互作用的学科[9],在天体物理、地球物理、宇航工程、电磁学以及工程技术中都有广泛的应用.Bég 等[10]研究了导电金属流体在感应磁场作用下的流动,发现增加磁参数会使速度提高、感应磁场降低.Hayat 等[11]采用有限差分格式数值计算了随时间变化的黏性纳米流体在感应磁场中的流动, 并讨论了 Brown 运动和热泳运动.但在现有的关于非均匀磁场的数值模拟研究中, 尚未清楚阐明感应磁场的物理机制.

考虑到黏弹性磁纳米流体在流动过程中的复杂特性[12],将分数阶导数引入本构关系能更灵活地描述黏弹性流体的性质.杨旭等[13]基于分数阶微积分理论,采用空间分数阶导数建立了圆管内分数阶非 Newton 流体本构模型,为非 Newton 流体的记忆特征提供了一种建模方法.Zhao 等[14]通过在本构关系中引入分数阶 Maxwell 剪切应力和 Cattaneo 热流模型,研究了 Maxwell 流体在平面上的非稳态 Marangoni 对流换热.然而上述模型中分数阶导数参数固定,导致了有限的记忆特性和非局部特性,且无法准确描述一些复杂的动力学过程,例如复合材料的流变特性[15].分布阶本构模型是分数阶导数在参数值范围内的积分,具有不同的时间和空间特征[16],作为更有效的工具引起了很多学者的关注.Yang 等[17]建立了空间分布阶本构关系来研究边界层中的流动和传热.Long 等[18]基于 Maxwell 流体流动和 Cattaneo 传热的传统本构关系,建立了分布阶导数的

非稳态 Marangoni 对流边界层流动和传热模型.Liu 等[19]将分布阶导数引入 Maxwell 流体的本构模型,并分析了相关参数的影响.

基于上述研究发现,在感应磁场作用下,非稳态时间分布阶的 Maxwell 磁纳米流体流动的问题很少有人研究.本文将时间分布阶 Maxwell 本构关系代入动量方程,并与磁扩散方程建立流动和感应磁场的控制方程,然后结合有限差分方法与 L1 算法获得控制方程的数值解,最后分析相关参数对流动和感应磁场的影响.

# 1 数学模型

时间分布阶 Maxwell 流体的本构方程为

$$\sigma_{xy} + \int_0^1 \omega_1(\alpha)\lambda_1^\alpha \frac{\partial^\alpha \sigma_{xy}}{\partial t^\alpha}\mathrm{d}\alpha = \mu\frac{\partial u}{\partial y}, \tag{1}$$

其中, $\sigma_{xy}$ 是随时间变化的剪切应力,$\lambda_1$ 和 $\mu$ 分别指的是松弛时间参数和黏度,$\alpha$ 是分数阶导数参数,$\omega_1(\alpha)$ 是控制分数阶导数的权重系数且满足条件 $\omega_1(\alpha) \geqslant 0, \int_0^1 \omega_1(\alpha)\mathrm{d}\alpha < \infty$ . $\frac{\partial^\alpha}{\partial t^\alpha}$ 是 Caputo 定义的分数阶导数[20],如下所示:

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\eta)^{-\alpha}\frac{\partial f(\eta)}{\partial \eta}\mathrm{d}\eta, \qquad 0 < \alpha < 1,$$

其中, $\Gamma(\cdot)$ 为 Gamma 函数.

考虑线性拉伸板上的二维不可压缩非稳态 Maxwell 磁纳米流体的边界层流动问题,如图 1 所示.建立二维直角坐标系,其中 $x$ 轴与平板平行,$y$ 轴垂直于平板.施加非均匀磁场 $H_e = H_0(\cos(x/L)+1)$,假设$(u, v)$和$(H_1, H_2)$ 分别是沿着板和垂直于板的速度和感应磁场的分量,并假设感应磁场的法向分量 $H_2$ 在壁处消失、平行分量 $H_1$ 在边界层边缘接近给定值.则时间分布阶 Maxwell 磁纳米流体的边界层流动和磁扩散控制方程为

