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考虑挠曲电与温度效应的 Mindlin-Medick 板理论及其应用*

鲁　双，　李东波，　陈晶博，　席　勃

（西安建筑科技大学 理学院，西安 710055）

摘要： 基于 Hamilton 变分原理，推导了挠曲电纳米板的二维场方程和边界条件，然后将本构关系和几何关系代入场方程中，得到了相应的控制方程。研究了非均匀温度变化引起的挠曲电纳米板面内拉伸变形、厚度伸缩变形、对称厚度-剪切变形及其耦合的挠曲电极化。位移场和电势场用双重 Fourier 级数解求解。结果表明，所有场都对温度载荷敏感，这为利用温度场控制挠曲电纳米板的力学和电学行为提供了前景。对比分析了温度场和机械场对位移场的影响，拓展了考虑挠曲电效应和温度效应的 Mindlin-Medick 板结构分析理论，其可为微纳米尺度器件的结构设计提供参考。

关　键　词： Mindlin-Medick 理论；　挠曲电效应；　温度效应；　温度调控；　机械调控
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The Mindlin-Medick Plate Theory and Its Application Under Flexoelectricity and Temperature Effects

LU Shuang，　LI Dongbo，　CHEN Jingbo，　XI Bo
(*School of Science*, *Xi'an University of Architecture and Technology*, *Xi'an* 710055, *P.R.China*)

Abstract: Based on the Hamiltonian variational principle, the 2D field equations and boundary conditions for flexoelectricity were derived, and the corresponding governing equations were obtained through substitution of the constitutive relation and geometric equations into the field equation. The in-plane tensile deformation, thickness-stretch deformation, symmetric thickness-shear deformation, and their coupled flexoelectric polarization of flexoelectric nanoplates caused by inhomogeneous temperature changes, were studied. The displacement fields and electric potential fields were solved with the double Fourier series method. The results demonstrate that, all fields are sensitive to the temperature load, which raises the prospect of controlling the mechanical and electrical behaviors of flexoelectric nanoplates by means of the temperature field. The effects of the thermal field and mechanical field on the displacement field were compared and examined. The work extends the Mindlin-Medick plate structure analysis theory in view of the flexoelectric and temperature effects, and provides a reference for the structural design of micro- and nano-scale devices.

Key words: Mindlin-Medick theory; flexoelectric effect; temperature effect; temperature control; mechanical control

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作者简介： 鲁双（1998—），女，硕士生（E-mail：lushuang@xauat.edu.cn）；
李东波（1982—），男，教授，博士，博士生导师（通讯作者. E-mail：ldb@xauat.edu.cn）.
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0　引　　言

常见的力电耦合效应,如压电效应、挠曲电效应、铁电效应、电致伸缩效应等,广泛存在于各类介电材料中.其中压电效应是最常见的一种力电耦合效应,其在俘能器[1-4]、传感器[5-6]、驱动器[7-8]等智能器件的设计中应用广泛.然而,随着纳米技术的发展,压电器件的材料制约了其进一步发展,主要表现在 3 个方面: 1)压电效应只存在于非中心对称晶体中; 2)随着器件的小型化和智能化,压电理论已经不能很好地解释与材料或结构尺度相关的不寻常的力电耦合现象; 3)压电器件要求其服役温度低于材料的 Curie 温度,进一步限制了压电器件的使用环境.相比于压电效应,挠曲电效应存在于所有的介电材料中[9],且随着材料或结构的尺寸减小,挠曲电效应会变得更加显著[10].同时,由挠曲电材料制成的智能器件,其工作温度不受 Curie 温度影响.因此,挠曲电效应受到了研究人员的关注.

经典压电学理论给出了极化与均匀应变的关系,而挠曲电效应则描述了极化与非均匀应变如应变梯度之间的耦合关系[11].1968 年,Mindlin 首次提出极化梯度的概念[12],成功将力电耦合效应从压电材料拓展到挠曲电材料.在 Mindlin 理论的基础上,Majdoub 等[10,13]考虑了纳米悬臂梁对应变梯度的响应,发现挠曲电效应可以显著增强纳米结构中的俘能效率,并对纳米结构中的压电和弹性行为产生影响.Hu 和 Shen[14-16]提出了一种连续介质力学力电耦合理论框架,更全面地考虑了纳米电介质的挠曲电效应、表面效应和静电力,给出了详细的控制方程和边界条件,为挠曲电效应的研究提供了理论基础.关于挠曲电效应较新的综述研究可参考文献[11],其提供了更多与挠曲电效应相关的参考文献.

对挠曲电器件变形和电场行为的调控包括接触式和非接触式,其中机械调控[9,17-18]属于接触式调控,磁场[19-21]和温度场[22-26]等调控属于非接触式调控.机械调控利用施加于器件上的机械力产生应变梯度,从而产生电极化;而温度场调控则是利用热弹性效应产生应变梯度,进而产生电极化.温度效应广泛存在于挠曲电器件中,并通过热弹性效应影响其性能.Hadjesfandiari[23]通过引入高阶应变梯度,建立了非均质各向异性固体中尺寸相关的热弹性方程.Samani 等[24]利用挠曲电效应和纳米梁尺寸效应的非经典理论,研究了挠曲电 Timoshenko 梁在热场和机械场作用下的屈曲行为.Chu 等[25]综合考虑非局域效应和应变梯度效应,分析了功能梯度挠曲电纳米梁在温度场作用下的热致非线性动力学问题.前人研究的主要是一维问题,目前关于温度效应的挠曲电二维问题的成果相对较少.

近来,随着微纳米尺度二维材料的快速发展,挠曲电纳米板的器件应用也越来越广泛,例如挠曲电传感器、致动器等.本文基于挠曲电理论[27-28]和温度效应建立了 Mindlin-Medick 板的理论模型,综合考虑厚度伸缩变形、面内拉伸变形和对称厚度剪切变形及其耦合的挠曲电极化,分别研究了温度调控和机械调控下挠曲电纳米板的力电耦合行为,以期为挠曲电器件的设计提供参考.

1　挠曲电纳米板的数学框架

1.1　挠曲电纳米板的几何方程

考虑如图 1 所示的挠曲电纳米板,直角坐标系建立在板的中面上.根据 Mindlin-Medick 假设,将挠曲电纳米板的位移场 $u_i(\boldsymbol{x},t)$、电势场 $\varphi(\boldsymbol{x},t)$ 和温差场 $\theta(\boldsymbol{x},t)$ 展开成关于厚度坐标 x_3 的幂级数:

$$\begin{cases}u_1(x_1,x_2,x_3,t)=u_1^{(0)}(x_1,x_2,t)+x_3^2u_1^{(2)}(x_1,x_2,t),\\ u_2(x_1,x_2,x_3,t)=u_2^{(0)}(x_1,x_2,t)+x_3^2u_2^{(2)}(x_1,x_2,t),\\ u_3(x_1,x_2,x_3,t)=x_3u_3^{(1)}(x_1,x_2,t),\\ \varphi(x_1,x_2,x_3,t)=\varphi^{(0)}(x_1,x_2,t)+x_3\varphi^{(1)}(x_1,x_2,t)+x_3^2\varphi^{(2)}(x_1,x_2,t),\\ \theta(x_1,x_2,x_3,t)=\theta^{(0)}(x_1,x_2,t)+x_3^2\theta^{(2)}(x_1,x_2,t),\end{cases}\tag{1}$$

式中, $u_1^{(0)}$ 和 $u_2^{(0)}$ 代表面内拉伸变形, $u_3^{(1)}$ 代表 x_1Ox_2 平面外的厚度伸缩变形, $u_1^{(2)}$ 和 $u_2^{(2)}$ 代表对称厚度剪切变形. $\varphi^{(0)}$, $\varphi^{(1)}$ 和 $\varphi^{(2)}$ 分别表示零阶、一阶和二阶电势.温度变化 $\theta^{(0)}$ 和 $\theta^{(2)}$ 为已知函数,分别表示挠曲电纳米板的面内温差和厚度方向的温差.式(1)中只保留了低阶项,该截断是无穷幂级数理论的一种特殊情况[29].对于温差的定义为:令给定的温度为 Θ, 单位为 K,参考温度为 Θ_0,则温差为 $\theta=\Theta-\Theta_0$.

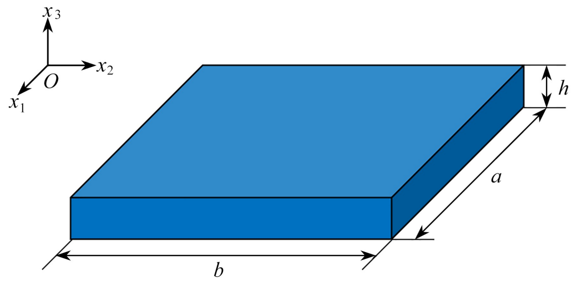


图 1 挠曲电纳米板模型及其坐标系

Fig. 1 The flexoelectric nanoplate model and its coordinate system

应变张量 S_{ij} 与位移 u_i、应变梯度 η_{ijk} 与应变 S_{ij} 以及电场 E_i 与电势 φ 之间满足如下梯度关系[30]:

$$S_{ij}=0.5(u_{i,j}+u_{j,i}),\ \eta_{ijk}=S_{ij,k},\ E_i=-\varphi_{,i}. \tag{2}$$

