http://www.applmathmech.cn

对边简支十次对称二维准晶板弯曲问题的辛分析*

范俊杰[1,2,3]，　李联和[1,2,3]，　阿拉坦仓[1,2,3]

(1. 内蒙古师范大学 数学科学学院, 呼和浩特 010022;
2. 内蒙古自治区应用数学中心, 呼和浩特 010022;
3. 无穷维哈密顿系统及其算法应用教育部重点实验室, 呼和浩特 010022)

摘要： 该文讨论了对边简支十次对称二维准晶中厚板弹性问题的辛方法.将十次对称二维准晶弹性理论基本方程转化为 Hamilton 对偶方程,采用分离变量方法,获得了相应 Hamilton 算子矩阵的辛特征值及辛特征函数系.证明了 Hamilton 算子矩阵的辛特征函数系在 Cauchy 主值意义下的完备性,在此基础上,基于 Hamilton 系统的辛特征函数展开,给出了十次对称二维准晶板弯曲问题的解析表达式.

关　键　词： 十次对称二维准晶； 辛方法； Hamilton 正则方程； 完备性
中图分类号： O343　　**文献标志码：** A　　DOI：10.21656/1000-0887.430267

Symplectic Analysis on the Bending Problem of Decagonal Symmetric 2D Quasicrystal Plates With 2 Opposite Edges Simply Supported

FAN Junjie[1,2,3],　LI Lianhe[1,2,3],　ALATANCANG[1,2,3]

(1. *College of Mathematics Science*, *Inner Mongolia Normal University*, *Hohhot* 010022, *P.R.China*;
2. *Center for Applied Mathematics Inner Mongolia*, *Hohhot* 010022, *P.R.China*;
3.*Key Laboratory of Infinite-Dimensional Hamiltonian System and Its Algorithm Application*, *Ministry of Education*, *Hohhot* 010022, *P.R.China*)

Abstract: The symplectic method for the elastic problem of decagonal symmetric 2D quasicrystal plates with 2 opposite edges simply supported, was discussed. The basic equations of the elastic theory for decagonal symmetric 2D quasicrystals were transformed into the Hamilton dual equations. With the method of separation of variables, the symplectic eigenvalues of the corresponding Hamilton operator matrix and the symplectic eigenfunction system were obtained. The completeness of the symplectic eigenfunction system of the Hamilton operator matrix in the sense of the Cauchy principal value was proved. Based on the symplectic eigenfunction expan-

* **收稿日期：** 2022-08-29; **修订日期：** 2022-10-21
基金项目： 国家自然科学基金项目(11962026;12002175;12162027;62161045);内蒙古自然科学基金项目(2020MS-01018;2021MS01013;2022ZD05;2023QN01007);内蒙古自治区高等学校科学技术研究项目(NJZY22519)
作者简介： 范俊杰(1987—),男,讲师,博士生(E-mail: tracyfan2432@126.com);
李联和(1978—),男,教授,博士,博士生导师(E-mail: nmglilianhe@163.com);
阿拉坦仓(1963—),男,教授,博士,博士生导师(通讯作者. E-mail: alatanca@imu.edu.cn).
引用格式： 范俊杰, 李联和, 阿拉坦仓. 对边简支十次对称二维准晶板弯曲问题的辛分析[J]. 应用数学和力学, 2023, **44**(7): 834-846.

sion of the Hamilton system, the analytical solution to the bending problem of the decagonal symmetric 2D quasicrystal plate was given.

Key words: decagonal symmetric 2D quasicrystal; symplectic method; Hamilton canonical equation; completeness

0　引　　言

自 Shechtman 等[1]在 1984 年首次发现准晶体之后,人们对准晶的电子结构、光学、磁性、热和弹性理论进行了大量研究[2].准晶体是具有准周期原子排列和旋转对称性的新的固体结构.准晶中有两种不同类型的低频元激发,即声子场和相位子场.

近年来,一些定量描述和分析准晶弹性理论的数学方法也得到了很大发展,并取得了一系列有价值的结果.Fan[3]和他的团队发展了经典弹性理论中的消元法,将复杂的准晶弹性方程简化为一个或几个高阶偏微分方程,然后通过 Fourier 变换法或复变函数法求解.此外,广义摄动方法[4]、Green 函数方法[5]、积分变换法[6]、复变函数法[7-8]、半逆解法[9]、势理论方法[10]和 Stroh 形式法[11]也成功地被推广和应用到准晶弹性理论问题的研究中.

辛方法由钟万勰[12]在 20 世纪 90 年代提出,用于求解弹性板和梁等问题.该方法的主要特点是求解过程在具有对偶变量的辛空间中进行,而不是在具有一种变量的 Euclid 空间中进行.在 Hamilton 系统的框架内,可以通过变量分离和辛特征函数展开的方法获得所考虑问题的精确解,而无需对解形式进行任何先验假设,这显示了辛方法的独特优势.辛方法理论基础是无穷维 Hamilton 算子特征函数系的完备性,Alatancang 等[13-14]在该领域取得了一些成果.目前,辛方法已应用于各种研究领域,如弹性[15-18]、黏弹性[19-20]、纳米力学[21-22]、断裂力学[23]、压电[24-25]、功能梯度效应[26]、波传播[27]等.Zhou 等[28]研究了有限尺寸一维六方压电准晶双材料中 V 形界面缺口的Ⅲ型断裂行为.Yang 等[29]分析了具有复杂结构的压电石英晶体的Ⅲ型断裂行为.Wang 等[30-31]研究了二维准晶的平面弹性问题.Qiao 等[32]将辛方法推广到八次对称二维准晶的平面弹性问题.

本文首次研究了十次对称二维准晶中厚板问题的辛方法及其数学理论基础.通过引入适当的状态函数,将十次对称二维准晶中厚板弹性的位移平衡方程转化为一组由一阶常微分方程组成的 Hamilton 对偶方程,并给出了相应 Hamilton 算子矩阵的特征值.基于辛特征函数系的完备性定理,得到了给定边界条件下二维十次对称准晶中厚板弯曲问题弹性场的解析表达式,并与已有结论进行了比较.

1　十次对称二维准晶板弯曲问题的基本方程

假设 z 方向为十次对称二维准晶周期方向, xoy 平面是准周期平面.对于十次对称二维准晶,由准晶弹性理论[3],有变形的几何方程

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\ w_{kj}=\frac{\partial w_k}{\partial x_j},\tag{1}$$

不考虑体力情况下的平衡方程

$$\frac{\partial \sigma_{ij}}{\partial x_j}=0,\ \frac{\partial H_{kj}}{\partial x_j}=0,\tag{2}$$

广义 Hooke 定律

$$\begin{cases}\sigma_{xx}=C_{11}\varepsilon_{xx}+C_{12}\varepsilon_{yy}+R(w_{xx}+w_{yy}),\ \sigma_{yy}=C_{12}\varepsilon_{xx}+C_{11}\varepsilon_{yy}-R(w_{xx}+w_{yy}),\\ \sigma_{zz}=C_{13}\varepsilon_{xx}+C_{13}\varepsilon_{yy}+C_{33}\varepsilon_{zz},\ \sigma_{xy}=\sigma_{yx}=2C_{66}\varepsilon_{xy}+R(w_{yx}-w_{xy}),\\ \sigma_{yz}=\sigma_{zy}=2C_{44}\varepsilon_{yz},\ \sigma_{xz}=\sigma_{zx}=2C_{44}\varepsilon_{xz},\\ H_{xx}=K_1w_{xx}+K_2w_{yy}+R(\varepsilon_{xx}-\varepsilon_{yy}),\ H_{yy}=K_1w_{yy}+K_2w_{xx}+R(\varepsilon_{xx}-\varepsilon_{yy}),\\ H_{xy}=K_1w_{xy}-K_2w_{yx}-2R\varepsilon_{xy},\ H_{yx}=K_1w_{yx}-K_2w_{yx}+2R\varepsilon_{xy},\\ H_{xz}=K_3w_{xz},\ H_{yz}=K_3w_{yz},\end{cases}\tag{3}$$

其中 $C_{66}=(C_{11}-C_{12})/2$；$\sigma_{ij}(\sigma_{ij}=\sigma_{ji})$，$\varepsilon_{ij}(\varepsilon_{ij}=\varepsilon_{ji})$，$u_i$ 和 C_{ij} 分别是声子场的应力、应变、位移和弹性常数；$H_{ij}(H_{ij}\neq H_{ji})$，$w_{ij}(w_{ij}\neq w_{ji})$，$w_i$ 和 K_i 是相位子场的应力、应变、位移和弹性常数；R 是声子场和相位子场的耦合弹性常数.这里记 $(x,y,z)=(x_1,x_2,x_3)$.