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \ \frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0, \tag{2}$$

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \int_0^1 \omega_1(\alpha)\lambda_1^\alpha \frac{\partial^{\alpha+1} u}{\partial t^{\alpha+1}}\mathrm{d}\alpha + \int_0^1 \omega_1(\alpha)\lambda_1^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\left(u\frac{\partial u}{\partial x}\right)\mathrm{d}\alpha + \int_0^1 \omega_1(\alpha)\lambda_1^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\left(v\frac{\partial u}{\partial y}\right)\mathrm{d}\alpha = \\
&\quad -\frac{\mu_e H_e}{4\pi\rho_{nf}}\frac{\mathrm{d}H_e}{\mathrm{d}x} + \frac{\mu_e}{4\pi\rho_{nf}}\int_0^1 \omega_1(\alpha)\lambda_1^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\left(H_1\frac{\partial H_1}{\partial x}\right)\mathrm{d}\alpha + \\
&\quad \frac{\mu_e}{4\pi\rho_{nf}}\int_0^1 \omega_1(\alpha)\lambda_1^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\left(H_2\frac{\partial H_1}{\partial y}\right)\mathrm{d}\alpha + \left(\frac{\mu_{nf}}{\rho_{nf}}\right)\frac{\partial^2 u}{\partial y^2},
\end{aligned} \tag{3}$$

$$\frac{\partial H_1}{\partial t} + u\frac{\partial H_1}{\partial x} + v\frac{\partial H_1}{\partial y} - H_2\frac{\partial u}{\partial y} - H_1\frac{\partial u}{\partial x} = \alpha_m\frac{\partial^2 H_1}{\partial y^2}. \tag{4}$$

满足下列初始条件和边界条件:

$$\begin{cases}
t = 0: \quad u = 0, \ v = 0, \ H_1 = 0, \ H_2 = 0, \\
t > 0: \quad u = u_w = ax, \ v = 0, \ \dfrac{\partial H_1}{\partial y} = H_2 = 0, \qquad y = 0, \\
\qquad\quad u \to 0, \ H_1 \to H_e = H_0\left(\cos\dfrac{x}{L} + 1\right), \qquad y \to \infty.
\end{cases} \tag{5}$$

式(2)—(5)中,$\mu_{nf}$ 和 $\rho_{nf}$ 分别是纳米流体的黏度和密度,$\alpha_m = \frac{1}{4\pi\mu_e\sigma_{nf}}$ 是磁扩散系数,$\mu_e$ 和 $\sigma_{nf}$ 分别是磁导率和纳米流体的电阻率,$\alpha$ 为拉伸参数,$L$ 为特征长度(板长).

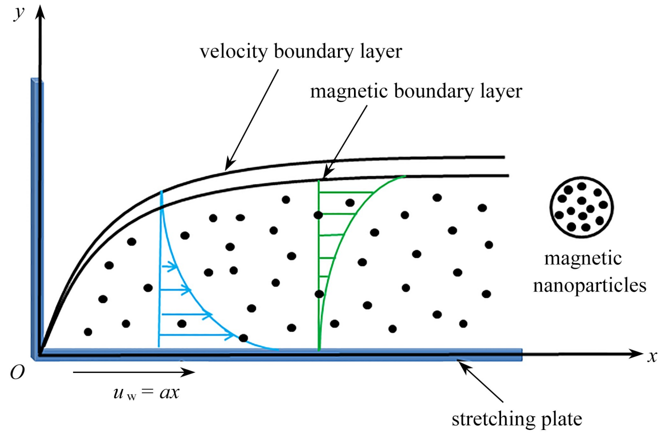


**图 1　物理模型示意图**

Fig. 1　Schematic diagram of the physical model

纳米流体的物理性质参数为

$$\mu_{nf}=\frac{\mu_f}{(1-\phi)^{2.5}},\ \rho_{nf}=\rho_f(1-\phi)+\rho_s\phi,\ \sigma_{nf}=\sigma_f\left[1+\frac{3\left(\frac{\sigma_s}{\sigma_f}-1\right)\phi}{\left(\frac{\sigma_s}{\sigma_f}+2\right)-\left(\frac{\sigma_s}{\sigma_f}-1\right)\phi}\right], \tag{6}$$