将式(1)代入式(2),得到相应的应变分量 S_{ij}、应变梯度分量 η_{ijk} 和电场分量 E_i 的非零项分别为

$$\begin{cases} S_{11}=u_{1,1}^{(0)}+x_3^2u_{1,1}^{(2)},\ S_{12}=0.5(u_{1,2}^{(0)}+x_3^2u_{1,2}^{(2)}+u_{2,1}^{(0)}+x_3^2u_{2,1}^{(2)}), \\ S_{13}=x_3u_1^{(2)}+0.5x_3u_{3,1}^{(1)},\ S_{22}=u_{2,2}^{(0)}+x_3^2u_{2,2}^{(2)},\ S_{23}=x_3u_2^{(2)}+0.5x_3u_{3,2}^{(1)},\ S_{33}=u_3^{(1)}, \end{cases} \tag{3}$$

$$\begin{cases} \eta_{111}=u_{1,11}^{(0)}+x_3^2u_{1,11}^{(2)},\ \eta_{112}=u_{1,12}^{(0)}+x_3^2u_{1,12}^{(2)},\ \eta_{113}=2x_3u_{1,1}^{(2)},\ \eta_{123}=x_3u_{1,2}^{(2)}+x_3u_{2,1}^{(2)}, \\ \eta_{121}=0.5(u_{1,21}^{(0)}+x_3^2u_{1,21}^{(2)}+u_{2,11}^{(0)}+x_3^2u_{2,11}^{(2)}),\ \eta_{122}=0.5(u_{1,22}^{(0)}+x_3^2u_{1,22}^{(2)}+u_{2,12}^{(0)}+x_3^2u_{2,12}^{(2)}), \\ \eta_{131}=x_3u_{1,1}^{(2)}+0.5x_3u_{3,11}^{(1)},\ \eta_{133}=u_1^{(2)}+0.5u_{3,1}^{(1)},\ \eta_{132}=x_3u_{1,2}^{(2)}+0.5x_3u_{3,12}^{(1)}, \\ \eta_{221}=u_{2,21}^{(0)}+x_3^2u_{2,21}^{(2)},\ \eta_{222}=u_{2,22}^{(0)}+x_3^2u_{2,22}^{(2)},\ \eta_{223}=2x_3u_{2,2}^{(2)},\ \eta_{231}=x_3u_{2,1}^{(2)}+0.5x_3u_{3,21}^{(1)}, \\ \eta_{232}=x_3u_{2,2}^{(2)}+0.5x_3u_{3,22}^{(1)},\ \eta_{233}=u_2^{(2)}+0.5u_{3,2}^{(1)},\ \eta_{331}=u_{3,1}^{(1)},\ \eta_{332}=u_{3,2}^{(1)}, \end{cases} \tag{4}$$

$$E_1=-\varphi_{,1}^{(0)}-x_3\varphi_{,1}^{(1)}-x_3^2\varphi_{,1}^{(2)},\ E_2=-\varphi_{,2}^{(0)}-x_3\varphi_{,2}^{(1)}-x_3^2\varphi_{,2}^{(2)},\ E_3=-\varphi^{(1)}-2x_3\varphi^{(2)}. \tag{5}$$

1.2 挠曲电纳米板的 Hamilton 变分原理

挠曲电纳米板的应变能 U、外体力 f_i 所做的虚功 δW 和动能 K 分别为[31]

$$\delta U=\int_\Omega (T_{ij}\delta S_{ij}+\tau_{ijk}\delta\eta_{ijk}-D_i\delta E_i)\mathrm{d}V, \tag{6}$$

$$\delta W=\int_\Omega (f_i\delta u_i)\mathrm{d}V, \tag{7}$$

$$K=\frac{1}{2}\int_\Omega \rho\dot{u}_i^2\mathrm{d}V, \tag{8}$$

式中, T_{ij} 为应力张量,τ_{ijk} 为高阶应力张量,D_i 为电位移,Ω 为挠曲电纳米板所占据的体积,$\mathrm{d}V$ 为体积微元.

对于挠曲电纳米板,其 Hamilton 变分原理可表述为[32]

$$\delta\int_0^T (K-U)\mathrm{d}t+\int_0^T \delta W\mathrm{d}t=0. \tag{9}$$

将式(1)、(3)—(5)代入式(6)—(8),再代入到变分表达式(9),应用变分法基本原理[33]和分部积分,得到挠曲电纳米板的场方程以及板边界 Γ 上的线积分等式分别为

$$T_{11,1}^{(0)}+T_{12,2}^{(0)}-\tau_{111,11}^{(0)}-\tau_{112,21}^{(0)}-\tau_{121,12}^{(0)}-\tau_{122,22}^{(0)}+f_1^{(0)}=m^{(0)}\ddot{u}_1^{(0)}+m^{(2)}\ddot{u}_1^{(2)}, \tag{10a}$$

$$\begin{aligned} &-2T_{13}^{(1)}+T_{12,2}^{(2)}+T_{11,1}^{(2)}-\tau_{111,11}^{(2)}-\tau_{112,21}^{(2)}+2\tau_{113,1}^{(1)}-\tau_{121,12}^{(2)}-\tau_{122,22}^{(2)}+2\tau_{123,2}^{(1)}+2\tau_{131,1}^{(1)}+ \\ &\quad 2\tau_{132,2}^{(1)}-2\tau_{133}^{(0)}+f_1^{(2)}=m^{(2)}\ddot{u}_1^{(0)}+m^{(4)}\ddot{u}_1^{(2)}, \end{aligned} \tag{10b}$$

$$T_{12,1}^{(0)}+T_{22,2}^{(0)}-\tau_{121,11}^{(0)}-\tau_{122,21}^{(0)}-\tau_{221,12}^{(0)}-\tau_{222,22}^{(0)}+f_2^{(0)}=m^{(0)}\ddot{u}_2^{(0)}+m^{(2)}\ddot{u}_2^{(2)}, \tag{10c}$$

$$\begin{aligned} &T_{12,1}^{(2)}+T_{22,2}^{(2)}-2T_{23}^{(1)}-\tau_{121,11}^{(2)}-\tau_{122,21}^{(2)}+2\tau_{123,1}^{(1)}-\tau_{221,12}^{(2)}-\tau_{222,22}^{(2)}+2\tau_{223,2}^{(1)}+2\tau_{231,1}^{(1)}+ \\ &\quad 2\tau_{232,2}^{(1)}-2\tau_{233}^{(0)}+f_2^{(2)}=m^{(2)}\ddot{u}_2^{(0)}+m^{(4)}\ddot{u}_2^{(2)}, \end{aligned} \tag{10d}$$

$$\begin{aligned} &T_{13,1}^{(1)}+T_{23,2}^{(1)}-T_{33}^{(0)}-\tau_{131,11}^{(1)}-\tau_{132,21}^{(1)}+\tau_{133,1}^{(0)}-\tau_{231,12}^{(1)}- \\ &\quad \tau_{232,22}^{(1)}+\tau_{233,2}^{(0)}+\tau_{331,1}^{(0)}+\tau_{332,2}^{(0)}+f_3^{(1)}=m^{(2)}\ddot{u}_3^{(1)}, \end{aligned} \tag{10e}$$

$$D_{1,1}^{(0)}+D_{2,2}^{(0)}=0, \tag{10f}$$

$$D_{1,1}^{(1)} + D_{2,2}^{(1)} - D_3^{(0)} = 0, \tag{10g}$$

$$D_{2,2}^{(2)} + D_{1,1}^{(2)} - 2D_3^{(1)} = 0, \tag{10h}$$

和

$$\begin{aligned}
&\int_0^T \oint_\Gamma [(-T_{11}^{(0)}n_1 - T_{12}^{(0)}n_2 + \tau_{111,1}^{(0)}n_1 + \tau_{112,2}^{(0)}n_1 + \tau_{121,1}^{(0)}n_2 + \tau_{122,2}^{(0)}n_2)\delta u_1^{(0)} + \\
&(-T_{11}^{(2)}n_1 - T_{12}^{(2)}n_2 - 2\tau_{113}^{(1)}n_1 + \tau_{111,1}^{(2)}n_1 + \tau_{121,1}^{(2)}n_2 + \tau_{112,2}^{(2)}n_1 + \tau_{122,2}^{(2)}n_2 - 2\tau_{123}^{(1)}n_2 - 2\tau_{132}^{(1)}n_2 - \\
&2\tau_{131}^{(1)}n_1)\delta u_1^{(2)} + (-T_{12}^{(0)}n_1 - T_{22}^{(0)}n_2 + \tau_{121,1}^{(0)}n_1 + \tau_{122,2}^{(0)}n_1 + \tau_{221,1}^{(0)}n_2 + \tau_{222,2}^{(0)}n_2)\delta u_2^{(0)} + \\
&(-T_{12}^{(2)}n_1 - T_{22}^{(2)}n_2 - 2\tau_{123}^{(1)}n_1 + \tau_{121,1}^{(2)}n_1 + \tau_{122,2}^{(2)}n_1 + \tau_{221,1}^{(2)}n_2 - 2\tau_{223}^{(1)}n_2 + \tau_{222,2}^{(2)}n_2 - 2\tau_{231}^{(1)}n_1 - \\
&2\tau_{232}^{(1)}n_2)\delta u_2^{(2)} + (-T_{13}^{(1)}n_1 - T_{23}^{(1)}n_2 - \tau_{133}^{(0)}n_1 + \tau_{131,1}^{(1)}n_1 + \tau_{132,2}^{(1)}n_1 + \tau_{231,1}^{(1)}n_2 + \tau_{232,2}^{(1)}n_2 - \\
&\tau_{233}^{(0)}n_2 - \tau_{331}^{(0)}n_1 - \tau_{332}^{(0)}n_2)\delta u_3^{(1)} - (\tau_{111}^{(0)}n_1 + \tau_{112}^{(0)}n_2)\delta u_{1,1}^{(0)} - (\tau_{111}^{(2)}n_1 + \tau_{112}^{(2)}n_2)\delta u_{1,1}^{(2)} - \\
&(\tau_{121}^{(0)}n_1 + \tau_{122}^{(0)}n_2)\delta u_{1,2}^{(0)} - (\tau_{122}^{(2)}n_2 + \tau_{121}^{(2)}n_1)\delta u_{1,2}^{(2)} - (\tau_{121}^{(0)}n_1 + \tau_{122}^{(0)}n_2)\delta u_{2,1}^{(0)} - \\
&(\tau_{121}^{(2)}n_1 + \tau_{122}^{(2)}n_2)\delta u_{2,1}^{(2)} - (\tau_{221}^{(0)}n_1 + \tau_{222}^{(0)}n_2)\delta u_{2,2}^{(0)} - (\tau_{221}^{(2)}n_1 + \tau_{222}^{(2)}n_2)\delta u_{2,2}^{(2)} - \\
&(\tau_{131}^{(1)}n_1 + \tau_{132}^{(1)}n_2)\delta u_{3,1}^{(1)} - (\tau_{231}^{(1)}n_1 + \tau_{232}^{(1)}n_2)\delta u_{3,2}^{(1)} - (D_1^{(0)}n_1 + D_2^{(0)}n_2)\delta\varphi^{(0)} - \\
&(D_1^{(1)}n_1 + D_2^{(1)}n_2)\delta\varphi^{(1)} - (D_1^{(2)}n_1 + D_2^{(2)}n_2)\delta\varphi^{(2)}]\mathrm{d}s\mathrm{d}t = 0,
\end{aligned} \tag{11}$$