基于 Mindlin 板理论，关于十次对称二维准晶板弯曲问题假设如下[33]：

$$\begin{cases}u_x(x,y,z)=-z\phi_x(x,y),\ u_y(x,y,z)=-z\phi_y(x,y),\ u_z(x,y,z)=u_z(x,y),\\ w_x(x,y,z)=-zv_x(x,y),\ w_y(x,y,z)=-zv_y(x,y),\end{cases}\tag{4}$$

其中 $u_z(x,y)$ 为中面挠度，$\phi_x(x,y)$，$\phi_y(x,y)$ 和 $v_x(x,y)$，$v_y(x,y)$ 分别是声子场和相位子场中 xoz 与 yoz 平面内的转角.

将式(4)代入式(1)中，得到

$$\begin{cases}\varepsilon_{xx}=-z\dfrac{\partial\phi_x}{\partial x},\ \varepsilon_{yy}=-z\dfrac{\partial\phi_y}{\partial y},\ \varepsilon_{xy}=-\dfrac{z}{2}\left(\dfrac{\partial\phi_x}{\partial y}+\dfrac{\partial\phi_y}{\partial x}\right),\\ \varepsilon_{xz}=\dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial x}-\phi_x\right),\ \varepsilon_{yz}=\dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial y}-\phi_y\right),\\ w_{xx}=-z\dfrac{\partial v_x}{\partial x},\ w_{yy}=-z\dfrac{\partial v_y}{\partial y},\ w_{xy}=-z\dfrac{\partial v_x}{\partial y},\\ w_{yx}=-z\dfrac{\partial v_y}{\partial x},\ w_{xz}=-v_x,\ w_{yz}=-v_y.\end{cases}\tag{5}$$

弯矩 M_{xx}，M_{yy} 和 M_{xy}，剪力 Q_x 和 Q_y，广义弯矩 N_{xx}，N_{yy}，N_{xy} 和 N_{yx} 可以表示为

$$M_{ij}=\int_{-h/2}^{h/2}\sigma_{ij}z\mathrm{d}z,\ N_{ij}=\int_{-h/2}^{h/2}H_{ij}z\mathrm{d}z,\ Q_x=\int_{-h/2}^{h/2}\sigma_{xz}\mathrm{d}z,\ Q_y=\int_{-h/2}^{h/2}\sigma_{yz}\mathrm{d}z,\tag{6}$$

其中 h 是板的厚度.

将式(3)和(5)代入式(6)，得到弯矩、广义弯矩和剪力的表达式为

$$\begin{aligned}
M_{xx}&=-\tau\left(C_{11}\frac{\partial\phi_x}{\partial x}+C_{12}\frac{\partial\phi_y}{\partial y}+R\frac{\partial v_x}{\partial x}+R\frac{\partial v_y}{\partial y}\right),\\
M_{yy}&=-\tau\left(C_{12}\frac{\partial\phi_x}{\partial x}+C_{11}\frac{\partial\phi_y}{\partial y}-R\frac{\partial v_x}{\partial x}-R\frac{\partial v_y}{\partial y}\right),\\
M_{xy}&=-\tau\left(C_{66}\frac{\partial\phi_x}{\partial y}+C_{66}\frac{\partial\phi_y}{\partial x}-R\frac{\partial v_x}{\partial y}+R\frac{\partial v_y}{\partial x}\right),\\
N_{xx}&=-\tau\left(R\frac{\partial\phi_x}{\partial x}-R\frac{\partial\phi_y}{\partial y}+K_1\frac{\partial v_x}{\partial x}+K_2\frac{\partial v_y}{\partial y}\right),\\
N_{yy}&=-\tau\left(R\frac{\partial\phi_x}{\partial x}-R\frac{\partial\phi_y}{\partial y}+K_2\frac{\partial v_x}{\partial x}+K_1\frac{\partial v_y}{\partial y}\right),\\
N_{xy}&=-\tau\left(K_1\frac{\partial v_x}{\partial y}-K_2\frac{\partial v_y}{\partial x}-R\frac{\partial\phi_x}{\partial y}-R\frac{\partial\phi_y}{\partial x}\right),\\
N_{yx}&=-\tau\left(K_1\frac{\partial v_x}{\partial y}-K_2\frac{\partial v_y}{\partial x}+R\frac{\partial\phi_x}{\partial y}+R\frac{\partial\phi_y}{\partial x}\right),
\end{aligned}$$

其中 $\tau=h^3/12$.

十次对称二维准晶板的力和力矩平衡方程可以表示为

$$\begin{cases}\dfrac{\partial N_{xx}}{\partial x}+\dfrac{\partial N_{xy}}{\partial y}=0,\ \dfrac{\partial N_{yy}}{\partial y}+\dfrac{\partial N_{yx}}{\partial x}=0,\\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x=0,\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y=0,\\ \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+q=0,\end{cases}\tag{7}$$

其中 q 是板在单位面积上的横向荷载.

2　十次对称二维准晶板弯曲问题的 Hamilton 对偶方程

引入状态函数

$$\begin{cases}\varphi_1 = M_{yy} + \tau\left(\dfrac{2R^2}{K_1} - 2C_{66}\right)\dfrac{\partial\phi_x}{\partial x},\ \varphi_2 = N_{yy} + \tau(K_2 - K_1)\dfrac{\partial v_x}{\partial x},\\ \varphi_3 = M_{xy} - \tau\left(\dfrac{2R^2}{K_1} - 2C_{66}\right)\dfrac{\partial\phi_y}{\partial x},\ \varphi_4 = N_{xy} - \tau(K_2 - K_1)\dfrac{\partial v_y}{\partial x}.\end{cases} \tag{8}$$

则由式(7)和(8)可得到 Hamilton 正则方程为

$$\frac{\partial \boldsymbol{Z}(x,y)}{\partial y} = \boldsymbol{HZ}(x,y) + \boldsymbol{f}(x,y), \tag{9}$$

这里 $\boldsymbol{H}$ 为 Hamilton 算子,且

$$\boldsymbol{H} = \begin{pmatrix}\boldsymbol{0} & \boldsymbol{F}\\ \boldsymbol{G} & \boldsymbol{0}\end{pmatrix}, \tag{10}$$

$$\boldsymbol{F} = \begin{pmatrix} 1/C & 1 & 0 & 0 & 0\\ 1 & 0 & \dfrac{\partial}{\partial x} & \dfrac{2R}{K_1}\dfrac{\partial}{\partial x} & 0\\ 0 & \dfrac{\partial}{\partial x} & -\dfrac{K_1}{\eta_1} & -\dfrac{R}{\eta_1} & 0\\ 0 & \dfrac{2R}{K_1}\dfrac{\partial}{\partial x} & -\dfrac{R}{\eta_1} & -\dfrac{C_{66}}{\eta_1} & \dfrac{\partial}{\partial x}\\ 0 & 0 & 0 & \dfrac{\partial}{\partial x} & 0\end{pmatrix},\ \boldsymbol{G} = \begin{pmatrix} -C\dfrac{\partial^2}{\partial x^2} & 0 & C\dfrac{\partial}{\partial x} & 0 & 0\\ 0 & -\dfrac{K_1}{\eta_2} & -\dfrac{\partial}{\partial x} & 0 & -\dfrac{R}{\eta_2}\\ C\dfrac{\partial}{\partial x} & -\dfrac{\partial}{\partial x} & -C & 0 & -\dfrac{2R}{K_1}\dfrac{\partial}{\partial x}\\ 0 & 0 & 0 & 0 & -\dfrac{\partial}{\partial x}\\ 0 & -\dfrac{R}{\eta_2} & -\dfrac{2R}{K_1}\dfrac{\partial}{\partial x} & -\dfrac{\partial}{\partial x} & -\dfrac{C_{11}}{\eta_2}\end{pmatrix},$$

$$C = \frac{5h}{6}C_{66},\ \eta_1 = \tau(C_{66}K_1 - R^2),\ \eta_2 = \tau(C_{11}K_1 - R^2),$$

$$\boldsymbol{Z} = (u_z, \varphi_1, \phi_x, v_x, \varphi_2, Q_y, \phi_y, \varphi_3, \varphi_4, v_y)^{\mathrm{T}},$$

$$\boldsymbol{f} = (0,0,0,0,0,-q,0,0,0,0)^{\mathrm{T}}.$$

下面将采用辛方法求解对边简支矩形十次对称二维准晶中厚板问题,板的坐标和尺寸如图 1 所示.此问题的边界条件可以表示为

$$u_z(x)\big|_{x=0,a} = 0,\ \phi_y(x)\big|_{x=0,a} = 0,\ M_x(x)\big|_{x=0,a} = 0,\ v_y(x)\big|_{x=0,a} = 0,\ N_{xx}(x)\big|_{x=0,a} = 0. \tag{11}$$