其中 $\phi$ 是磁性纳米颗粒的体积分数,$\mu_f$ 是流体的黏度,$\rho_f$ 和 $\rho_s$ 分别是流体和磁性纳米颗粒的密度,$\sigma_f$ 和 $\sigma_s$ 分别是流体和磁性纳米颗粒的电阻率.磁性纳米颗粒铁、钴、$Fe_2O_3$ 和 $Fe_3O_4$ 主要的物理性质如表 1 所示[21-22].

**表 1　磁性纳米颗粒的物理性质**

Table 1　Physical properties of magnetic nanoparticles

| | $\rho$ /(kg/m$^3$) | $\sigma$ /($\Omega$·m)$^{-1}$ |
|---|---|---|
| $Fe_3O_4$ | 5 200 | 25 000 |
| $Fe_2O_3$ | 5 180 | $10^{-5.99}$ |
| Fe | 7 870 | $9.93\times10^6$ |
| Co | 8 900 | $6.24\times10^6$ |

对方程(2)—(5)进行无量纲化:

$$\begin{cases} u^*=\dfrac{u}{aL},\ v^*=\dfrac{v}{aL\sqrt{Re}},\ x^*=\dfrac{x}{L},\ y^*=\dfrac{y}{L\sqrt{Re}},\ t^*=ta,\ H_1^*=\dfrac{H_1}{H_0}, \\ H_2^*=\dfrac{H_2}{H_0\sqrt{Re}},\ \lambda_1^*=\lambda_1 a,\ \lambda_2^*=\lambda_2 a,\ M=\dfrac{\mu_e H_0^2}{4\pi\rho_f a^2L^2},\ Pr_m=\dfrac{\mu_f}{\rho_f\alpha_{mf}}, \end{cases} \tag{7}$$

其中 $M$ 是磁参数,$Pr$ 是磁 Prandtl 数, $Re$ 是 Reynolds 数,得到无量纲控制方程如下(为了方便,后面将省略标记“ * ”):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\ \frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0, \tag{8}$$

$$\frac{\partial u}{\partial t}+\int_0^1\omega_1(\alpha)\lambda_1^\alpha\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}\mathrm{d}\alpha+u\frac{\partial u}{\partial x}+\int_0^1\omega_1(\alpha)\lambda_1^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\left(u\frac{\partial u}{\partial x}\right)\mathrm{d}\alpha+$$
$$v\frac{\partial u}{\partial y}+\int_0^1\omega_1(\alpha)\lambda_1^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\left(v\frac{\partial u}{\partial y}\right)\mathrm{d}\alpha=$$
$$M\frac{\rho_f}{\rho_{nf}}(\cos x+1)\sin x+M\frac{\rho_f}{\rho_{nf}}H_1\frac{\partial H_1}{\partial x}+M\frac{\rho_f}{\rho_{nf}}H_2\frac{\partial H_1}{\partial y}+\frac{1}{(1-\phi)^{2.5}}\frac{\rho_f}{\rho_{nf}}\frac{\partial^2 u}{\partial y^2}+$$

$$M\frac{\rho_{\mathrm{f}}}{\rho_{\mathrm{nf}}}\int_0^1\omega_1(\alpha)\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(H_2\frac{\partial H_1}{\partial y}\right)\mathrm{d}\alpha+M\frac{\rho_{\mathrm{f}}}{\rho_{\mathrm{nf}}}\int_0^1\omega_1(\alpha)\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(H_1\frac{\partial H_1}{\partial x}\right)\mathrm{d}\alpha, \tag{9}$$

$$\frac{\partial H_1}{\partial t}+u\frac{\partial H_1}{\partial x}+v\frac{\partial H_1}{\partial y}-H_1\frac{\partial u}{\partial x}-H_2\frac{\partial u}{\partial y}=\frac{1}{Pr_{\mathrm{m}}}\frac{\sigma_{\mathrm{f}}}{\sigma_{\mathrm{nf}}}\frac{\partial^2 H_1}{\partial y^2}. \tag{10}$$