式中, Γ 表示围成板中面的边缘曲线, ds 表示边缘曲线 Γ 上的线微元, 变量上的点表示对时间的导数. 其中 n 阶应力、n 阶电位移、n 阶高阶应力、n 阶外力和 n 阶质量密度的定义为

$$[T_{ij}^{(n)}, D_i^{(n)}, \tau_{ijk}^{(n)}, f_i^{(n)}, m^{(n)}] = \int_{-h/2}^{h/2} [(T_{ij}, D_i, \tau_{ijk}, f_i, \rho)x_3^n]\mathrm{d}x_3. \tag{12}$$

1.3　挠曲电纳米矩形板的完整边界条件

挠曲电纳米板的完整边界条件可由板边界上的线积分等式(11)得到, 注意到式(11)的变分项 δ(·) 中, 在边界上关于位移或者电势的切向导数不与其法向导数独立, 因此需要进一步处理变分项 δ(·) 中包含切向导数的项. 对于图 2 所示的挠曲电纳米矩形板, 将边界上外法线的方向余弦 n_1, n_2 代入式(11), 根据变分法基本原理, 得到相应的边界条件如下.

对于边界 $x_2 = 0, b$, 其中 $n_1 = 0$, $n_2 = -1\ (x_2 = 0)$ 或 $n_2 = 1\ (x_2 = b)$:

$$\begin{cases}
-T_{12}^{(0)} + \tau_{121,1}^{(0)} + \tau_{122,2}^{(0)} + \tau_{112,1}^{(0)} = 0 \quad \text{or} \quad u_1^{(0)} = \bar{u}_1^{(0)}, \\
-T_{12}^{(2)} + \tau_{121,1}^{(2)} + \tau_{122,2}^{(2)} + \tau_{112,1}^{(2)} - 2\tau_{123}^{(1)} - 2\tau_{132}^{(1)} = 0 \quad \text{or} \quad u_1^{(2)} = \bar{u}_1^{(2)}, \\
-T_{22}^{(0)} + \tau_{122,1}^{(0)} + \tau_{221,1}^{(0)} + \tau_{222,2}^{(0)} = 0 \quad \text{or} \quad u_2^{(0)} = \bar{u}_2^{(0)},\ D_2^{(0)} = 0 \quad \text{or} \quad \varphi^{(0)} = \bar{\varphi}^{(0)}, \\
-T_{22}^{(2)} + \tau_{221,1}^{(2)} + \tau_{222,2}^{(2)} + \tau_{122,1}^{(2)} - 2\tau_{223}^{(1)} - 2\tau_{232}^{(1)} = 0 \quad \text{or} \quad u_2^{(2)} = \bar{u}_2^{(2)}, \\
-T_{23}^{(1)} + \tau_{231,1}^{(1)} + \tau_{232,2}^{(1)} - \tau_{233}^{(0)} - \tau_{332}^{(0)} + \tau_{132,1}^{(1)} = 0 \quad \text{or} \quad u_3^{(1)} = \bar{u}_3^{(1)}, \\
D_2^{(1)} = 0 \quad \text{or} \quad \varphi^{(1)} = \bar{\varphi}^{(1)},\ D_2^{(2)} = 0 \quad \text{or} \quad \varphi^{(2)} = \bar{\varphi}^{(2)}, \\
\tau_{122}^{(2)} = 0 \quad \text{or} \quad u_{1,2}^{(2)} = \bar{u}_{1,2}^{(2)},\ \tau_{222}^{(0)} = 0 \quad \text{or} \quad u_{2,2}^{(0)} = \bar{u}_{2,2}^{(0)}, \\
\tau_{222}^{(2)} = 0 \quad \text{or} \quad u_{2,2}^{(2)} = \bar{u}_{2,2}^{(2)},\ \tau_{122}^{(0)} = 0 \quad \text{or} \quad u_{1,2}^{(0)} = \bar{u}_{1,2}^{(0)},\ \tau_{232}^{(1)} = 0 \quad \text{or} \quad u_{3,2}^{(1)} = \bar{u}_{3,2}^{(1)},
\end{cases} \tag{13}$$

式中, $\bar{u}_i^{(n)}$ 为边界上给定的位移值, $\bar{\varphi}^{(n)}$ 为边界上给定的电势值, 其余类似.

对于边界 $x_1 = 0, a$, 其中 $n_2 = 0$, $n_1 = -1\ (x_1 = 0)$ 或 $n_1 = 1\ (x_1 = a)$:

$$\begin{cases}
T_{11}^{(0)} - \tau_{111,1}^{(0)} - \tau_{112,2}^{(0)} - \tau_{121,2}^{(0)} = 0 \quad \text{or} \quad u_1^{(0)} = \bar{u}_1^{(0)}, \\
T_{11}^{(2)} - \tau_{111,1}^{(2)} - \tau_{112,2}^{(2)} - \tau_{121,2}^{(2)} + 2\tau_{113}^{(1)} + 2\tau_{131}^{(1)} = 0 \quad \text{or} \quad u_1^{(2)} = \bar{u}_1^{(2)}, \\
T_{12}^{(0)} - \tau_{121,1}^{(0)} - \tau_{122,2}^{(0)} - \tau_{221,2}^{(0)} = 0 \quad \text{or} \quad u_2^{(0)} = \bar{u}_2^{(0)},\ \tau_{111}^{(0)} = 0 \quad \text{or} \quad u_{1,1}^{(0)} = \bar{u}_{1,1}^{(0)}, \\
T_{12}^{(2)} - \tau_{121,1}^{(2)} - \tau_{122,2}^{(2)} - \tau_{221,2}^{(2)} + 2\tau_{123}^{(1)} + 2\tau_{231}^{(1)} = 0 \quad \text{or} \quad u_2^{(2)} = \bar{u}_2^{(2)}, \\
D_1^{(0)} = 0 \quad \text{or} \quad \varphi^{(0)} = \bar{\varphi}^{(0)},\ D_1^{(1)} = 0 \quad \text{or} \quad \varphi^{(1)} = \bar{\varphi}^{(1)},\ D_1^{(2)} = 0 \quad \text{or} \quad \varphi^{(2)} = \bar{\varphi}^{(2)}, \\
-T_{13}^{(1)} - \tau_{133}^{(0)} + \tau_{131,1}^{(1)} + \tau_{132,2}^{(1)} - \tau_{331}^{(0)} + \tau_{231,2}^{(1)} = 0 \quad \text{or} \quad u_3^{(1)} = \bar{u}_3^{(1)}, \\
\tau_{111}^{(2)} = 0 \quad \text{or} \quad u_{1,1}^{(2)} = \bar{u}_{1,1}^{(2)},\ \tau_{131}^{(1)} = 0 \quad \text{or} \quad u_{3,1}^{(1)} = \bar{u}_{3,1}^{(1)}, \\
\tau_{121}^{(0)} = 0 \quad \text{or} \quad u_{2,1}^{(0)} = \bar{u}_{2,1}^{(0)},\ \tau_{121}^{(2)} = 0 \quad \text{or} \quad u_{2,1}^{(2)} = \bar{u}_{2,1}^{(2)}.
\end{cases} \tag{14}$$

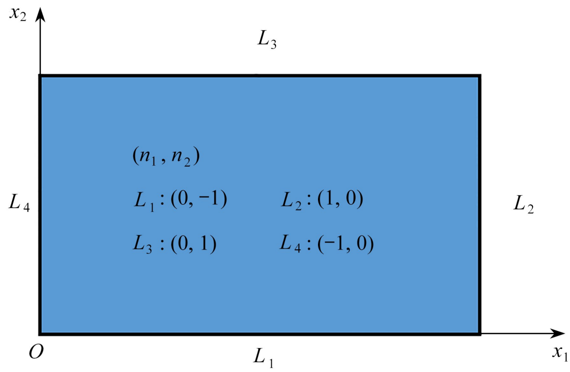


图 2 挠曲电纳米矩形板的边界

Fig. 2 The boundary of a flexoelectric nanorectangular plate

1.4 挠曲电纳米板的二维本构方程

取挠曲电纳米板的材料为立方晶系(m3m 点群),且不考虑非局部刚度常数 g_{ijklmn} 和热释电系数 p_i,挠曲电纳米板的本构关系为[16]

$$T_{ij}=c_{ijkl}S_{kl}-\lambda_{ij}\theta,\ \tau_{ijk}=-f_{lijk}E_l,\ D_i=\varepsilon_{ij}E_j+f_{ijkl}\eta_{jkl},\tag{15}$$

式中,c_{ijkl} 为弹性常数,λ_{ij} 为热弹性常数,f_{ijkl} 为挠曲电系数,ε_{ij} 为介电常数.