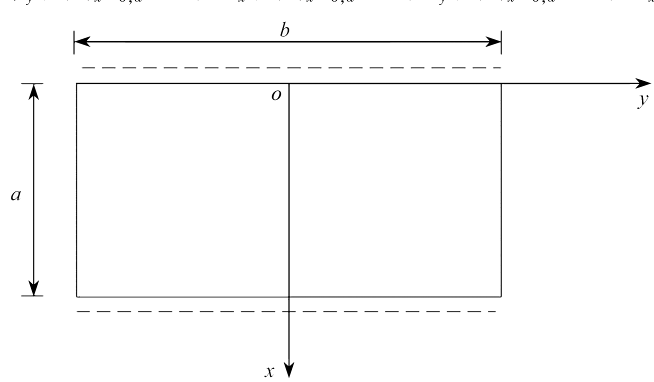

图 1　矩形准晶中厚板示意图

Fig. 1　Schematic diagram of a rectangular quasicrystal medium thickness plate

3 十次对称二维准晶板弯曲问题的辛分析

式(8)对应的齐次 Hamilton 方程为

$$\frac{\partial \boldsymbol{Z}(x,y)}{\partial y}=\boldsymbol{H}\boldsymbol{Z}(x,y), \tag{12}$$

设 $\boldsymbol{Z}=\boldsymbol{X}(x)\boldsymbol{Y}(y)$，并将其代入式(12)，得到

$$\frac{\mathrm{d}\boldsymbol{Y}(y)}{\mathrm{d}y}=\mu \boldsymbol{Y}(y), \tag{13}$$

$$\boldsymbol{H}\boldsymbol{X}(x)=\mu \boldsymbol{X}(x), \tag{14}$$

其中，μ 是 Hamilton 算子矩阵 $\boldsymbol{H}$ 的特征值，

$$\boldsymbol{X}(x)=[u_z(x),\varphi_1(x),\phi_x(x),v_x(x),\varphi_2(x),Q_y(x),\phi_y(x),\varphi_3(x),\varphi_4(x),v_y(x)]^{\mathrm{T}} \tag{15}$$

是满足边界条件(11)的特征向量.

由式(14)求得 Hamilton 算子矩阵 $\boldsymbol{H}$ 的特征多项式为

$$(\lambda^2+\mu^2)^5-\frac{Ck_1}{\eta_1}(\lambda^2+\mu^2)^4=0, \tag{16}$$

这表明 $\lambda_{1,2}=\pm\mu\mathrm{i}$ 是四重根，$\lambda_{3,4}=\pm\xi\mathrm{i}$ 是单根，其中 $\xi=\sqrt{\mu^2-CK_1/\eta_1}$.因此，满足 $\boldsymbol{H}$ 的特征函数可以表示为

$$\begin{aligned}X_i(x)=&(A_i+B_ix+C_ix^2+D_ix^3)\cos(\mu x)+\\&(E_i+F_ix+G_ix^2+H_ix^3)\sin(\mu x)+S_i\cos(\xi x)+R_i\sin(\xi x),\end{aligned} \tag{17}$$

其中 $A_i,B_i,C_i,D_i,E_i,F_i,G_i,H_i,S_i$ 和 $R_i(i=1,2,\cdots,10)$ 是待定常数.将式(17)代入式(14)中可以得到常数之间的关系.

3.1 Hamilton 算子 $\boldsymbol{H}$ 的特征值和特征向量

将式(17)代入边界条件(11)，为使式(17)存在非零解，令对应的系数行列式为零得

$$\sin^4(a\xi)\sin(a\xi)=0, \tag{18}$$

进而可得

$$\mu_n=\frac{n\pi}{a},\ \mu_{-n}=-\frac{n\pi}{a}, \tag{19}$$

$$\tilde{\mu}_n=\sqrt{\left(\frac{n\pi}{a}\right)^2+\frac{CK_1}{\eta_1}},\ \tilde{\mu}_{-n}=-\sqrt{\left(\frac{n\pi}{a}\right)^2+\frac{CK_1}{\eta_1}}. \tag{20}$$

式(19)是四重根，式(20)是单根.

将式(19)代入式(14)可得 μ_n,μ_{-n} 相应的特征函数为

$$\boldsymbol{X}_n^0(x)=[0,0,0,\cos(\mu_nx),0,0,0,0,0,\sin(\mu_nx)]^{\mathrm{T}},$$

$$\boldsymbol{X}_{-n}^0(x)=[0,0,0,\cos(\mu_nx),0,0,0,0,0,-\sin(\mu_nx)]^{\mathrm{T}}.$$

根据 $\boldsymbol{H}\boldsymbol{X}_n^k(x)=\mu_n\boldsymbol{X}_n^k(x)+\boldsymbol{X}_n^{k-1}(x)\ (k=1,2,3)$ 和边界条件(11)，可求得 $\mu_{\pm n}$ 的一阶、二阶和三阶 Jordan 形式特征函数 $\boldsymbol{X}_{\pm n}^1(x),\boldsymbol{X}_{\pm n}^2(x),\boldsymbol{X}_{\pm n}^3(x)$.将式(20)代入式(14)可得 $\tilde{\mu}_{\pm n}$ 的特征函数 $\tilde{\boldsymbol{X}}_{\pm n}(x)$.特征函数 $\boldsymbol{X}_{\pm n}^1(x)$，$\boldsymbol{X}_{\pm n}^2(x),\boldsymbol{X}_{\pm n}^3(x),\tilde{\boldsymbol{X}}_{\pm n}(x)$ 的具体表达式见附录 A .

3.2 特征函数系的辛正交性和完备性

下面讨论 Hamilton 算子 $\boldsymbol{H}$ 的特征函数系的辛正交性，给出了特征函数的展开系数，并讨论了特征函数系的完备性.

我们用 $\mathscr{X}$ 表示 Hilbert 空间 $L^2(0,a)\times L^2(0,a)\times L^2(0,a)\times L^2(0,a)\times L^2(0,a)\times L^2(0,a)\times L^2(0,a)\times L^2(0,a)\times L^2(0,a)\times L^2(0,a)$.

定义 1 算子的所有特征向量和 Jordan 形式特征函数的集合

$$\{\boldsymbol{X}_n^0,\boldsymbol{X}_n^1,\boldsymbol{X}_n^2,\boldsymbol{X}_n^3,\boldsymbol{X}_{-n}^0,\boldsymbol{X}_{-n}^1,\boldsymbol{X}_{-n}^2,\boldsymbol{X}_{-n}^3,\tilde{\boldsymbol{X}}_n,\tilde{\boldsymbol{X}}_{-n}\} \tag{21}$$

为广义特征函数系.

定义 2　对于 $\forall \boldsymbol{U}(x)=[u_1(x),\cdots,u_{10}(x)]^{\mathrm{T}}, \boldsymbol{V}(x)=[v_1(x),\cdots,v_{10}(x)]^{\mathrm{T}}\in\mathscr{X}$, 辛内积为 $\langle \boldsymbol{U}(x), \boldsymbol{V}(x)\rangle=(\boldsymbol{U}(x),\boldsymbol{J}\boldsymbol{V}(x))=\int_0^a \boldsymbol{U}(x)^{\mathrm{T}}\boldsymbol{J}\boldsymbol{V}(x)\,\mathrm{d}x$, 其中 $\boldsymbol{J}=\begin{pmatrix}\boldsymbol{0} & \boldsymbol{I}_{5\times5}\\ -\boldsymbol{I}_{5\times5} & \boldsymbol{0}\end{pmatrix}$ 是单位辛矩阵, $\boldsymbol{I}_{5\times5}$ 是 5×5 单位矩阵,且辛内积满足以下关系：

$$\langle \boldsymbol{U}(x),\boldsymbol{U}(x)\rangle=0,\ \langle \boldsymbol{U}(x),\boldsymbol{V}(x)\rangle=-\langle \boldsymbol{V}(x),\boldsymbol{U}(x)\rangle.$$

引理 1　由定义 1 给出的广义特征函数系满足如下辛正交关系：

(i) $\langle \boldsymbol{X}_n^0,\tilde{\boldsymbol{X}}_m\rangle=\langle \boldsymbol{X}_n^0,\boldsymbol{X}_m^i\rangle=0,\qquad i=0,1,2,$

$$\langle \boldsymbol{X}_n^0,\boldsymbol{X}_m^3\rangle=\begin{cases}\dfrac{aK_1\eta_2}{2R^2\mu_n^2}, & n=m,\\ 0, & n=-m;\end{cases}$$

(ii) $$\langle \boldsymbol{X}_n^1,\boldsymbol{X}_m^2\rangle=\begin{cases}\dfrac{aK_1\eta_2}{2R^2\mu_n^2}, & n=-m,\\ 0, & n=m,\end{cases}$$

$$\langle \boldsymbol{X}_n^1,\boldsymbol{X}_m^3\rangle=\begin{cases}\dfrac{a\eta_2(CK_1+4(\eta_2-\eta_1)\mu_n^2)}{4CR^2\mu_n^3}, & n=-m,\\ -\dfrac{a\eta_2(CK_1+4\eta_1\mu_n^2)}{4CR^2\mu_n^3}, & n=m,\end{cases}$$

$$\langle \boldsymbol{X}_n^1,\tilde{\boldsymbol{X}}_m\rangle=\begin{cases}-\dfrac{aCk_1}{2R\mu_n\tilde{\mu}_n}, & n=m,\\ \dfrac{aCk_1}{2R\mu_n\tilde{\mu}_n}, & n=-m;\end{cases}$$