无量纲化后的初值和边界条件为

$$\begin{cases} t=0: \quad u=0,\ v=0,\ H_1=0,\ H_2=0, \\ t>0: \quad u=x,\ v=0,\ \dfrac{\partial H_1}{\partial y}=H_2=0, \qquad y=0, \\ \qquad\qquad u\to 0,\ H_1\to\cos x+1, \qquad y\to\infty. \end{cases} \tag{11}$$

因此,建立了二维非稳态 Maxwell 磁纳米流体的控制方程(8)—(11),方程(11)为相应的初始条件和边界条件.

## 2 数值差分格式

采用有限差分法与 L1 算法[23]相结合来求解耦合的二维分布阶控制方程(8)—(11).对时间和空间进行网格划分,设 $h_x$, $h_y$ 分别为沿 $x$ 轴和 $y$ 轴的空间步长,$\tau$ 为时间步长.定义

$$\begin{cases} x_i=ih_x, & i=0,1,2,\cdots,M_x, \\ y_j=jh_y, & j=0,1,2,\cdots,M_y, \\ t_k=k\tau, & k=0,1,2,\cdots,N, \end{cases}$$

其中 $M_x,M_y,N$ 是网格划分的数量.

**定理 1** 设 $0<\alpha<1$, $f(x)=C^n[a,b]$, $n=\lceil\alpha\rceil$, 则 Caputo 分数阶导数为[24]

$$ {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha}f(t_n)=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}\frac{f'(t)\mathrm{d}\tau}{(t_n-\tau)^{\alpha}}=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\frac{f'(t_n-\tau)\mathrm{d}\tau}{\tau^{\alpha}}\approx $$

$$\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n-1}\frac{f(t_n-t_j)-f(t_n-t_{j+1})}{h}\int_{t_j}^{t_{j+1}}\tau^{-\alpha}\mathrm{d}\tau=$$

$$\frac{h^{-\alpha}}{\Gamma(1-\alpha)}\sum_{j=0}^{n-1}(f_{n-j}-f_{n-j-1})[(j+1)^{1-\alpha}-j^{1-\alpha}],$$

称之为 L1 算法,其基本思想是将被积函数中出现的 $f$ 的分数阶导数直接用数值微分公式逼近,还可改写为

$$ {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha}f(t_n)=\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}[f(t_{n-j})-f(t_{n-j-1})]+O(h^{2-\alpha})=$$

$$\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\left[f(t_n)-c_{n-1}f(t_0)-\sum_{j=1}^{n-1}(c_{j-1}-c_j)f(t_{n-j})\right]+O(h^{2-\alpha}), \tag{12}$$

其中 $c_j=(j+1)^{1-\alpha}-j^{1-\alpha}$, $j=0,1,2,\cdots,N$.

根据中点 Gauss 求积规则,使用多个分数项的加权和变换时间分布阶分数导数,可以获得分布项的数值离散格式:

$$\begin{cases} \displaystyle\int_0^1\omega_1(\alpha)\lambda_1^{\alpha}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\mathrm{d}\alpha\approx h_{\alpha}\sum_{r=0}^{R_1-1}\omega_1(\alpha_r)\lambda_1^{\alpha_r}\frac{\partial^{\alpha_r}u}{\partial t^{\alpha_r}}, \\ \displaystyle\int_0^1\omega_1(\alpha)\lambda_1^{\alpha}\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}\mathrm{d}\alpha\approx h_{\alpha}\sum_{r=0}^{R_1-1}\omega_1(\alpha_r)\lambda_1^{\alpha_r}\frac{\partial^{\alpha_r+1}u}{\partial t^{\alpha_r+1}}, \\ \displaystyle\int_0^1\omega_1(\alpha)\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(u\frac{\partial u}{\partial x}\right)\mathrm{d}\alpha\approx h_{\alpha}\sum_{r=0}^{R_1-1}\omega_1(\alpha_r)\lambda_1^{\alpha_r}\frac{\partial^{\alpha_r}}{\partial t^{\alpha_r}}\left(u\frac{\partial u}{\partial x}\right), \end{cases} \tag{13}$$