在 1.2 小节中所得二维二阶板方程的表达式(10),由于忽略了位移展开式中的高阶项,必然会引起截断误差,为此,这里引入修正系数对截断误差进行修正[26].即在应力本构式(15)的第 1 式中,通过应力释放进行修正,将 S_{13},S_{23} 和 S_{33} 分别替换为 k_1S_{13},k_2S_{23} 和 k_3S_{33},其中 k_1,k_2 和 k_3 为修正系数[26],且

$$k_1=k_2=\frac{\pi}{\sqrt{15}},\ k_3=\frac{\pi}{\sqrt{12}}.\tag{16}$$

沿厚度方向分别对三维本构方程与 x_3^n 的乘积进行积分,得到相应的二维本构方程.零阶、一阶及二阶的应力本构方程为

$$\begin{cases}T_{11}^{(0)}=c_{11}^{(0)}u_{1,1}^{(0)}+c_{11}^{(2)}u_{1,1}^{(2)}+c_{12}^{(0)}u_{2,2}^{(0)}+c_{12}^{(2)}u_{2,2}^{(2)}+c_{12}^{(0)}k_3u_3^{(1)}-\lambda_1^{(0)}\theta^{(0)}-\lambda_1^{(2)}\theta^{(2)},\\ T_{12}^{(0)}=c_{44}^{(0)}u_{1,2}^{(0)}+c_{44}^{(2)}u_{1,2}^{(2)}+c_{44}^{(0)}u_{2,1}^{(0)}+c_{44}^{(2)}u_{2,1}^{(2)},\\ T_{22}^{(0)}=c_{12}^{(0)}u_{1,1}^{(0)}+c_{12}^{(2)}u_{1,1}^{(2)}+c_{11}^{(0)}u_{2,2}^{(0)}+c_{11}^{(2)}u_{2,2}^{(2)}+c_{12}^{(0)}k_3u_3^{(1)}-\lambda_1^{(0)}\theta^{(0)}-\lambda_1^{(2)}\theta^{(2)},\\ T_{33}^{(0)}=c_{12}^{(0)}u_{1,1}^{(0)}+c_{12}^{(2)}u_{1,1}^{(2)}+c_{12}^{(0)}u_{2,2}^{(0)}+c_{12}^{(2)}u_{2,2}^{(2)}+c_{11}^{(0)}k_3u_3^{(1)}-\lambda_1^{(0)}\theta^{(0)}-\lambda_1^{(2)}\theta^{(2)},\\ T_{13}^{(1)}=2k_1c_{44}^{(2)}u_1^{(2)}+k_1c_{44}^{(2)}u_{3,1}^{(1)},\ T_{23}^{(1)}=2k_2c_{44}^{(2)}u_2^{(2)}+k_2c_{44}^{(2)}u_{3,2}^{(1)},\\ T_{11}^{(2)}=c_{11}^{(2)}u_{1,1}^{(0)}+c_{11}^{(4)}u_{1,1}^{(2)}+c_{12}^{(2)}u_{2,2}^{(0)}+c_{12}^{(4)}u_{2,2}^{(2)}+c_{12}^{(2)}k_3u_3^{(1)}-\lambda_1^{(2)}\theta^{(0)}-\lambda_1^{(4)}\theta^{(2)},\\ T_{12}^{(2)}=c_{44}^{(2)}u_{1,2}^{(0)}+c_{44}^{(4)}u_{1,2}^{(2)}+c_{44}^{(2)}u_{2,1}^{(0)}+c_{44}^{(4)}u_{2,1}^{(2)},\\ T_{22}^{(2)}=c_{12}^{(2)}u_{1,1}^{(0)}+c_{12}^{(4)}u_{1,1}^{(2)}+c_{11}^{(2)}u_{2,2}^{(0)}+c_{11}^{(4)}u_{2,2}^{(2)}+c_{12}^{(2)}k_3u_3^{(1)}-\lambda_1^{(2)}\theta^{(0)}-\lambda_1^{(4)}\theta^{(2)}.\end{cases}\tag{17}$$

零阶、一阶及二阶的电位移本构方程为

$$D_1^{(0)}=f_{11}^{(0)}u_{1,11}^{(0)}+f_{11}^{(2)}u_{1,11}^{(2)}+f_{14}^{(0)}u_{2,21}^{(0)}+f_{14}^{(2)}u_{2,21}^{(2)}+f_{14}^{(0)}u_{3,1}^{(1)}+f_{111}^{(0)}u_{1,22}^{(0)}+f_{111}^{(2)}u_{1,22}^{(2)}+ f_{111}^{(0)}u_{2,12}^{(0)}+f_{111}^{(2)}u_{2,12}^{(2)}+2f_{111}^{(0)}u_1^{(2)}+f_{111}^{(0)}u_{3,1}^{(1)}-\varepsilon_{11}^{(0)}\varphi_{,1}^{(0)}-\varepsilon_{11}^{(2)}\varphi_{,1}^{(2)},\tag{18a}$$

$$D_2^{(0)}=f_{14}^{(0)}u_{1,12}^{(0)}+f_{14}^{(2)}u_{1,12}^{(2)}+f_{11}^{(0)}u_{2,22}^{(0)}+f_{11}^{(2)}u_{2,22}^{(2)}+f_{14}^{(0)}u_{3,2}^{(1)}+f_{111}^{(0)}u_{1,21}^{(0)}+f_{111}^{(2)}u_{1,21}^{(2)}+ f_{111}^{(0)}u_{2,11}^{(0)}+f_{111}^{(2)}u_{2,11}^{(2)}+2f_{111}^{(0)}u_2^{(2)}+f_{111}^{(0)}u_{3,2}^{(1)}-\varepsilon_{11}^{(0)}\varphi_{,2}^{(0)}-\varepsilon_{11}^{(2)}\varphi_{,2}^{(2)},\tag{18b}$$

$$D_3^{(0)}=-\varepsilon_{11}^{(0)}\varphi^{(1)},\ D_1^{(1)}=-\varepsilon_{11}^{(2)}\varphi_{,1}^{(1)},\ D_2^{(1)}=-\varepsilon_{11}^{(2)}\varphi_{,2}^{(1)},\tag{18c}$$

$$D_3^{(1)}=2f_{14}^{(2)}u_{1,1}^{(2)}+2f_{14}^{(2)}u_{2,2}^{(2)}+2f_{111}^{(2)}u_{1,1}^{(2)}+f_{111}^{(2)}u_{3,11}^{(1)}+2f_{111}^{(2)}u_{2,2}^{(2)}+f_{111}^{(2)}u_{3,22}^{(1)}-2\varepsilon_{11}^{(2)}\varphi^{(2)},\tag{18d}$$

$$D_1^{(2)}=f_{11}^{(2)}u_{1,11}^{(0)}+f_{11}^{(4)}u_{1,11}^{(2)}+f_{14}^{(2)}u_{2,21}^{(0)}+f_{14}^{(4)}u_{2,21}^{(2)}+f_{14}^{(2)}u_{3,1}^{(1)}+f_{111}^{(2)}u_{1,22}^{(0)}+f_{111}^{(4)}u_{1,22}^{(2)}+ f_{111}^{(2)}u_{2,12}^{(0)}+f_{111}^{(4)}u_{2,12}^{(2)}+2f_{111}^{(2)}u_1^{(2)}+f_{111}^{(2)}u_{3,1}^{(1)}-\varepsilon_{11}^{(2)}\varphi_{,1}^{(0)}-\varepsilon_{11}^{(4)}\varphi_{,1}^{(2)},\tag{18e}$$

$$D_2^{(2)}=f_{14}^{(2)}u_{1,12}^{(0)}+f_{14}^{(4)}u_{1,12}^{(2)}+f_{11}^{(2)}u_{2,22}^{(0)}+f_{11}^{(4)}u_{2,22}^{(2)}+f_{14}^{(2)}u_{3,2}^{(1)}+f_{111}^{(2)}u_{1,21}^{(0)}+f_{111}^{(4)}u_{1,21}^{(2)}+$$

$$f_{111}^{(2)}u_{2,11}^{(0)}+f_{111}^{(4)}u_{2,11}^{(2)}+2f_{111}^{(2)}u_2^{(2)}+f_{111}^{(2)}u_{3,2}^{(1)}-\varepsilon_{11}^{(2)}\varphi_{,2}^{(0)}-\varepsilon_{11}^{(4)}\varphi_{,2}^{(2)}. \tag{18f}$$

零阶、一阶及二阶的高阶应力本构方程为

$$\begin{cases}\tau_{111}^{(0)}=f_{11}^{(0)}\varphi_{,1}^{(0)}+f_{11}^{(2)}\varphi_{,1}^{(2)},\ \tau_{112}^{(0)}=f_{14}^{(0)}\varphi_{,2}^{(0)}+f_{14}^{(2)}\varphi_{,2}^{(2)},\ \tau_{221}^{(0)}=f_{14}^{(0)}\varphi_{,1}^{(0)}+f_{14}^{(2)}\varphi_{,1}^{(2)},\\ \tau_{222}^{(0)}=f_{11}^{(0)}\varphi_{,2}^{(0)}+f_{11}^{(2)}\varphi_{,2}^{(2)},\ \tau_{223}^{(0)}=f_{14}^{(0)}\varphi^{(1)},\ \tau_{233}^{(0)}=f_{111}^{(0)}\varphi_{,2}^{(0)}+f_{111}^{(2)}\varphi_{,2}^{(2)},\\ \tau_{331}^{(0)}=f_{14}^{(0)}\varphi_{,1}^{(0)}+f_{14}^{(2)}\varphi_{,1}^{(2)},\ \tau_{332}^{(0)}=f_{14}^{(0)}\varphi_{,2}^{(0)}+f_{14}^{(2)}\varphi_{,2}^{(2)},\ \tau_{333}^{(0)}=f_{11}^{(0)}\varphi^{(1)},\\ \tau_{121}^{(0)}=f_{111}^{(0)}\varphi_{,2}^{(0)}+f_{111}^{(2)}\varphi_{,2}^{(2)},\ \tau_{122}^{(0)}=f_{111}^{(0)}\varphi_{,1}^{(0)}+f_{111}^{(2)}\varphi_{,1}^{(2)},\ \tau_{133}^{(0)}=f_{111}^{(0)}\varphi_{,1}^{(0)}+f_{111}^{(2)}\varphi_{,1}^{(2)},\\ \tau_{113}^{(1)}=2f_{14}^{(2)}\varphi^{(2)},\ \tau_{131}^{(1)}=2f_{111}^{(2)}\varphi^{(2)},\ \tau_{232}^{(1)}=2f_{111}^{(2)}\varphi^{(2)},\ \tau_{223}^{(1)}=2f_{14}^{(2)}\varphi^{(2)},\\ \tau_{111}^{(2)}=f_{11}^{(2)}\varphi_{,1}^{(0)}+f_{11}^{(4)}\varphi_{,1}^{(2)},\ \tau_{112}^{(2)}=f_{14}^{(2)}\varphi_{,2}^{(0)}+f_{14}^{(4)}\varphi_{,2}^{(2)},\ \tau_{121}^{(2)}=f_{111}^{(2)}\varphi_{,2}^{(0)}+f_{111}^{(4)}\varphi_{,2}^{(2)},\\ \tau_{122}^{(2)}=f_{111}^{(2)}\varphi_{,1}^{(0)}+f_{111}^{(4)}\varphi_{,1}^{(2)},\ \tau_{221}^{(2)}=f_{14}^{(2)}\varphi_{,1}^{(0)}+f_{14}^{(4)}\varphi_{,1}^{(2)},\ \tau_{222}^{(2)}=f_{11}^{(2)}\varphi_{,2}^{(0)}+f_{11}^{(4)}\varphi_{,2}^{(2)}.\end{cases} \tag{19}$$