(iii) $$\langle \boldsymbol{X}_n^2,\boldsymbol{X}_m^2\rangle=\begin{cases}-\dfrac{a\eta_2(CK_1+2\eta_1\mu_n^2)}{2CR^2\mu_n^3}, & n=-m,\\ 0, & n=m,\end{cases}$$

$$\langle \boldsymbol{X}_n^2,\tilde{\boldsymbol{X}}_m\rangle=\begin{cases}-\dfrac{aCk_1}{2R\mu_n^2\tilde{\mu}_n}, & n=-m,\\ \dfrac{aCk_1}{2R\mu_n^2\tilde{\mu}_n}, & n=m,\end{cases}$$

$$\langle \boldsymbol{X}_m^2,\tilde{\boldsymbol{X}}_n\rangle=\begin{cases}\dfrac{aCk_1}{2R\mu_n^2\tilde{\mu}_n}, & n=-m,\\ -\dfrac{aCk_1}{2R\mu_n^2\tilde{\mu}_n}, & n=m,\end{cases}$$

$$\langle \boldsymbol{X}_n^2,\boldsymbol{X}_m^3\rangle=\begin{cases}-\dfrac{a\eta_2(C^2K_1^2+4CK_1(\eta_2-2\eta_1)\mu_n^2+16\eta_2(\eta_2-\eta_1)\mu_n^4)}{8C^2K_1R^2\mu_n^4}, & n=-m,\\ \dfrac{a\eta_2(CK_1+4\eta_1\mu_n^2)}{4CR^2\mu_n^4}, & n=m;\end{cases}$$

(iv) $$\langle \boldsymbol{X}_n^3,\boldsymbol{X}_m^3\rangle=\begin{cases}-\dfrac{a\eta_2\alpha}{4C^3K_1^2R^4\mu_n^5}, & n=-m,\\ 0, & n=m,\end{cases}$$

$$\langle \boldsymbol{X}_n^3, \tilde{\boldsymbol{X}}_m \rangle = \begin{cases} \dfrac{a(3CK_1 + 8\eta_2\mu_n^2)}{8R\mu_n^3\tilde{\mu}_n}, & n = -m, \\ -\dfrac{a(3CK_1 + 8\eta_2\mu_n^2)}{8R\mu_n^3\tilde{\mu}_n}, & n = m, \end{cases}$$

其中

$$\alpha = C^3C_{11}K_1^4 + 3C^2K_1^2R^2(2\eta_1 + \eta_2)\mu_n^2 + 16CK_1R^2\eta_1\eta_2\mu_n^4 - 16R^2(\eta_1 - \eta_2)^2\eta_2\mu_n^6, \qquad n = \pm 1, \pm 2, \cdots.$$

定理 1 在 Cauchy 主值意义下,Hamilton 算子的特征函数系(21)在 Hilbert 空间 $\mathscr{X}$ 中是完备的.

证明 对于 $\forall \boldsymbol{F} = [f_1(x), f_2(x), f_3(x), f_4(x), f_5(x), f_6(x), f_7(x), f_8(x), f_9(x), f_{10}(x)]^{\mathrm{T}} \in \mathscr{X}$, 在 Cauchy 主值意义下由广义特征函数系(21)给出的辛-Fourier 展开式为

$$\boldsymbol{F} = \sum_{n=1}^{+\infty}\left(\sum_{i=0}^{3} c_n^i \boldsymbol{X}_n^i + \sum_{i=0}^{3} c_{-n}^i \boldsymbol{X}_{-n}^i + c_n\tilde{\boldsymbol{X}}_n + c_{-n}\tilde{\boldsymbol{X}}_{-n}\right). \tag{22}$$

根据引理 1,可知

$$c_n^3 = \frac{\langle \boldsymbol{F}, \boldsymbol{X}_n^0 \rangle}{\langle \boldsymbol{X}_n^3, \boldsymbol{X}_n^0 \rangle},\ c_n^2 = \frac{\langle \boldsymbol{F}, \boldsymbol{X}_{-n}^1 \rangle - c_n^3\langle \boldsymbol{X}_n^3, \boldsymbol{X}_{-n}^1 \rangle - c_{-n}^3\langle \boldsymbol{X}_{-n}^3, \boldsymbol{X}_{-n}^1 \rangle}{\langle \boldsymbol{X}_n^2, \boldsymbol{X}_{-n}^1 \rangle},$$

$$c_n^1 = \frac{\langle \boldsymbol{F}, \boldsymbol{X}_{-n}^2 \rangle - c_n^2\langle \boldsymbol{X}_n^2, \boldsymbol{X}_{-n}^2 \rangle - c_n^3\langle \boldsymbol{X}_n^3, \boldsymbol{X}_{-n}^2 \rangle - c_{-n}^3\langle \boldsymbol{X}_{-n}^3, \boldsymbol{X}_{-n}^2 \rangle}{\langle \boldsymbol{X}_n^1, \boldsymbol{X}_{-n}^2 \rangle},$$

$$c_n^0 = \frac{\langle \boldsymbol{F}, \boldsymbol{X}_n^3 \rangle - c_n^1\langle \boldsymbol{X}_n^1, \boldsymbol{X}_n^3 \rangle - c_n^2\langle \boldsymbol{X}_n^2, \boldsymbol{X}_n^3 \rangle - c_{-n}^1\langle \boldsymbol{X}_{-n}^1, \boldsymbol{X}_n^3 \rangle - c_{-n}^2\langle \boldsymbol{X}_{-n}^2, \boldsymbol{X}_n^3 \rangle - c_{-n}^3\langle \boldsymbol{X}_{-n}^3, \boldsymbol{X}_n^3 \rangle}{\langle \boldsymbol{X}_n^0, \boldsymbol{X}_n^3 \rangle},$$

$$c_n = \frac{\langle \boldsymbol{F}, \tilde{\boldsymbol{X}}_{-n} \rangle - (c_n^1 - c_{-n}^1)\langle \boldsymbol{X}_n^1, \tilde{\boldsymbol{X}}_{-n} \rangle - (c_n^2 + c_{-n}^2)\langle \boldsymbol{X}_n^2, \tilde{\boldsymbol{X}}_{-n} \rangle - (c_n^3 - c_{-n}^3)\langle \boldsymbol{X}_n^3, \tilde{\boldsymbol{X}}_{-n} \rangle}{\langle \tilde{\boldsymbol{X}}_n, \tilde{\boldsymbol{X}}_{-n} \rangle},$$

c_n^i 具体结果见附录 B.将上述系数代入式(22),整理得

$$\begin{aligned}
\boldsymbol{F} = {} & \sum_{n=1}^{+\infty}\left(\sum_{i=0}^{3} c_n^i \boldsymbol{X}_n^i + \sum_{i=0}^{3} c_{-n}^i \boldsymbol{X}_{-n}^i + c_n\tilde{\boldsymbol{X}}_n + c_{-n}\tilde{\boldsymbol{X}}_{-n}\right) = \\
& \sum_{n=1}^{+\infty}\Bigg(\left(\frac{2}{a}\int_0^a \sin\frac{n\pi x}{a} f_1(x)\,\mathrm{d}x\right)\sin\frac{n\pi x}{a}, \left(\frac{2}{a}\int_0^a \sin\frac{n\pi x}{a} f_2(x)\,\mathrm{d}x\right)\sin\frac{n\pi x}{a}, \\
& \left(\frac{2}{a}\int_0^a \cos\frac{n\pi x}{a} f_3(x)\,\mathrm{d}x\right)\cos\frac{n\pi x}{a}, \left(\frac{2}{a}\int_0^a \cos\frac{n\pi x}{a} f_4(x)\,\mathrm{d}x\right)\cos\frac{n\pi x}{a}, \\
& \left(\frac{2}{a}\int_0^a \sin\frac{n\pi x}{a} f_5(x)\,\mathrm{d}x\right)\sin\frac{n\pi x}{a}, \left(\frac{2}{a}\int_0^a \sin\frac{n\pi x}{a} f_6(x)\,\mathrm{d}x\right)\sin\frac{n\pi x}{a}, \\
& \left(\frac{2}{a}\int_0^a \sin\frac{n\pi x}{a} f_7(x)\,\mathrm{d}x\right)\sin\frac{n\pi x}{a}, \left(\frac{2}{a}\int_0^a \cos\frac{n\pi x}{a} f_8(x)\,\mathrm{d}x\right)\cos\frac{n\pi x}{a}, \\
& \left(\frac{2}{a}\int_0^a \cos\frac{n\pi x}{a} f_9(x)\,\mathrm{d}x\right)\cos\frac{n\pi x}{a}, \left(\frac{2}{a}\int_0^a \sin\frac{n\pi x}{a} f_{10}(x)\,\mathrm{d}x\right)\sin\frac{n\pi x}{a}\Bigg)^{\mathrm{T}},
\end{aligned}$$

其中,$f_i(x)\,(i = 1,2,3,\cdots,10)$ 是按 Hilbert 空间 $L^2(0,a)$ 中的完全正交基 $\left\{\sin\dfrac{n\pi x}{a}\right\}_{n=1}^{+\infty}$ 或 $\left\{\cos\dfrac{n\pi x}{a}\right\}_{n=1}^{+\infty}$ 展开的 Fourier 级数.因此,在 Cauchy 主值意义下,Hamilton 算子矩阵 $\boldsymbol{H}$ 的广义特征函数系统在 Hilbert 空间 $\mathscr{X}$ 中是完备的.