其中 $h_\alpha = \dfrac{1}{R_1}$ 表示区间[0,1]中分数阶 $\alpha_r$ 的步长,且 $\alpha_r = \dfrac{rh_\alpha + (r+1)h_\alpha}{2}$ $(r = 0,1,\cdots,R_1 - 1)$,其余项的离散格式 $v\dfrac{\partial u}{\partial y}, H_1\dfrac{\partial H_1}{\partial x}, H_2\dfrac{\partial H_1}{\partial y}$ 和方程(13)类似.

设在网格点 $(x_i, y_j, t_k)$ 处速度的精确解和数值解分别表示为 $u(x_i, y_j, t_k)$ 和 $u_{i,j}^k$,则对于控制方程中整数阶导数的离散格式分别采用向后差分和中心差分.

对于离散区域[0,1]中 $\alpha_r$ 阶的时间分数阶导数,利用定理 1 中的 L1 数值离散格式(13),可得到

$$\frac{\partial^{\alpha_r+1}u_{i,j}^k}{\partial t^{\alpha_r+1}} = \frac{\tau^{-\alpha_r-1}}{\Gamma(2-\alpha_r)}\left(u_{i,j}^k - u_{i,j}^{k-1} - c_{r,k-1}\tau\frac{\partial u_{i,j}^0}{\partial t}\right) - \frac{\tau^{-\alpha_r-1}}{\Gamma(2-\alpha_r)}\sum_{s=1}^{k-1}(c_{r,s-1} - c_{r,s})(u_{i,j}^{k-s} - u_{i,j}^{k-s-1}) + O(\tau), \tag{14}$$

$$\frac{\partial^{\alpha_r}}{\partial t^{\alpha_r}}\left(u_{i,j}^{k-1}\frac{\partial u_{i,j}^k}{\partial x}\right) = \frac{\tau^{-\alpha_r}u_{i,j}^{k-1}}{h_x\Gamma(2-\alpha_r)}(u_{i,j}^k - u_{i-1,j}^k) - \frac{\tau^{-\alpha_r}}{h_x\Gamma(2-\alpha_r)}\sum_{s=1}^{k-1}(c_{r,s-1} - c_{r,s})u_{i,j}^{k-s-1}(u_{i,j}^{k-s} - u_{i-1,j}^{k-s}) + O(\tau^{2-\alpha_r} + h_x), \tag{15}$$

其中

$$c_{r,0} = 1,\ c_{r,s} = (s+1)^{1-\alpha_r} - s^{1-\alpha_r},\ s = 0,1,\cdots,N.$$

将离散格式(12)、(14)和(15)代入控制方程(8)—(10)和初边值条件(11),得到求解问题的数值差分格式.

## 3 结果分析

利用第 2 节中的有限差分方法得到动量方程和磁扩散方程的数值解,并说明相关参数对速度和感应磁场的影响.构造解析解验证了差分格式的收敛性,如图 2 所示.