式(17)—(19)中高阶材料参数的定义为

$$[c_{pq}^{(n)},\varepsilon_{ij}^{(n)},f_{11}^{(n)},f_{14}^{(n)},f_{111}^{(n)},\lambda_p^{(n)}]=\int_{-h/2}^{h/2}[(c_{pq},\varepsilon_{ij},f_{11},f_{14},f_{111},\lambda_p)x_3^n]\mathrm{d}x_3,$$
$$i,j=1,2,3;\ p,q=1,2,\cdots,6. \tag{20}$$

1.5　挠曲电纳米板二阶理论的控制方程

将式(17)—(19)代入式(10),得到以基本未知量 $u_1^{(0)},u_1^{(2)},u_2^{(0)},u_2^{(2)},u_3^{(1)},\varphi^{(0)},\varphi^{(1)}$ 和 $\varphi^{(2)}$ 表示的控制方程:

$$c_{11}^{(0)}u_{1,11}^{(0)}+c_{11}^{(2)}u_{1,11}^{(2)}+c_{12}^{(0)}u_{2,21}^{(0)}+c_{12}^{(2)}u_{2,21}^{(2)}+c_{12}^{(0)}k_3u_{3,1}^{(1)}-\lambda_1^{(0)}\theta_{,1}^{(0)}-\lambda_1^{(2)}\theta_{,1}^{(2)}+c_{44}^{(0)}u_{1,22}^{(0)}+ c_{44}^{(2)}u_{1,22}^{(2)}+c_{44}^{(0)}u_{2,12}^{(0)}+c_{44}^{(2)}u_{2,12}^{(2)}-f_{11}^{(0)}\varphi_{,111}^{(0)}-f_{11}^{(2)}\varphi_{,111}^{(2)}-f_{14}^{(0)}\varphi_{,221}^{(0)}-f_{14}^{(2)}\varphi_{,221}^{(2)}- f_{111}^{(0)}\varphi_{,212}^{(0)}-f_{111}^{(2)}\varphi_{,212}^{(2)}-f_{111}^{(0)}\varphi_{,122}^{(0)}-f_{111}^{(2)}\varphi_{,122}^{(2)}+f_1^{(0)}=m^{(0)}\ddot{u}_1^{(0)}+m^{(2)}\ddot{u}_1^{(2)}, \tag{21a}$$

$$c_{44}^{(0)}u_{1,21}^{(0)}+c_{44}^{(2)}u_{1,21}^{(2)}+c_{44}^{(0)}u_{2,11}^{(0)}+c_{44}^{(2)}u_{2,11}^{(2)}+c_{12}^{(0)}u_{1,12}^{(0)}+c_{12}^{(2)}u_{1,12}^{(2)}+c_{11}^{(0)}u_{2,22}^{(0)}+c_{11}^{(2)}u_{2,22}^{(2)}+ c_{12}^{(0)}k_3u_{3,2}^{(1)}-\lambda_1^{(0)}\theta_{,2}^{(0)}-\lambda_1^{(2)}\theta_{,2}^{(2)}-f_{111}^{(0)}\varphi_{,211}^{(0)}-f_{111}^{(2)}\varphi_{,211}^{(2)}-f_{111}^{(0)}\varphi_{,121}^{(0)}-f_{111}^{(2)}\varphi_{,121}^{(2)}- f_{14}^{(0)}\varphi_{,112}^{(0)}-f_{14}^{(2)}\varphi_{,112}^{(2)}-f_{11}^{(0)}\varphi_{,222}^{(0)}-f_{11}^{(2)}\varphi_{,222}^{(2)}+f_2^{(0)}=m^{(0)}\ddot{u}_2^{(0)}+m^{(2)}\ddot{u}_2^{(2)}, \tag{21b}$$

$$2c_{44}^{(2)}k_1u_{1,1}^{(2)}+c_{44}^{(2)}k_1u_{3,11}^{(1)}+2c_{44}^{(2)}k_2u_{2,2}^{(2)}+c_{44}^{(2)}k_2u_{3,22}^{(1)}-c_{12}^{(0)}u_{1,1}^{(0)}-c_{12}^{(2)}u_{1,1}^{(2)}-c_{12}^{(0)}u_{2,2}^{(0)}- c_{12}^{(2)}u_{2,2}^{(2)}-c_{11}^{(0)}k_3u_3^{(1)}+\lambda_1^{(0)}\theta^{(0)}+\lambda_1^{(2)}\theta^{(2)}-f_{111}^{(2)}\varphi_{,11}^{(2)}+f_{111}^{(0)}\varphi_{,11}^{(0)}-f_{111}^{(2)}\varphi_{,22}^{(2)}+ f_{111}^{(0)}\varphi_{,22}^{(0)}+f_{14}^{(0)}\varphi_{,11}^{(0)}+f_{14}^{(2)}\varphi_{,11}^{(2)}+f_{14}^{(0)}\varphi_{,22}^{(0)}+f_{14}^{(2)}\varphi_{,22}^{(2)}+f_3^{(1)}=m^{(2)}\ddot{u}_3^{(1)}, \tag{21c}$$

$$-4c_{44}^{(2)}k_1u_1^{(2)}-2c_{44}^{(2)}k_1u_{3,1}^{(1)}+c_{44}^{(2)}u_{1,22}^{(0)}+c_{44}^{(4)}u_{1,22}^{(2)}+c_{44}^{(2)}u_{2,12}^{(0)}+c_{44}^{(4)}u_{2,12}^{(2)}+c_{11}^{(2)}u_{1,11}^{(0)}+ c_{11}^{(4)}u_{1,11}^{(2)}+c_{12}^{(2)}u_{2,21}^{(0)}+c_{12}^{(4)}u_{2,21}^{(2)}+c_{12}^{(2)}k_3u_{3,1}^{(1)}-\lambda_1^{(2)}\theta_{,1}^{(0)}-\lambda_1^{(4)}\theta_{,1}^{(2)}-f_{11}^{(2)}\varphi_{,111}^{(0)}-f_{11}^{(4)}\varphi_{,111}^{(2)}- f_{14}^{(2)}\varphi_{,221}^{(0)}-f_{14}^{(4)}\varphi_{,221}^{(2)}+4f_{14}^{(2)}\varphi_{,1}^{(2)}-f_{111}^{(2)}\varphi_{,212}^{(0)}-f_{111}^{(4)}\varphi_{,212}^{(2)}-f_{111}^{(2)}\varphi_{,122}^{(0)}-f_{111}^{(4)}\varphi_{,122}^{(2)}+ 2f_{111}^{(2)}\varphi_{,1}^{(2)}-2f_{111}^{(0)}\varphi_{,1}^{(0)}+f_1^{(2)}=m^{(2)}\ddot{u}_1^{(0)}+m^{(4)}\ddot{u}_1^{(2)}, \tag{21d}$$

$$c_{44}^{(2)}u_{1,21}^{(0)}+c_{44}^{(4)}u_{1,21}^{(2)}+c_{44}^{(2)}u_{2,11}^{(0)}+c_{44}^{(4)}u_{2,11}^{(2)}+c_{12}^{(2)}u_{1,12}^{(0)}+c_{12}^{(4)}u_{1,12}^{(2)}+c_{11}^{(2)}u_{2,22}^{(0)}+c_{11}^{(4)}u_{2,22}^{(2)}+ c_{12}^{(2)}k_3u_{3,2}^{(1)}-\lambda_1^{(2)}\theta_{,2}^{(0)}-\lambda_1^{(4)}\theta_{,2}^{(2)}-4c_{44}^{(2)}k_2u_2^{(2)}-2c_{44}^{(2)}k_2u_{3,2}^{(1)}-f_{111}^{(2)}\varphi_{,211}^{(0)}-f_{111}^{(4)}\varphi_{,211}^{(2)}- f_{111}^{(2)}\varphi_{,121}^{(0)}-f_{111}^{(4)}\varphi_{,121}^{(2)}-f_{14}^{(2)}\varphi_{,112}^{(0)}-f_{11}^{(4)}\varphi_{,222}^{(2)}-f_{14}^{(4)}\varphi_{,112}^{(2)}-f_{11}^{(2)}\varphi_{,222}^{(0)}+4f_{14}^{(2)}\varphi_{,2}^{(2)}+ 2f_{111}^{(2)}\varphi_{,2}^{(2)}-2f_{111}^{(0)}\varphi_{,2}^{(0)}+f_2^{(2)}=m^{(2)}\ddot{u}_2^{(0)}+m^{(4)}\ddot{u}_2^{(2)}, \tag{21e}$$