3.3 十次对称二维准晶板弯曲的辛解析解

由定理 1 和解的叠加原理,式(9)的通解可以表示为

$$\boldsymbol{Z}(x,y) = \sum_{n=1}^{+\infty}\left(\sum_{i=0}^{3} \boldsymbol{Y}_n^i(y)\boldsymbol{X}_n^i + \sum_{i=0}^{3} \boldsymbol{Y}_{-n}^i(y)\boldsymbol{X}_{-n}^i + \boldsymbol{Y}_n(y)\tilde{\boldsymbol{X}}_n + \boldsymbol{Y}_{-n}(y)\tilde{\boldsymbol{X}}_{-n}\right), \tag{23}$$

式(9)中向量 $\boldsymbol{f} = (0,0,0,0,0,-q,0,0,0,0)^{\mathrm{T}}$ 可以表示为

$$\boldsymbol{f}(x,y)=\sum_{n=1}^{+\infty}\left(\sum_{i=0}^{3}G_n^i(y)\boldsymbol{X}_n^i+\sum_{i=0}^{3}G_{-n}^i(y)\boldsymbol{X}_{-n}^i+G_n(y)\tilde{\boldsymbol{X}}_n+G_{-n}(y)\tilde{\boldsymbol{X}}_{-n}\right),\tag{24}$$

其中

$$G_n^2(y)=-G_{-n}^2(y)=\frac{qR}{n\pi\eta_2}(1-\cos n\pi)=\frac{2qR}{n\pi\eta_2}\sin^2\left(\frac{n\pi}{2}\right),$$

其余系数全为零.

将式(23)和式(24)代入式(9)得到微分方程组

$$\begin{cases}\dfrac{\mathrm{d}Y_{\pm n}(y)}{\mathrm{d}y}=\tilde{\mu}_{\pm n}Y_{\pm n}(y),\ \dfrac{\mathrm{d}Y_{\pm n}^3(y)}{\mathrm{d}y}=\mu_{\pm n}Y_{\pm n}^3(y),\\ \dfrac{\mathrm{d}Y_{\pm n}^i(y)}{\mathrm{d}y}=\mu_{\pm n}Y_{\pm n}^i(y)+Y_{\pm n}^{i+1}(y),\qquad i=0,1,\\ \dfrac{\mathrm{d}Y_{\pm n}^2(y)}{\mathrm{d}y}=\mu_{\pm n}Y_{\pm n}^2(y)+Y_{\pm n}^3(y)-G_{\pm n}^2(y).\end{cases}\tag{25}$$

解微分方程组(25)得

$$\begin{cases}Y_{\pm n}(y)=d_{\pm n}\mathrm{e}^{\tilde{\mu}_{\pm n}y},\ Y_{\pm n}^3(y)=d_{\pm n}^3\mathrm{e}^{\mu_{\pm n}y},\\ Y_{\pm n}^2(y)=(d_{\pm n}^2+d_{\pm n}^3y)\mathrm{e}^{\mu_{\pm n}y}+\dfrac{1}{\mu_{\pm n}}G_{\pm n}^2,\\ Y_{\pm n}^1(y)=\left(d_{\pm n}^1+d_{\pm n}^2y+\dfrac{1}{2}d_{\pm n}^3y^2\right)\mathrm{e}^{\mu_{\pm n}y}-\dfrac{1}{\mu_{\pm n}^2}G_{\pm n}^2,\\ Y_{\pm n}^0(y)=\left(d_{\pm n}^0+d_{\pm n}^1y+\dfrac{1}{2}d_{\pm n}^2y^2+\dfrac{1}{6}d_{\pm n}^3y^3\right)\mathrm{e}^{\mu_{\pm n}y}+\dfrac{1}{\mu_{\pm n}^3}G_{\pm n}^2,\end{cases}\tag{26}$$

其中 $d_{\pm n}^i$ 是未知常数,可以通过 $y=\pm b/2$ 处的边界条件来确定.

为了求解未知常数 $d_{\pm n}^i$, 考虑矩形区域中 y 方向简单支撑的混合边界条件:

$$\begin{cases}u_z(x,y)\big|_{y=\pm b/2}=0,\ \phi_x(x,y)\big|_{y=\pm b/2}=0,\ M_y(x,y)\big|_{y=\pm b/2}=0,\\ v_x(x,y)\big|_{y=\pm b/2}=0,\ N_{yy}(x,y)\big|_{y=\pm b/2}=0.\end{cases}\tag{27}$$

结合式(23)、(24)、(26)、(27)和状态函数(8),可求得十次对称二维准晶板的挠度 $u_z(x,y)$, 弯矩 $M_{xx}(x,y)$, $M_{yy}(x,y)$ 和广义弯矩 $N_{xx}(x,y)$, $N_{yy}(x,y)$ 的解析解,其中系数 $d_{\pm n}^i$ 见附录 C.

挠度 $u_z(x,y)$ 的解析表达式为

$$u_z(x,y)=\frac{qK_1a^4}{\eta_2}\sum_{n=1}^{+\infty}\frac{2\sin^2\left(\dfrac{n\pi}{2}\right)\sin(\mu_nx)}{Ca^4n\pi(1+\mathrm{e}^{b\mu_n})^2\mu_n^4}S_n(y),\tag{28}$$

其中

$$\begin{aligned}S_n(y)=\Bigg(&2\frac{\eta_2}{K_1}\mu_n^2(\mathrm{e}^{(b/2-y)\mu_n}+\mathrm{e}^{(b/2+y)\mu_n}-1-\mathrm{e}^{b\mu_n})(1+\mathrm{e}^{b\mu_n})+\\&\bigg(C(\mathrm{e}^{(b/2-y)\mu_n}+\mathrm{e}^{(b/2+y)\mu_n})\left(2-\frac{b}{2}\mu_n\right)+\\&(\mathrm{e}^{(3b/2-y)\mu_n}+\mathrm{e}^{(3b/2+y)\mu_n})\left(2+\frac{b}{2}\mu_n\right)-2(\mathrm{e}^{b\mu_n}+1)^2\bigg)+\\&C\mu_n\mathrm{e}^{(b/2-y)\mu_n}(1-\mathrm{e}^{2y\mu_n})(1+\mathrm{e}^{b\mu_n})y\Bigg).\end{aligned}$$

弯矩 M_{yy}, M_{xx} 和广义弯矩 N_{xx}, N_{yy} 的解析解表达式为

$$M_{yy}(x,y)=qa^2\sum_{n=1}^{+\infty}\frac{\tau\sin^2\left(\dfrac{n\pi}{2}\right)\sin(\mu_nx)}{a^2n\pi(1+\mathrm{e}^{b\mu_n})^2\eta_2\mu_n^2}h_n(y),\tag{29}$$

$$M_{xx}(x,y)=qa^2\sum_{n=1}^{+\infty}\frac{\tau\sin^2\left(\frac{n\pi}{2}\right)\sin(\mu_n x)}{a^2 n\pi(1+\mathrm{e}^{b\mu_n})^2\eta_2\mu_n^2}\tilde{h}_n(y),\tag{30}$$

$$N_{xx}(x,y)=qa^2\sum_{n=1}^{+\infty}\frac{(K_1-K_2)\tau R\sin^2\left(\frac{n\pi}{2}\right)\sin(\mu_n x)}{2a^2 n\pi(1+\mathrm{e}^{b\mu_n})^3\eta_2\mu_n^2}k_n(y),\tag{31}$$

$$N_{yy}(x,y)=-qa^2\sum_{n=1}^{+\infty}\frac{(K_1-K_2)\tau R\sin^2\left(\frac{n\pi}{2}\right)\sin(\mu_n x)}{2a^2 n\pi(1+\mathrm{e}^{b\mu_n})^3\eta_2\mu_n^2}\tilde{k}_n(y),\tag{32}$$