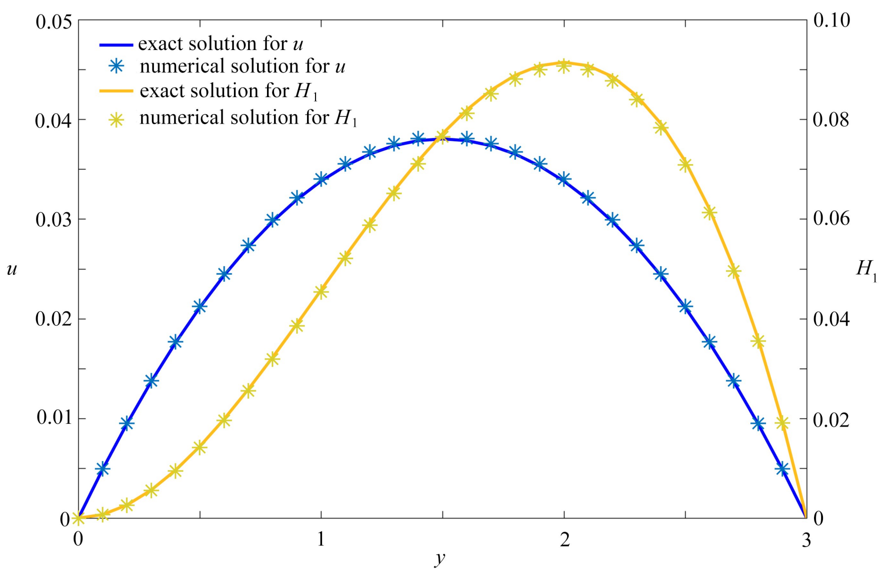


**图 2**　数值解和解析解的比较

Fig. 2　Comparison between numerical and analytical solutions

图 3 和图 4 描绘了磁性纳米颗粒 $Fe_2O_3$、$Fe_3O_4$、Fe 和 Co 对速度和感应磁场的影响.从图 3 可以看出,密度相近的 $Fe_2O_3$和 $Fe_3O_4$纳米颗粒对速度的影响几乎相同,而密度最大的 Co 使流速受到最大限制,速度边界层厚度最薄.从物理角度看,当磁性纳米颗粒的体积分数相同时,密度越大,颗粒间的间隙越小,从而阻碍了

流体的运动.图 4 表明,添加的 $Fe_2O_3$ 纳米颗粒电阻率最小,则感应磁场最大,磁边界层厚度最厚;对于电阻率相近的 Fe 和 Co,感应磁场的分布几乎相同.由于磁性纳米颗粒的电阻率与电导率呈反比,所以当电阻率增大时,感应磁场减小.

图 5 和图 6 为变化的磁参数 $M$ 对 $Fe_3O_4$ 纳米流体速度和感应磁场的影响.图 5 中的曲线表明流动的模式一致,流速值随着磁参数的增加而降低.在非均匀磁场作用下,边界层内产生与流体运动方向相反的磁场力,所以当 $M > 0$ 时,磁场力对流体的速度起抑制作用,使得边界层厚度变薄.当 $M = 0$ 时,无磁场力,$Fe_3O_4$ 纳米流体速度由于内摩擦阻力作用逐渐减小.如图 6 所示,在边界层内感应磁场随 $M$ 的增大而增大.因为 $M$ 增大时磁导率增加,磁扩散效应显著,从而感应磁场增加.