$$f_{11}^{(0)}u_{1,111}^{(0)}+f_{11}^{(2)}u_{1,111}^{(2)}+f_{14}^{(0)}u_{2,211}^{(0)}+f_{14}^{(2)}u_{2,211}^{(2)}+f_{14}^{(0)}u_{3,11}^{(1)}+f_{111}^{(0)}u_{1,221}^{(0)}+f_{111}^{(2)}u_{1,221}^{(2)}+f_{111}^{(0)}u_{2,121}^{(0)}+ f_{111}^{(2)}u_{2,121}^{(2)}+2f_{111}^{(0)}u_{1,1}^{(2)}+f_{111}^{(0)}u_{3,11}^{(1)}-\varepsilon_{11}^{(0)}\varphi_{,11}^{(0)}-\varepsilon_{11}^{(2)}\varphi_{,11}^{(2)}+f_{14}^{(0)}u_{1,122}^{(0)}+f_{14}^{(2)}u_{1,122}^{(2)}+f_{11}^{(0)}u_{2,222}^{(0)}+ f_{11}^{(2)}u_{2,222}^{(2)}+f_{14}^{(0)}u_{3,22}^{(1)}+f_{111}^{(0)}u_{1,212}^{(0)}+f_{111}^{(2)}u_{1,212}^{(2)}+f_{111}^{(0)}u_{2,112}^{(0)}+f_{111}^{(2)}u_{2,112}^{(2)}+2f_{111}^{(0)}u_{2,2}^{(2)}+f_{111}^{(0)}u_{3,22}^{(1)}- \varepsilon_{11}^{(0)}\varphi_{,22}^{(0)}-\varepsilon_{11}^{(2)}\varphi_{,22}^{(2)}=0, \tag{21f}$$

$$-\varepsilon_{11}^{(2)}\varphi_{,11}^{(1)}-\varepsilon_{11}^{(2)}\varphi_{,22}^{(1)}+\varepsilon_{11}^{(0)}\varphi^{(1)}=0, \tag{21g}$$

$$f_{14}^{(2)}u_{1,122}^{(0)}+f_{14}^{(4)}u_{1,122}^{(2)}+f_{11}^{(2)}u_{2,222}^{(0)}+f_{11}^{(4)}u_{2,222}^{(2)}+f_{14}^{(2)}u_{3,22}^{(1)}+f_{111}^{(2)}u_{1,212}^{(0)}+f_{111}^{(4)}u_{1,212}^{(2)}+f_{111}^{(2)}u_{2,112}^{(0)}+ f_{111}^{(4)}u_{2,112}^{(2)}+2f_{111}^{(2)}u_{2,2}^{(2)}+f_{111}^{(2)}u_{3,22}^{(1)}-\varepsilon_{11}^{(2)}\varphi_{,22}^{(0)}-\varepsilon_{11}^{(4)}\varphi_{,22}^{(2)}+f_{11}^{(2)}u_{1,111}^{(0)}+f_{11}^{(4)}u_{1,111}^{(2)}+f_{14}^{(2)}u_{2,211}^{(0)}+ f_{14}^{(4)}u_{2,211}^{(2)}+f_{14}^{(2)}u_{3,11}^{(1)}+f_{111}^{(2)}u_{1,221}^{(0)}+f_{111}^{(4)}u_{1,221}^{(2)}+f_{111}^{(2)}u_{2,121}^{(0)}+f_{111}^{(4)}u_{2,121}^{(2)}+2f_{111}^{(2)}u_{1,1}^{(2)}+f_{111}^{(2)}u_{3,11}^{(1)}-$$

$$\varepsilon_{11}^{(2)}\varphi_{,11}^{(0)} - \varepsilon_{11}^{(4)}\varphi_{,11}^{(2)} - 4f_{14}^{(2)}u_{1,1}^{(2)} - 4f_{14}^{(2)}u_{2,2}^{(2)} - 4f_{111}^{(2)}u_{1,1}^{(2)} - 2f_{111}^{(2)}u_{3,11}^{(1)} - 4f_{111}^{(2)}u_{2,2}^{(2)} - 2f_{111}^{(2)}u_{3,22}^{(1)} + 4\varepsilon_{11}^{(2)}\varphi^{(2)} = 0. \tag{21h}$$

由式(21g)可知,一阶电势 $\varphi^{(1)}$ 与位移场 $\boldsymbol{u}$、电势场 $\varphi^{(0)}$ 和 $\varphi^{(2)}$ 及温差场 $\theta^{(0)}$ 和 $\theta^{(2)}$ 解耦,故之后的讨论不再考虑一阶电势 $\varphi^{(1)}$.

2 算例分析

考虑一个如图 1 所示长度为 a,宽度为 b,厚度为 h 的挠曲电纳米矩形板,其局部区域上作用有温差场 $\theta^{(0)}(x_1,x_2)$,$\theta^{(2)}(x_1,x_2)$ 和均布机械载荷 $f_3^{(1)}(x_1,x_2)$.对于本文研究的二维纳米矩形板问题,假设板的边缘接地,且无电荷积累.对于给定的边 $x_1 = 0,a$, 相应的边界条件为

$$\begin{cases} u_2^{(0)} = 0,\ u_2^{(2)} = 0,\ u_3^{(1)} = 0,\ u_{1,1}^{(0)} = 0,\ u_{1,1}^{(2)} = 0, \\ \tau_{121}^{(0)} = 0,\ \tau_{121}^{(2)} = 0,\ \tau_{131}^{(1)} = 0,\ \varphi^{(0)} = 0,\ \varphi^{(2)} = 0, \\ T_{11}^{(0)} - \tau_{111,1}^{(0)} - \tau_{112,2}^{(0)} - \tau_{121,2}^{(0)} = 0, \\ T_{11}^{(2)} - \tau_{111,1}^{(2)} - \tau_{112,2}^{(2)} - \tau_{121,2}^{(2)} + 2\tau_{113}^{(1)} + 2\tau_{131}^{(1)} = 0; \end{cases} \tag{22}$$

对于给定的边 $x_2 = 0,b$, 相应的边界条件为

$$\begin{cases} u_1^{(0)} = 0,\ u_1^{(2)} = 0,\ u_3^{(1)} = 0,\ u_{2,2}^{(0)} = 0,\ u_{2,2}^{(2)} = 0, \\ \tau_{122}^{(0)} = 0,\ \tau_{122}^{(2)} = 0,\ \tau_{232}^{(1)} = 0,\ \varphi^{(0)} = 0,\ \varphi^{(2)} = 0, \\ -T_{22}^{(0)} + \tau_{122,1}^{(0)} + \tau_{221,1}^{(0)} + \tau_{222,2}^{(0)} = 0, \\ -T_{22}^{(2)} + \tau_{221,1}^{(2)} + \tau_{222,2}^{(2)} + \tau_{122,1}^{(2)} - 2\tau_{223}^{(1)} - 2\tau_{232}^{(1)} = 0. \end{cases} \tag{23}$$

假设 $u_1^{(0)}$,$u_1^{(2)}$,$u_2^{(0)}$,$u_2^{(2)}$,$u_3^{(1)}$,$\varphi^{(0)}$,$\varphi^{(2)}$,$\theta^{(0)}$,$\theta^{(2)}$ 和 $f_3^{(1)}$ 有以下满足给定边界条件式(22)、(23)的双重 Fourier 级数解:

$$\begin{cases} \{u_1^{(0)},u_1^{(2)}\} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{U_{mn}^{(0)},U_{mn}^{(2)}\}\cos(\xi_m x_1)\sin(\zeta_n x_2), \\ \{u_2^{(0)},u_2^{(2)}\} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{V_{mn}^{(0)},V_{mn}^{(2)}\}\sin(\xi_m x_1)\cos(\zeta_n x_2), \\ \{u_3^{(1)},\varphi^{(0)},\varphi^{(2)},\theta^{(0)},\theta^{(2)},f_3^{(1)}\} = \\ \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{W_{mn}^{(1)},\ \Phi_{mn}^{(0)},\ \Phi_{mn}^{(2)},\ \Theta_{mn}^{(0)},\ \Theta_{mn}^{(2)},\ F_{mn}^{(1)}\}\sin(\xi_m x_1)\sin(\zeta_n x_2), \\ \xi_m = \dfrac{m\pi}{a},\ \zeta_n = \dfrac{n\pi}{b}, \qquad m,n = 1,2,3,\cdots, \end{cases} \tag{24}$$

式中,$\Theta_{mn}^{(0)}$,$\Theta_{mn}^{(2)}$ 和 $F_{mn}^{(1)}$ 已知,$U_{mn}^{(0)}$,$U_{mn}^{(2)}$,$V_{mn}^{(0)}$,$V_{mn}^{(2)}$,$W_{mn}^{(1)}$,$\Phi_{mn}^{(0)}$ 和 $\Phi_{mn}^{(2)}$ 为待定常数.式(22)、(23)中的边界条件满足式(24).在数值计算中,选取挠曲电材料为 Si.在中心为 (x_0,y_0),长为 $2c$、宽为 $2d$ 的矩形区域作用有局部载荷,如图 3 所示.相应的横向均布力 $f_3^{(1)}$ 为

$$f_3^{(1)} = \begin{cases} P, & x \in x_0 \pm c,\ y \in y_0 \pm d, \\ 0, & \text{others}. \end{cases} \tag{25}$$

对于给定的 $\theta^{(0)}$,$\theta^{(2)}$ 和 $f_3^{(1)}$, 其 Fourier 系数 $\Theta_{mn}^{(0)}$,$\Theta_{mn}^{(2)}$ 和 $F_{mn}^{(1)}$ 分别为

$$\{\Theta_{mn}^{(0)},\Theta_{mn}^{(2)},F_{mn}^{(1)}\} = \{\theta^{(0)},\theta^{(2)},f_3^{(1)}\}\frac{4}{ab}\frac{1}{\xi_m\zeta_n}\times [\cos\xi_m(x_0-c) - \cos\xi_m(x_0+c)][\cos\zeta_n(y_0-d) - \cos\zeta_n(y_0+d)]. \tag{26}$$