其中 $h_n(y)$，$\tilde{h}_n(y)$，$k_n(y)$，$\tilde{k}_n(y)$ 见附录 D．

当不考虑相位子场时，式(28)和式(29)与文献[34]中经典板的解完全相同．

4 数值结果

为讨论所提出方法的有效性，本节给出了十次对称二维准晶板在不同长宽比下的挠度 $u_z(qa^4K_1/\eta_2)$ 在中点处 $(a/2,0)$ 的数值解，其中材料参数为[33]

$$C_{11}=23.43\ \mathrm{GPa},\ C_{12}=5.741\ \mathrm{GPa},\ K_1=12.2\ \mathrm{GPa},\ K_2=2.4\ \mathrm{GPa},\ R=-0.11\ \mathrm{GPa}.$$

如表 1 所示，本文获得的级数解具有较好的收敛性．

表 1 不同宽度和厚度比下中点处的挠度

Table 1 Deflections at the midpoint under different width-to-thickness ratios

b/a	h/a	n	$u_z(qa^4K_1/\eta_2)$
1.0	0.2	5	0.004 846 34
		15	0.004 842 8
		25	0.004 843 01
		35	0.004 842 96
		45	0.004 842 98
		55	0.004 842 97
		65	0.004 842 97
1.5	0.2	5	0.008 795 16
		15	0.008 791 62
		25	0.008 791 83
		35	0.008 791 78
		45	0.008 791 8
		55	0.008 791 79
		65	0.008 791 79
2.0	0.2	5	0.011 338 6
		15	0.011 335 1
		25	0.011 335 3
		35	0.011 335 2
		45	0.011 335 3
		55	0.011 335 2
		65	0.011 335 2

5 结 论

本文将辛方法应用于十次对称二维准晶中厚板弯曲的问题研究，通过引入适当的对偶变量，将问题表述为 Hamilton 正则系统．证明了相应 Hamilton 算子矩阵的广义特征函数系统在 Cauchy 主值意义下具有辛正交性和完备性，保证了 Hamilton 系统分离变量的可行性，并导出了精确解析解．随后，对解析解进行了退化比较

研究，并给出数值结果，以证明解析解的收敛性和准确性.本文方法的优点在于不需要提前假设任何函数，方法直观合理，为解决准晶板的弹性问题提供了一种系统的方法.此外，该方法有望应用于准晶板的屈曲和振动等问题的研究中.

致谢 本文作者衷心感谢内蒙古师范大学基本科研业务费（2022JBZD010；2022JBXC013）、内蒙古农业大学基础学科科研启动基金项目（JC2020002）和内蒙古师范大学研究生科研创新基金项目（CXJJB22011）对本文的资助.

附 录 A

$$\boldsymbol{X}_n^1(x)=\sin(\mu_n x)\left[\frac{K_1}{2R\mu_n^2},0,\frac{K_1\cot(\mu_n x)}{2R\mu_n},0,0,0,\frac{K_1}{2R\mu_n},0,0,0\right]^{\mathrm T},$$

$$\boldsymbol{X}_{-n}^1(x)=\sin(\mu_n x)\left[-\frac{K_1}{2R\mu_n^2},0,-\frac{K_1\cot(\mu_n x)}{2R\mu_n},0,0,0,\frac{K_1}{2R\mu_n},0,0,0\right]^{\mathrm T},$$

$$\boldsymbol{X}_n^2(x)=\sin(\mu_n x)\left[-\frac{CK_1+2\eta_2\mu_n^2}{2CR\mu_n^3},-\frac{\eta_2}{R\mu_n},-\frac{K_1\cot(\mu_n x)}{2R\mu_n^2},0,0,-\frac{\eta_2}{R},0,0,0,0\right]^{\mathrm T},$$

$$\boldsymbol{X}_{-n}^2(x)=\sin(\mu_n x)\left[-\frac{CK_1+2\eta_2\mu_n^2}{2CR\mu_n^3},-\frac{\eta_2}{R\mu_n},-\frac{K_1\cot(\mu_n x)}{2R\mu_n^2},0,0,\frac{\eta_2}{R},0,0,0,0\right]^{\mathrm T},$$

$$\begin{aligned}\boldsymbol{X}_n^3(x)=\sin(\mu_n x)\Bigg[&\frac{3C^2K_1^2+4C\eta_2\mu_n^2K_1+16(\eta_2-\eta_1)\eta_2\mu_n^4}{8C^2R\mu_n^4K_1},\frac{\eta_2(CK_1+2\eta_2\mu_n^2)}{CK_1R\mu_n^2},\\&\frac{3CK_1+8\eta_2\mu_n^2}{8CR\mu_n^3}\cot(\mu_n x),0,\frac{-K_1\eta_2}{2R^2\mu_n^2},\frac{\eta_2(CK_1+4(\eta_2-\eta_1)\mu_n^2)}{2CR\mu_nk_1},\frac{CK_1+8\eta_2\mu_n^2}{-8CR\mu_n^3},\\&\frac{\eta_2(CK_1+4\eta_1\mu_n^2)}{-2CK_1R\mu_n^2}\cot(\mu_n x),\frac{K_1\eta_2\cot(\mu_n x)}{2R^2\mu_n^2},\frac{2C_{11}K_1-R^2}{4R^2\mu_n^3}\Bigg]^{\mathrm T},\end{aligned}$$

$$\begin{aligned}\boldsymbol{X}_{-n}^3(x)=\sin(\mu_n x)\Bigg[&-\frac{3C^2K_1^2+4C\eta_2\mu_n^2K_1+16(\eta_2-\eta_1)\eta_2\mu_n^4}{8C^2R\mu_n^4K_1},-\frac{\eta_2(CK_1+2\eta_2\mu_n^2)}{CK_1R\mu_n^2},\\&-\frac{3CK_1+8\eta_2\mu_n^2}{8CR\mu_n^3}\cot(\mu_n x),0,\frac{K_1\eta_2}{2R^2\mu_n^2},\frac{\eta_2(CK_1+4(\eta_2-\eta_1)\mu_n^2)}{2CR\mu_nk_1},-\frac{CK_1+8\eta_2\mu_n^2}{8CR\mu_n^3},\\&-\frac{\eta_2(CK_1+4\eta_1\mu_n^2)}{2CK_1R\mu_n^2}\cot(\mu_n x),\frac{K_1\eta_2}{2R^2\mu_n^2}\cot(\mu_n x),\frac{2C_{11}K_1-R^2}{4R^2\mu_n^3}\Bigg]^{\mathrm T},\end{aligned}$$

$$\begin{aligned}\tilde{\boldsymbol{X}}_n(x)=\Bigg[&0,0,\cos(\xi_n x),\frac{R(4\eta_1\tilde{\mu}_n^2-3CK_1)}{CK_1^2}\cos(\xi_n x),0,-\frac{C\xi_n}{\tilde{\mu}_n}\sin(\xi_n x),\\&\frac{\xi_n}{\tilde{\mu}_n}\sin(\xi_n x),-\frac{C}{\tilde{\mu}_n}\cos(\xi_n x),0,\frac{R\xi_n(4\eta_1\tilde{\mu}_n^2-CK_1)}{CK_1^2\tilde{\mu}_n}\sin(\xi_n x)\Bigg]^{\mathrm T},\end{aligned}$$

$$\begin{aligned}\tilde{\boldsymbol{X}}_{-n}(x)=\Bigg[&0,0,\cos(\xi_n x),\frac{R(4\eta_1\tilde{\mu}_n^2-3CK_1)}{CK_1^2}\cos(\xi_n x),0,\frac{C\xi_n}{\tilde{\mu}_n}\sin(\xi_n x),\\&-\frac{\xi_n}{\tilde{\mu}_n}\sin(\xi_n x),\frac{C}{\tilde{\mu}_n}\cos(\xi_n x),0,\frac{R\xi_n(4\eta_1\tilde{\mu}_n^2-CK_1)}{-CK_1^2\tilde{\mu}_n}\sin(\xi_n x)\Bigg]^{\mathrm T},\end{aligned}$$

其中 $\mu_n=\dfrac{n\pi}{a},\ \xi_n=\dfrac{n\pi}{a},\ \tilde{\mu}_n=\sqrt{\dfrac{CK_1}{\eta_1}+\xi_n^2}$.