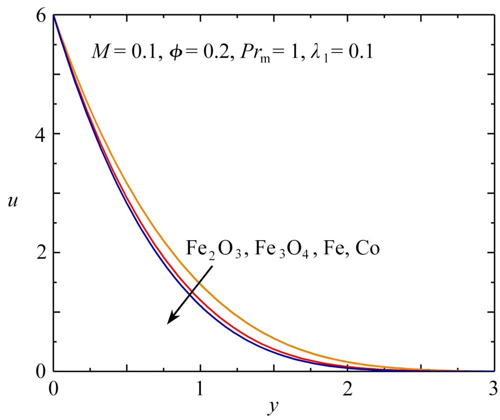


**图 3** 不同磁性纳米颗粒对速度的影响

Fig. 3 Effects of different magnetic nanoparticles on the velocity

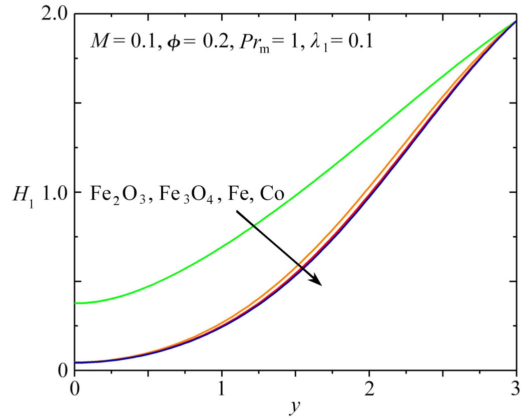


**图 4** 不同磁性纳米颗粒对感应磁场的影响

Fig. 4 Effects of different magnetic nanoparticles on the induced magnetic field

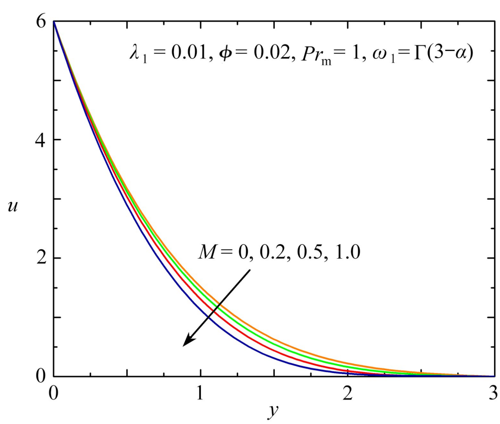


**图 5** 不同 $M$ 对速度的影响

Fig. 5 Effects of different $M$ values on the velocity

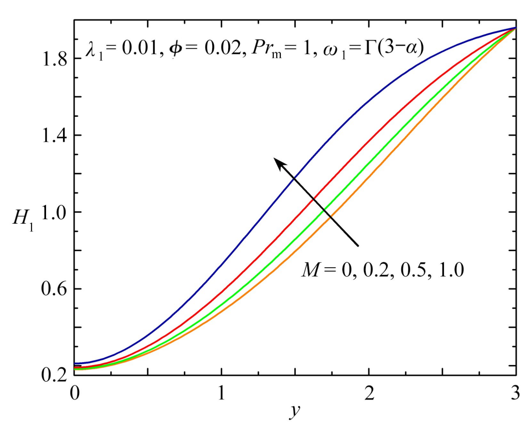


**图 6** 不同 $M$ 对感应磁场的影响

Fig. 6 Effects of different $M$ values on the induced magnetic field

图 7 和图 8 显示了速度和感应磁场随磁性纳米颗粒 $Fe_3O_4$ 体积分数 $\phi$ 的变化.当 $\phi$ 增加时,速度增大,边界层厚度变厚,感应磁场减小.由于拉伸板的运动导致固体颗粒运动增加,从而速度随着磁纳米颗粒的加入而增加.而在外加磁场的作用下,随着 $\phi$ 的增加,速度的增大导致磁对流项增大,使得磁扩散效应减小,从而在边界层内感应磁场减小.图 9 和图 10 显示了松弛时间参数 $\lambda_1$ 对 $Fe_3O_4$ 纳米流体中速度和感应磁场的影响.当选择权重系数 $\omega_1(\alpha) = \Gamma(3 - \alpha)$,无量纲时间 $t = 1$ 时,可以观察到 $\lambda_1$ 越大,速度和感应磁场越小,这是由于流体的黏性增加,阻碍了流体的运动进而导致磁场减小,这和应力松弛现象是一致的.