本文主要研究由不均匀的温度变化和机械载荷造成的挠曲电纳米板的面内拉伸变形、对称厚度剪切变形和厚度伸缩变形,因此在控制方程(21)中,$f_1^{(0)}$,$f_2^{(0)}$,$f_1^{(2)}$ 和 $f_2^{(2)}$ 不再考虑.且对于静态问题,位移关于时间的导数项也将消失.将式(24)代入式(21),得到关于 $U_{mn}^{(0)}$,$U_{mn}^{(2)}$,$V_{mn}^{(0)}$,$V_{mn}^{(2)}$,$W_{mn}^{(1)}$,$\Phi_{mn}^{(0)}$ 和 $\Phi_{mn}^{(2)}$ 的非齐次线性方程组:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} \\ K_{12} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} \\ K_{13} & K_{23} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} \\ K_{14} & K_{24} & K_{34} & K_{44} & K_{45} & K_{46} & K_{47} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} \\ K_{16} & K_{26} & K_{36} & K_{46} & K_{56} & K_{66} & K_{67} \\ K_{17} & K_{27} & K_{37} & K_{47} & K_{57} & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} U_{mn}^{(0)} \\ U_{mn}^{(2)} \\ V_{mn}^{(0)} \\ V_{mn}^{(2)} \\ W_{mn}^{(1)} \\ \Phi_{mn}^{(0)} \\ \Phi_{mn}^{(2)} \end{bmatrix} = \begin{bmatrix} \lambda_1^{(0)} \xi_m \Theta_{mn}^{(0)} + \lambda_1^{(2)} \xi_m \Theta_{mn}^{(2)} \\ \lambda_1^{(2)} \xi_m \Theta_{mn}^{(0)} + \lambda_1^{(4)} \xi_m \Theta_{mn}^{(2)} \\ \lambda_1^{(0)} \zeta_n \Theta_{mn}^{(0)} + \lambda_1^{(2)} \zeta_n \Theta_{mn}^{(2)} \\ \lambda_1^{(2)} \zeta_n \Theta_{mn}^{(0)} + \lambda_1^{(4)} \zeta_n \Theta_{mn}^{(2)} \\ -\lambda_1^{(0)} \Theta_{mn}^{(0)} - \lambda_1^{(2)} \Theta_{mn}^{(2)} - F_{mn}^{(1)} \\ 0 \\ 0 \end{bmatrix}, \tag{27}$$

式中,系数矩阵 K_{ij} 为

$$\begin{cases} K_{11} = -(c_{11}^{(0)} \xi_m^2 + c_{44}^{(0)} \zeta_n^2),\ K_{12} = -(c_{11}^{(2)} \xi_m^2 + c_{44}^{(2)} \zeta_n^2),\ K_{13} = -(c_{12}^{(0)} + c_{44}^{(0)}) \xi_m \zeta_n, \\ K_{14} = -(c_{12}^{(2)} + c_{44}^{(2)}) \xi_m \zeta_n,\ K_{15} = c_{12}^{(0)} k_3 \xi_m, \\ K_{16} = f_{11}^{(0)} \xi_m^3 + (2f_{111}^{(0)} + f_{14}^{(0)}) \xi_m \zeta_n^2,\ K_{17} = (f_{14}^{(2)} + 2f_{111}^{(2)}) \xi_m \zeta_n^2 + f_{11}^{(2)} \xi_m^3, \\ K_{22} = -(4c_{44}^{(2)} k_1 + c_{11}^{(4)} \xi_m^2 + c_{44}^{(4)} \zeta_n^2),\ K_{23} = -(c_{44}^{(2)} + c_{12}^{(2)}) \xi_m \zeta_n,\ K_{24} = -(c_{12}^{(4)} + c_{44}^{(4)}) \xi_m \zeta_n, \\ K_{25} = (c_{12}^{(2)} k_3 - 2c_{44}^{(2)} k_1) \xi_m,\ K_{26} = (f_{14}^{(2)} + 2f_{111}^{(2)}) \xi_m \zeta_n^2 + f_{11}^{(2)} \xi_m^3 - 2f_{111}^{(0)} \xi_m, \\ K_{27} = (f_{14}^{(4)} + 2f_{111}^{(4)}) \xi_m \zeta_n^2 + f_{11}^{(4)} \xi_m^3 + 2(f_{111}^{(2)} + 2f_{14}^{(2)}) \xi_m,\ K_{33} = -(c_{44}^{(0)} \xi_m^2 + c_{11}^{(0)} \zeta_n^2), \\ K_{34} = -(c_{44}^{(2)} \xi_m^2 + c_{11}^{(2)} \zeta_n^2),\ K_{35} = c_{12}^{(0)} k_3 \zeta_n,\ K_{36} = (f_{14}^{(0)} + 2f_{111}^{(0)}) \xi_m^2 \zeta_n + f_{11}^{(0)} \zeta_n^3, \\ K_{37} = f_{11}^{(2)} \zeta_n^3 + (2f_{111}^{(2)} + f_{14}^{(2)}) \xi_m^2 \zeta_n,\ K_{44} = -(c_{44}^{(4)} \xi_m^2 + c_{11}^{(4)} \zeta_n^2 + 4c_{44}^{(2)} k_2), \\ K_{45} = (c_{12}^{(2)} k_3 - 2c_{44}^{(2)} k_2) \zeta_n,\ K_{46} = (2f_{111}^{(2)} + f_{14}^{(2)}) \xi_m^2 \zeta_n - 2f_{111}^{(0)} \zeta_n + f_{11}^{(2)} \zeta_n^3, \\ K_{47} = f_{11}^{(4)} \zeta_n^3 + (2f_{111}^{(4)} + f_{14}^{(4)}) \xi_m^2 \zeta_n + (2f_{111}^{(2)} + 4f_{14}^{(2)}) \zeta_n, \\ K_{51} = c_{12}^{(0)} \xi_m,\ K_{52} = (c_{12}^{(2)} - 2c_{44}^{(2)} k_1) \xi_m,\ K_{53} = c_{12}^{(0)} \zeta_n,\ K_{54} = (c_{12}^{(2)} - 2c_{44}^{(2)} k_2) \zeta_n, \\ K_{55} = -(c_{44}^{(2)} k_1 \xi_m^2 + c_{44}^{(2)} k_2 \zeta_n^2 + c_{11}^{(0)} k_3),\ K_{56} = -(f_{14}^{(0)} + f_{111}^{(0)})(\xi_m^2 + \zeta_n^2), \\ K_{57} = (f_{111}^{(2)} - f_{14}^{(2)})(\xi_m^2 + \zeta_n^2),\ K_{66} = \varepsilon_{11}^{(0)} (\xi_m^2 + \zeta_n^2), \\ K_{67} = \varepsilon_{11}^{(2)} (\xi_m^2 + \zeta_n^2),\ K_{77} = \varepsilon_{11}^{(4)} (\xi_m^2 + \zeta_n^2) + 4\varepsilon_{11}^{(2)}. \end{cases} \tag{28}$$

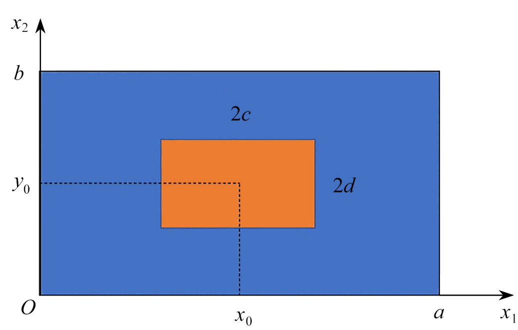


图 3　加载区域

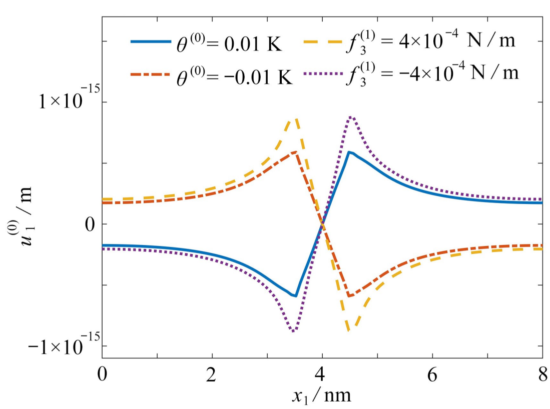
Fig. 3　The loading area

在算例计算中,挠曲电纳米板的弹性常数 c_{ijkl}、热弹性常数 λ_{ij}、挠曲电系数 f_{ijkl} 和介电常数 ε_{ij} 取自文献[32]: $c_{11} = 1.657 \times 10^{11}$ N/m^2, $c_{12} = 6.39 \times 10^{10}$ N/m^2, $c_{44} = 7.956 \times 10^{10}$ N/m^2, $\lambda_1 = 4.291\ 6 \times 10^5$ N/(m^2·K), $f_{11} = 1.3 \times 10^{-9}$ C/m, $f_{14} = 4.0 \times 10^{-10}$ C/m, $f_{111} = 4.0 \times 10^{-10}$ C/m, $\varepsilon_{11} = 1.036 \times 10^{-10}$ C/(mV).纳米板的几何尺寸为长度 $a = 800$ nm,宽度 $b = 1\ 400$ nm,厚度 $h = 50$ nm.局部加载区域的中心坐标为 $x_0 = 0.5a$, $y_0 = 0.5b$,长度 $c = 50$ nm,宽度 $d = 100$ nm.数值结果表明,当级数求和项数取较大值时($m > 200$, $n > 200$), $u_1^{(0)}$, $u_1^{(2)}$, $u_2^{(0)}$, $u_2^{(2)}$, $u_3^{(1)}$ 和 $\varphi^{(2)}$ 的数值结果与 $m = 200$, $n = 200$ 时结果相同,这说明对于其 Fourier 级数展开式,在该问题下级数求和项数取 $m = 200$, $n = 200$ 时的精度是足够的,能够保证结果收敛.但是对于 $\varphi^{(0)}$ 来说,其收

敛性较差,当取 $m = 1\ 000$,$n = 1\ 000$ 时,才能保证 $\varphi^{(0)}$ 收敛.