附 录 B

$$c_n^3=\frac{2R^2n^2\pi^2}{a^3K_1\eta_2}\int_0^a\left(f_9(x)\cos\frac{n\pi x}{a}-f_5(x)\sin\frac{n\pi x}{a}\right)\mathrm dx,$$

$$\begin{aligned}c_n^2=-\frac{R}{aCK_1^2\eta_2}\int_0^a&(Ck_1^2(f_8(x)\mu_n\cos(\mu_n x)+f_6(x)\sin(\mu_n x)+f_2(x)\mu_n\sin(\mu_n x))+\\&4f_9(x)R(2\eta_1-\eta_2)\mu_n^3\cos(\mu_n x)+f_5(x)(2Ck_1R\mu_n+4R\eta_2\mu_n^3)\sin(\mu_n x))\mathrm dx,\end{aligned}$$

$$c_n^1 = -\frac{R}{2aC^2K_1^3\eta_2}\int_0^a(((16R\eta_1\eta_2\mu_n^4 - C^2K_1^2R - 4CK_1R\eta_2\mu_n^2)f_9(x) + 4CK_1^2\eta_2\mu_n^2 f_8(x))\cos(\mu_n x) + (CK_1^2(4\eta_2\mu_n^2 + 2CK_1)f_2(x) + R(12CK_1\eta_2\mu_n^2 + 16\eta_1\eta_2\mu_n^4 + C^2K_1^2)f_5(x) + (4C^2K_1^2\eta_2\mu_n(f_7(x) + f_1(x)\mu_n)))\sin(\mu_n x))\mathrm{d}x,$$

$$c_n^0 = \frac{-1}{2aC^3K_1^4\eta_2\mu_n}\int_0^a(C^3K_1^3(2K_1C_{11} - R^2)f_9(x)\cos(\mu_n x) - 2C^3K_1^4\eta_2\mu_n(f_4(x)\cos(\mu_n x) + f_{10}(x)\sin(\mu_n x)) + (2C^2K_1^3R\eta_2 + 8CK_1^2R\eta_1\eta_2\mu_n^4)f_2(x)\sin(\mu_n x) + (2C^2K_1^3R\eta_2 + 8C^2K_1^2R\eta_1\eta_2\mu_n^3)f_3(x)\cos(\mu_n x) + (6C^2K_1^2R^2\eta_2\mu_n^2 + 32CK_1R^2\eta_1\eta_2\mu_n^4 + 32R^2\eta_1^2\eta_2\mu_n^6)f_5(x)\sin(\mu_n x) - (2C^2K_1^3R\eta_2\mu_n^2 + 8C^2K_1^2R\eta_1\eta_2\mu_n^4)f_1(x)\sin(\mu_n x))\mathrm{d}x,$$

$$c_n = \frac{1}{a}\int_0^a\left(f_3(x)\cos(\mu_n x) + \left(\frac{f_2(x)}{C} - f_1(x) + \frac{R}{CK_1}\left(3 + \frac{4\eta_1\mu_n^2}{CK_1}\right)f_5(x)\right)\mu_n\sin(\mu_n x)\right)\mathrm{d}x,$$

其中 $n = \pm 1, \pm 2, \cdots$.

附 录 C

$$d_n^0 = d_{-n}^0 = -\frac{\mathrm{e}^{b\mu_n/2}qR(8 - 4b\mu_n + b^2\mu_n^2 + \mathrm{e}^{b\mu_n}(16 - 6b^2\mu_n^2) + \mathrm{e}^{2b\mu_n}(8 + 4b\mu_n + b^2\mu_n^2))\sin^2\left(\frac{n\pi}{2}\right)}{2n\pi(1 + \mathrm{e}^{b\mu_n})^3\eta_2\mu_n^3},$$

$$d_n^1 = -d_{-n}^1 = \frac{2\mathrm{e}^{b\mu_n/2}qR(2 - b\mu_n + \mathrm{e}^{b\mu_n}(2 + b\mu_n))\sin^2\left(\frac{n\pi}{2}\right)}{(1 + \mathrm{e}^{b\mu_n})^2 n\pi\eta_2\mu_n^2},$$

$$d_n^2 = d_{-n}^2 = -\frac{4\mathrm{e}^{b\mu_n/2}qR}{(1 + \mathrm{e}^{b\mu})n\pi\eta_2\mu_n}\sin^2\left(\frac{n\pi}{2}\right),\ d_n^3 = d_{-n}^3 = d_n = d_{-n} = 0.$$

附 录 D

$$h_n(y) = \mathrm{e}^{-y\mu_n}(C_{11}K_1\mathrm{e}^{b\mu_n/2}(-2 + \mathrm{e}^{(b+2y)\mu_n}(b - 2y)\mu_n + 2y\mu_n - \mathrm{e}^{2y\mu_n}(b + 2y)\mu_n + \eta_2(2 - b\mu_n) + \mathrm{e}^{b\mu_n}(-2 + 2y\mu_n + \eta_2(2 + b\mu_n))) + 4(R^2 + C_{12}k_1)\mathrm{e}^{y\mu_n}(1 + \mathrm{e}^{b\mu_n})^2 - \mathrm{e}^{3b\mu_n/2+2y\mu_n}(2R^2(2 + b\mu_n - 2y\mu_n) + C_{12}K_1(4 + b\mu_n - 2y\mu_n)) + \mathrm{e}^{(b+4y)\mu_n/2}(2R^2(-2 + b\mu_n - 2y\mu_n) + C_{12}K_1(-4 + b\mu_n - 2y\mu_n)) + \mathrm{e}^{b\mu_n/2}(2R^2(-2y\mu_n + \eta_2(-2 + b\mu_n)) - C_{12}K_1(2 + 2y\mu_n + \eta_2(2 - b\mu_n))) - \mathrm{e}^{3b\mu_n/2}(2R^2(2y\mu_n + \eta_2(2 + b\mu_n)) + C_{12}K_1(2 + 2y\mu_n + \eta_2(2 + b\mu_n)))),$$

$$\tilde{h}_n(y) = \mathrm{e}^{-y\mu_n}(C_{12}K_1\mathrm{e}^{b\mu_n/2}(-2 + \mathrm{e}^{(b+2y)\mu_n}(b - 2y)\mu_n + 2y\mu_n - \mathrm{e}^{2y\mu_n}(b + 2y)\mu_n + \eta_2(2 - b\mu_n) + \mathrm{e}^{b\mu_n}(-2 + 2y\mu_n + \eta_2(2 + b\mu_n))) + 4(C_{11}K_1 - R^2)\mathrm{e}^{y\mu_n}(1 + \mathrm{e}^{b\mu_n})^2 - \mathrm{e}^{3b\mu_n/2+2y\mu_n}(C_{11}K_1(4 + b\mu_n - 2y\mu_n) - 2R^2(2 + b\mu_n - 2y\mu_n)) + \mathrm{e}^{(b+4y)\mu_n/2}(2R^2(2 - b\mu_n + 2y\mu_n) + C_{11}K_1(-4 + b\mu_n - 2y\mu_n)) + \mathrm{e}^{b\mu_n/2}(2R^2(2y\mu_n - \eta_2(-2 + b\mu_n)) - C_{11}K_1(2 + 2y\mu_n + \eta_2(2 - b\mu_n))) - \mathrm{e}^{3b\mu_n/2}(C_{11}K_1(2 + 2y\mu_n + \eta_2(2 + b\mu_n)) - 2R^2(2y\mu_n + \eta_2(2 + b\mu_n)))),$$

$$k_n(y) = \mathrm{e}^{(b-2y)\mu_n/2}(8 - 4b\mu_n - 8y\mu_n + b^2\mu_n^2 + \mathrm{e}^{2(b+y)\mu_n}(b - 2y)^2\mu_n^2 + 4y^2\mu_n^2 + \mathrm{e}^{2y\mu_n}(b + 2y)^2\mu_n^2 - 2\mathrm{e}^{(b+2y)\mu_n}(3b^2 - 4y^2)\mu_n^2 - 4\eta_2(-2 + b\mu_n)(-1 + y\mu_n) - 2\mathrm{e}^{b\mu_n}(-8 + 8y\mu_n + 3b^2\mu_n^2 - 4y^2\mu_n^2 + \eta_2(8 - 8y\mu_n)) + \mathrm{e}^{2b\mu_n}(8 + 4b\mu_n - 8y\mu_n + b^2\mu_n^2 + 4y^2\mu_n^2 + 4\eta_2(2 + b\mu_n)(-1 + y\mu_n))),$$

$$\tilde{k}_n(y) = \mathrm{e}^{-y\mu_n}(-8(1 + \mathrm{e}^{b\mu_n})^3\mathrm{e}^{y\mu_n} - 2\mathrm{e}^{3b\mu_n/2+2y\mu_n}(-8 + 8y\mu_n + 3b^2\mu_n^2 - 4y^2\mu_n^2) + 2\mathrm{e}^{3b\mu/2}(8 + 8y\eta_2\mu_n - 3b^2\mu_n^2 + 4y^2\mu_n^2) + \mathrm{e}^{5b\mu_n/2+2y\mu}(8 - 8y\mu_n + b^2\mu_n^2 + 4y^2\mu_n^2 - 4b\mu_n(-1 + y\mu_n)) + \mathrm{e}^{(b+4y)\mu_n/2}(8 - 8y\mu_n + b^2\mu_n^2 + 4y^2\mu_n^2 + 4b\mu_n(-1 + y\mu_n)) + \mathrm{e}^{b\mu_n/2}(8 + 8y\eta_2\mu_n + b^2\mu_n^2 + 4y^2\mu_n^2 - 4b\mu_n(1 + y\eta_2\mu_n)) + \mathrm{e}^{5b\mu_n/2}(8 + 8y\eta_2\mu_n + b^2\mu_n^2 + 4y^2\mu_n^2 + 4b\mu_n(1 + y\eta_2\mu_n))).$$

参考文献(References):

[1] SHECHTMAN D, BLECH I, GRATIAS D, et al. Metallic phase with long-range orientational order and no translational symmetry[J]. *Physical Review Letters*, 1984, **53**(20): 1951-1953.