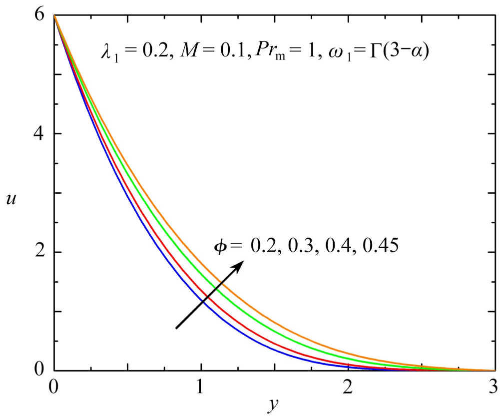


**图 7**　不同 $\phi$ 对速度的影响

Fig. 7　Effects of different $\phi$ values on the velocity

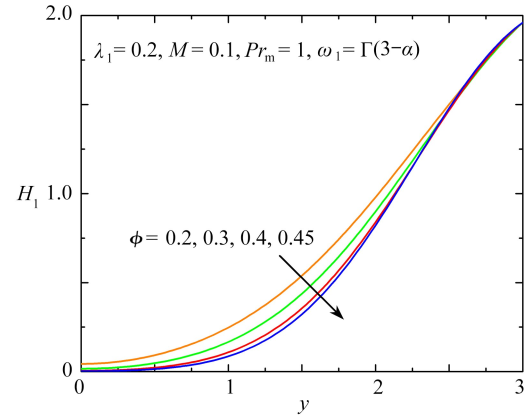


**图 8**　不同 $\phi$ 对感应磁场的影响

Fig. 8　Effects of different $\phi$ values on the induced magnetic field

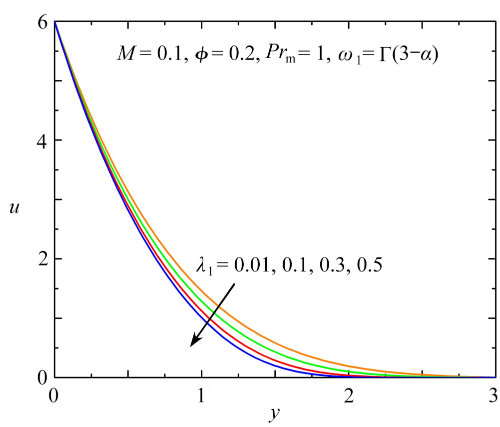


**图 9**　不同 $\lambda_1$ 对速度的影响

Fig. 9　Effects of different $\lambda_1$ values on the velocity

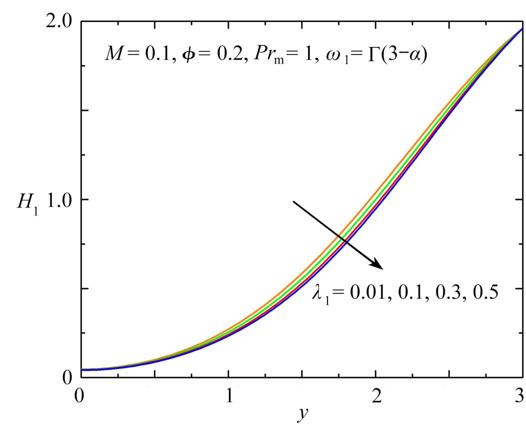


**图 10**　不同 $\lambda_1$ 对感应磁场的影响

Fig. 10　Effects of different $\lambda_1$ values on the induced magnetic field

## 4　结　　论

磁性纳米颗粒可以改善聚合物的性能,线性拉伸板上非稳态流动与磁扩散的研究为其提供了良好的理论基础.本文利用分布阶本构模型,研究了黏弹性基磁纳米流体在非均匀磁场下的二维不可压缩边界层流动,并考虑了高 Reynolds 数下的感应磁场分布.将时间分布阶 Maxwell 方程引入动量方程,建立了速度和磁扩散控制方程组.使用有限差分法和 L1 算法耦合求解偏微分方程组的数值解,并验证了收敛性,结合图像分析了相关物理参数对速度和感应磁场的影响.所得主要结论如下:Maxwell 流体的速度和感应磁场在添加 $Fe_2O_3$ 纳米颗粒时达到最大,此时速度边界层和磁边界层的厚度最厚;由于磁场力的影响,增大的磁参数阻碍了流体的运动,增强了磁扩散;随着 $Fe_3O_4$ 纳米流体体积分数增加,速度增大,感应磁场减小;此外,较高的松弛时间参数会使 Maxwell 流体的流动和磁场的扩散减弱.因此,本文从数值解的角度研究了添加磁性纳米颗粒的熔融聚合物在非均匀磁场下的边界层拉伸流动与磁扩散,对制备出高性能的聚合物材料具有重要意义.

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