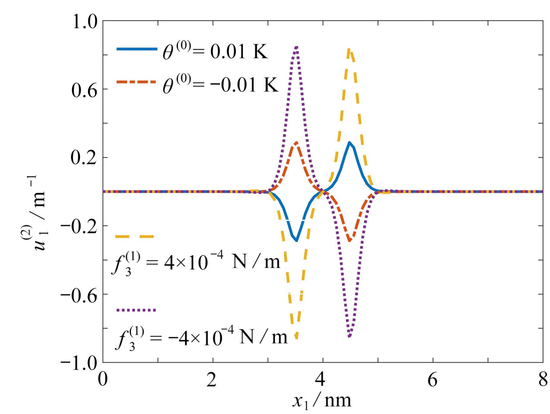
由于本文所用的是线性理论,故采用很小的温差载荷时结果是有效的.图 4 研究了温度载荷和机械载荷分别对截面 $x_2 = b/2$ 上的位移场 $u_1^{(0)}$,$u_1^{(2)}$ 和 $u_3^{(1)}$,及电势场 $\varphi^{(0)}$ 和 $\varphi^{(2)}$ 的影响.如图 4(a)、4(b)所示,面内拉伸变形的最大值和最小值均在 $\theta^{(0)}$ 局部作用区域,不同于面内拉伸变形的全局分布,对称厚度剪切变形仅分布在局部加载区域内.由式(1)可知,位移场 $u_1(x_1,x_2,x_3,t)$ 是由面内拉伸变形 $u_1^{(0)}$ 和对称厚度剪切变形 $u_1^{(2)}$ 叠加所得,由于热胀冷缩效应,$u_1^{(0)}$ 和 $u_1^{(2)}$ 具有相同的符号.在挠曲电纳米板的中面,即 $x_3 = 0$ 时,式(1)的第 1 式简化为 $u_1(x_1,x_2,x_3,t) = u_1^{(0)}(x_1,x_2,t)$,因此沿 x_1 方向的变形 u_1 在板中面处取最小值,且沿板的厚度方向变形程度逐渐增大,最终在板的局部加载区域的上下表面处取得最大值.与作用温度载荷时的热胀冷缩效应不同,施加机械载荷时挠曲电纳米板表现出明显的 Poisson 效应,此时 $u_1^{(0)}$ 和 $u_1^{(2)}$ 符号相反,变形 u_1 在板中面处取最大值,在板的局部加载区域的上下表面处取得最小值.与位移场 $u_1(x_1,x_2,x_3,t)$ 类似,电势场 $\varphi(x_1, x_2,x_3,t)$ 是由零阶电势 $\varphi^{(0)}(x_1,x_2,t)$ 和二阶电势 $\varphi^{(2)}(x_1,x_2,t)$ 叠加所得.在局部加载区域,板的上下表面会产生电势垒或电势阱,如图 4(c)、4(d)所示.厚度伸缩变形 $u_3^{(1)}$ 主要发生在矩形加载区域,其在中面上下两侧均沿厚度方向拉伸(或收缩),如图 4(e)所示.综上,利用局部的温度场或者机械场,可以实现对挠曲电纳米板的变形和电势的调控.

为了进一步研究温度场和机械场对厚度伸缩变形 $u_3^{(1)}$ 的协同影响,在挠曲电纳米板的局部区域施加一个恒温场 $\theta^{(0)} = 0.001$ K 和一个变化的分布力 $f_3^{(1)}$,结果如图 5 所示.



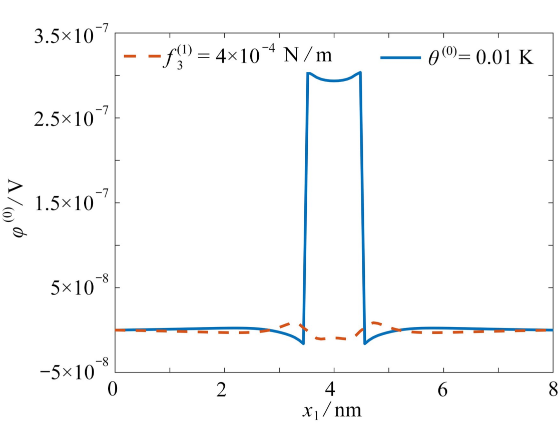
(a) 比较 $\theta^{(0)}$ 和 $f_3^{(1)}$ 对面内拉伸变形 $u_1^{(0)}$ 的影响

(a) Comparison of the effects of $\theta^{(0)}$ and $f_3^{(1)}$ on in-plane extensional deformation $u_1^{(0)}$



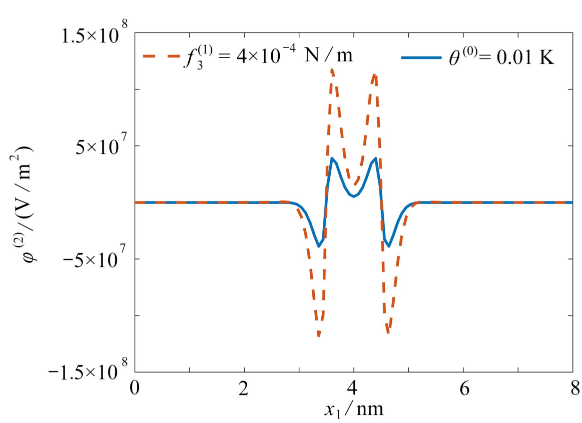
(b) 比较 $\theta^{(0)}$ 和 $f_3^{(1)}$ 对对称厚度剪切变形 $u_1^{(2)}$ 的影响

(b) Comparison of the effects of $\theta^{(0)}$ and $f_3^{(1)}$ on symmetric thickness-shear deformation $u_1^{(2)}$



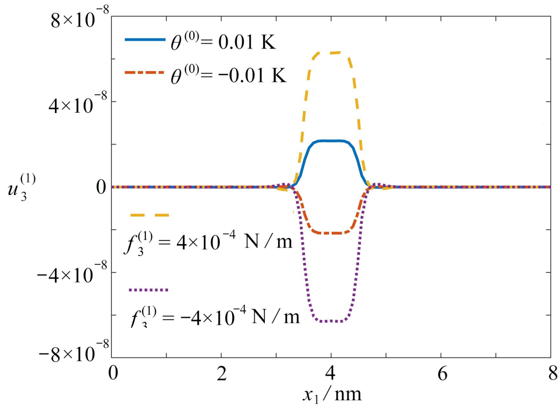
(c) 比较 $\theta^{(0)}$ 和 $f_3^{(1)}$ 对零阶电势 $\varphi^{(0)}$ 的影响

(c) Comparison of the effects of $\theta^{(0)}$ and $f_3^{(1)}$ on zero-order potential $\varphi^{(0)}$



(d) 比较 $\theta^{(0)}$ 和 $f_3^{(1)}$ 对二阶电势 $\varphi^{(2)}$ 的影响

(d) Comparison of the effects of $\theta^{(0)}$ and $f_3^{(1)}$ on 2nd-order potential $\varphi^{(2)}$



（e）比较 $\theta^{(0)}$ 和 $f_3^{(1)}$ 对厚度伸缩变形 $u_3^{(1)}$ 的影响

（e）Comparison of the effects of $\theta^{(0)}$ and $f_3^{(1)}$ on thickness-stretch deformation $u_3^{(1)}$

图 4　比较 $\theta^{(0)}$ 和 $f_3^{(1)}$ 对位移场和电势场的影响

Fig. 4　Comparison of the effects of $\theta^{(0)}$ and $f_3^{(1)}$ on the displacement field and potential field

当不考虑 $f_3^{(1)}$ 的作用时,整个板均产生拉伸变形.当 $f_3^{(1)} = -1\times10^{-5}$ N/m 时, $u_3^{(1)}$ 仅为无 $f_3^{(1)}$ 时的 30%,这表明机械载荷抵消了一部分 $\theta^{(0)}$ 作用下的热膨胀变形.当 $f_3^{(1)}$ 的绝对值继续增大,板的变形完全变为压缩变形,此时热膨胀变形完全被抵消了.结果表明,机械场与温度场的耦合关系可以用于调节挠曲电纳米板的力学行为.

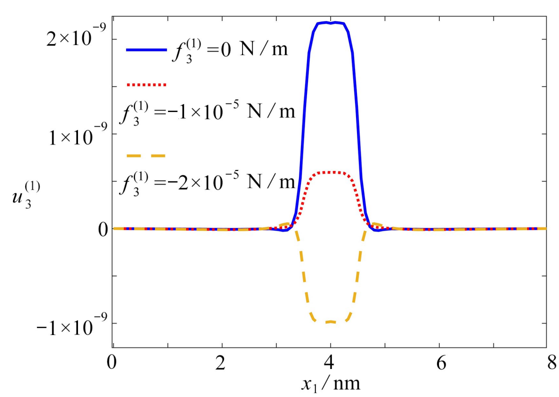


图 5　$\theta^{(0)}$ 和 $f_3^{(1)}$ 对厚度伸缩变形 $u_3^{(1)}$ 的协同影响

Fig. 5　Synergistic effects of $\theta^{(0)}$ and $f_3^{(1)}$ on thickness-stretch deformation $u_3^{(1)}$

3　结　　论

本文采用 Mindlin-Medick 理论和挠曲电理论,建立了挠曲电纳米板的理论模型.利用 Hamilton 原理求出纳米板的场方程和边界上的线积分等式,分别将二维本构方程和中面边界上外法线的方向余弦代入,得到了以基本未知量表示的挠曲电纳米板的控制方程和边界条件.然后利用双重 Fourier 级数解求解纳米板的位移场和电势场,分析局部热场和局部机械场对挠曲电纳米板变形和电场的影响,得到了以下结论:

1）挠曲电纳米板上作用有局部温度载荷时,由于热胀冷缩效应,对称厚度剪切变形会加剧面内拉伸变形的程度, u_1 在板中面处取最小值,且沿板的厚度方向变形程度逐渐增大,最终在板的局部加载区域的上下表面处取得最大值.施加机械载荷时挠曲电纳米板表现出明显的 Poisson 效应,对称厚度剪切变形则会抵消一部分面内拉伸变形的效果.

2）非接触式调控和接触式调控这两种方式为挠曲电纳米板的力电耦合行为研究提供了更多元的选择.对于考虑挠曲电效应的微纳米器件,接触式调控操作难度较大;而非接触式调控,如温度调控和磁场调控等,可操作性更强,具有很大的发展潜力.

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