[2] DUBOIS J M. *Useful Quasicrystals*[M]. Singapore: World Scientific Publishing, 2005.

[3] FAN T Y. *Mathematical Theory of Elasticity of Quasicrystals and Its Applications*[M]. Berlin: Springer, 2011.

[4] RICKER M, BACHTELER J, TREBIN H R. Elastic theory of icosahedral quasicrystals application to straight dislocations[J]. *The European Physical Journal B*: *Condensed Matter and Complex Systems*, 2001, **23**(3): 351-363.

[5] DE P, PELCOVITS R A. Linear elasticity theory of pentagonal quasicrystals[J]. *Physical Review B*, 1987, **35**(13): 8609-8620.

[6] YASLAN H C. Equations of anisotropic elastodynamics in 3D quasicrystals as a symmetric hyperbolic system: deriving the time-dependent fundamental solutions[J]. *Applied Mathematical Modelling*, 2013, **37**(18/19): 8409-8418.

[7] GAO Y, ZHAO Y T, ZHAO B S. Boundary value problems of holomorphic vector functions in 1D QCs[J]. *Physica B*: *Condensed Matter*, 2007, **394**(1): 56-61.

[8] ZHAO X F, LI X, DING S H. Two kinds of contact problems in dodecagonal quasicrystals of point group 12 mm[J]. *Acta Mechanica Solida Sinica*, 2016, **29**(2): 167-177.

[9] GUO J H, YU J, SI R. A semi-inverse method of a Griffith crack in one-dimensional hexagonal quasicrystals [J]. *Applied Mathematics and Computation*, 2013, **219**(14): 7445-7449.

[10] LI X Y. Elastic field in an infinite medium of one-dimensional hexagonal quasicrystal with a planar crack[J]. *International Journal of Solids and Structures*, 2014, **51**(6): 1442-1455.

[11] RADI E, MARIANO P M. Stationary straight cracks in quasicrystals[J]. *International Journal of Fracture*, 2010, **166**(1/2): 105-120.

[12] 钟万勰. 弹性力学求解新体系[M]. 大连: 大连理工大学出版社, 1995.(ZHONG Wanxie. *A New Systematic Methodology for Theory of Elasticity*[M]. Dalian: Dalian University of Technology Press, 1995.(in Chinese))

[13] ALATANCANG, WU D Y. Completeness in the sense of Cauchy principal value of the eigenfunction systems of infinite dimensional Hamiltonian operator[J]. *Science in China*(*Series A*): *Mathematics*, 2009, **52**(1): 173-180.

[14] ALATANCANG, HOU G L, HAI G J. Perturbation ofspectra for a class of 2×2 operator matrices[J]. *Acta Mathematicae Applicatae Sinica*(*English Series*), 2012, **28**(4): 711-720.

[15] 李锐, 田宇, 郑新然, 等. 求解弹性地基上自由矩形中厚板弯曲问题的辛-叠加方法[J]. 应用数学和力学, 2018, **39**(8): 875-891.(LI Rui, TIAN Yu, ZHENG Xinran, et al. A symplectic superposition method for bending problems of free-edge rectangular thick plates resting on elastic foundations[J]. *Applied Mathematics and Mechanics*, 2018, **39**(8): 875-891.(in Chinese))

[16] QIAO Y, HOU G, CHEN A. A complete symplectic approach for a class of partial differential equations arising from the elasticity[J]. *Applied Mathematical Modelling*, 2021, **89**(2): 1124-1139.

[17] 额布日力吐, 冯璐, 阿拉坦仓. 四边固支正交各向异性矩形薄板弯曲问题的辛叠加方法[J]. 应用数学和力学, 2018, **39**(3): 311-323.(EBURILITU, FENG Lu, ALATANCANG. Analytical bending solutions of clamped orthotropic rectangular thin plates with the symplectic superposition method[J]. *Applied Mathematics and Mechanics*, 2018, **39**(3): 311-323.(in Chinese))

[18] 张俊霖, 倪一文, 李庆东, 等. 吸湿老化影响下天然纤维增强复合圆柱壳屈曲分析的辛方法[J]. 应用数学和力学, 2021, **42**(12): 1238-1247.(ZHANG Junlin, NI Yiwen, LI Qingdong, et al. A symplectic approach for buckling analysis of natural fiber reinforced composite shells under hygrothermal aging[J]. *Applied Mathematics and Mechanics*, 2021, **42**(12): 1238-1247.(in Chinese))

[19] ZHANG W X, XU X S. The symplectic approach for two-dimensional thermo-viscoelastic analysis[J]. *International Journal of Engineering Science*, 2012, **50**(1): 56-69.

[20] LI X, YAO W A, HU X F, et al. Interfacial crack analysis between dissimilar viscoelastic media using symplectic analytical singular element[J]. *Engineering Fracture Mechanics*, 2019, **219**: 106628.

[21] XU X S, RONG D L, LIM C W, et al. An analytical symplectic approach to the vibration analysis of orthotropic graphene sheets[J]. *Acta Mechanica Sinica*, 2017, **33**(5): 912-925.

[22] FAN J H, RONG D L, ZHOU Z H, et al. Exact solutions for forced vibration of completely free orthotropic rectangular nanoplates resting on viscoelastic foundation[J]. *European Journal of Mechanics A: Solids*, 2019,**73**: 22-33.

[23] ZHOU Z H, YU X, YANG Z T, et al. An isogeometric-symplectic coupling approach for fracture analysis of magnetoelectroelastic bimaterials with crack terminating at the interface[J]. *Engineering Fracture Mechanics*, 2019, **216**: 106510.

[24] ZHOU Z H, NI Y W, ZHU S B, et al. An accurate and straightforward approach to thermo-electro-mechanical vibration of piezoelectric fiber-reinforced composite cylindrical shells[J]. *Composite Structures*, 2019, **207**: 292-303.

[25] FAN J J, LI L H, CHEN A. Symplectic method for the thin piezoelectric plates[J]. *Crystals*, 2022, **12**(5): 681.

[26] ZHANG K, GE M H, ZHAO C, et al. Free vibration of nonlocal Timoshenko beams made of functionally graded materials by symplectic method[J]. *Composites*(*Part B*): *Engineering*, 2019, **156**: 174-184.

[27] ZHANG K, DENG Z C, XU X J, et al. Symplectic analysis for wave propagation of hierarchical honeycomb structures[J]. *Acta Mechanica Solida Sinica*, 2015, **28**(3): 294-304.

[28] ZHOU Z H, YANG Z T, XU W, et al. Evaluation ofelectroelastic singularity of finite-size V-notched one-dimensional hexagonal quasicrystalline bimaterials with piezoelectric effect[J]. *Theoretical and Applied Fracture Mechanics*, 2019, **100**: 139-153.

[29] YANG Z T, YU X, XU C H, et al. A novel Hamiltonian-based isogeometric analysis of one-dimensional hexagonal piezoelectric quasicrystal with mode Ⅲ electrically permeable/impermeable cracks[J]. *Theoretical and Applied Fracture Mechanics*, 2020, **107**: 102552.

[30] WANG H, LI L H, HUANG J J, et al. Symplectic approach for the plane elasticity problem of quasicrystals with point group 10 mm[J]. *Applied Mathematical Modelling*, 2015, **39**(12): 3306-3316.

[31] WANG H, CHEN J R, ZHANG X Y, et al. On symplectic analysis for the plane elasticity problem of quasicrystals with point group 12 mm[J]. *Abstract and Applied Analysis*, 2014, **2014**: 367018.

[32] QIAO Y, HOU G, CHEN A. Symplectic approach for plane elasticity problems of two dimensional octagonal quasicrystals[J]. *Applied Mathematics and Computation*, 2021, **400**: 126043.

[33] LI L H, LIU G T. Decagonal quasicrystal plate with elliptic holes subjected to out-of-plane bending moments [J]. *Physics Letters A*, 2014, **378**(10): 839-844.

[34] HOU G L, QI G W, XU Y N, et al. The separable Hamiltonian system and complete biorthogonal expansion method of Mindlin plate bending problems[J]. *Science China: Physics, Mechanics & Astronomy*, 2013, **56**(1): 974-980.