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具有时变时滞的分数阶四元数神经网络的投影同步*

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摘要： 研究了具有时变时滞的分数阶四元数神经网络的投影同步问题.该文不将分数阶四元数神经网络系统转化成两个复值系统或四个实值系统,而是将四元数系统当做一个整体进行处理.在合适的控制器下,通过构造合适的 Lyapunov 函数,并利用一些不等式技巧,得到了具有时变时滞分数阶四元数时滞神经网络投影同步的充分性判据.最后,通过数值仿真实例验证了所得结论的有效性和可行性.

关　键　词： 分数阶四元数神经网络； 时变时滞； 投影同步； 线性矩阵不等式

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Projective Synchronization of Fractional Quaternion Neural Networks With Time-Varying Delays

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Abstract：The projective synchronization of fractional quaternion neural networks with time-varying delays was studied. Instead of transforming the fractional quaternion neural network system into 2 complex-valued systems or 4 real-valued systems，the means of treating the quaternion network system directly as a whole was applied. Under a rational controller，through the construction of a suitable Lyapunov function and with some inequality techniques，the sufficient criteria for the projective synchronization of fractional quaternion neural networks with time-varying delays were obtained. The numerical simulation example shows the validity and feasibility of the conclusions.

Key words：fractional quaternion neural network；time-varying delay；projection synchronization；linear matrix inequality

0　引　　言

近年来,神经网络在模式识别[1]、联想记忆、信号处理[2]和安全通信等领域的广泛应用,引起了许多研

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究者的关注[3]．由于分数阶模型相比传统的整数阶模型，有很好的遗传性和记忆性，能更准确地描述复杂系统的动力学行为，因此，许多学者开始研究分数阶神经网络，并得到了显著的研究成果[4-6]．

同步现象作为神经网络的一种重要的动态行为，是分数阶神经网络研究的一个热点问题，包括完全同步[7]、准同步[8]和 Mittag-Leffer 同步[9]．文献[10]研究了一类分数阶复值神经网络的准投影同步和完全同步问题，通过设计合适的线性反馈控制器和自适应控制，利用 Laplace 变换和 Mittag-Leffler 函数的性质建立了一个新的分数阶微分不等式．目前，已有许多学者对神经网络的投影问题进行了深入研究[11-13]．文献[14]利用复变函数和 Mittag-Leffler 函数的理论讨论了分数阶递归复值神经网络的拟投影同步．文献[15]研究了分数阶复值记忆神经网络的投影同步问题，根据分数阶多时滞系统的稳定性定理和比较原理，得到了保证驱动响应网络同步的一些判据．文献[16]研究了一类分数阶延迟神经网络的驱动响应同步问题．

以上关于神经网络的研究，都是关于实值或复值神经网络的，但在实际应用中，会遇到多维数据，实值神经网络和复值神经元无法很好地处理这些数据．而四元数由一个实部和三个虚部组成，可以有效地处理多维数据．因此，一些学者将四元数引入到经典的神经网络中，建立了四元数神经网络[17-18]．与实值神经网络和复值神经网络相比，四元数神经网络的存储容量大，可处理多维信息，应用于图像处理、计算机图形学、彩色夜视等领域[19-20]．文献[21]将分数阶四元数神经网络分解成四个实值部分，通过分数阶微分不等式，设计合适的控制器，研究了分数阶四元数神经网络的投影问题．文献[22-24]将四元数神经网络分解为两个复值系统或四个实值系统．虽然这种分离方法是可行和有效的，但它导致了模型维数增加，增强了理论分析的复杂性．文献[25]研究了具有时滞和参数不确定的四元数神经网络的鲁棒性问题．目前，将四元数神经网络同步性问题当作一个整体来研究尚不多见．

鉴于此，本文研究了具有时变时滞的四元数神经网络的投影同步问题．通过选取合适的 Lyapunov-Krasovskii 泛函，结合积分不等式，得到了网络投影同步的不等式判据．

注 1　H 表示四元数斜域，H^n 和 $H^{n\times n}$ 分别表示 n 维四元数空间和 $n\times n$ 四元数矩阵集．$\boldsymbol{A}^{\mathrm{T}}$ 和 $\boldsymbol{A}^*$ 分别表示矩阵 $\boldsymbol{A}$ 的转置矩阵和共轭转置矩阵．

1　预备知识

四元数可以写成 $\boldsymbol{q}=q_0+q_1\mathrm{i}+q_2\mathrm{j}+q_3\mathrm{k}$，其中 $q_0,q_1,q_2,q_3\in\mathbf{R}$，i，j，k 为虚数单位，满足下列条件：

$$\mathrm{i}^2=\mathrm{j}^2=\mathrm{k}^2=-1,\ \mathrm{ij}=-\mathrm{ji}=\mathrm{k},\ \mathrm{jk}=-\mathrm{kj}=\mathrm{i},\ \mathrm{ki}=-\mathrm{ik}=\mathrm{j}.$$

由此，可以看出四元数乘积不满足交换律．如果 $\boldsymbol{p}=p_0+p_1\mathrm{i}+p_2\mathrm{j}+p_3\mathrm{k}\in H$，$\boldsymbol{q}=q_0+q_1\mathrm{i}+q_2\mathrm{j}+q_3\mathrm{k}\in H$，则 $\boldsymbol{p}$ 与 $\boldsymbol{q}$ 的和定义为

$$\boldsymbol{p}+\boldsymbol{q}=(p_0+q_0)+(p_1+q_1)\mathrm{i}+(p_2+q_2)\mathrm{j}+(p_3+q_3)\mathrm{k};$$

$\boldsymbol{p}$ 与 $\boldsymbol{q}$ 的积定义为

$$\boldsymbol{pq}=(p_0q_0-p_1q_1-p_2q_2-p_3q_3)+(p_0q_1+p_1q_0-p_2q_3-p_3q_2)\mathrm{i}+\\(p_0q_2+p_2q_0-p_1q_3+p_3q_1)\mathrm{j}+(p_0q_3+p_3q_0+p_1q_2-p_2q_1)\mathrm{k}.$$

四元数 $\boldsymbol{q}$ 的共轭定义为

$$\bar{\boldsymbol{q}}=q_0-q_1\mathrm{i}-q_2\mathrm{j}-q_3\mathrm{k};$$

四元数 $\boldsymbol{q}$ 的模定义为

$$|\boldsymbol{q}|=\sqrt{\boldsymbol{q}\bar{\boldsymbol{q}}}=\sqrt{q_0{}^2+q_1^2+q_2^2+q_3^2}.$$

定义 1（分数阶积分）　设 $f(t)\in H^n$ 在 $[t_0,+\infty)$ 是分段连续的函数，函数 $f(t)$ 的分数阶积分定义为

$${}_{t_0}\mathrm{D}_t^{-\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s)\,\mathrm{d}s,$$

其中 $\alpha\in(0,1)$，$\Gamma(\cdot)$ 表示 Gamma 函数，$\Gamma(s)=\int_0^{+\infty}t^{s-1}\mathrm{e}^{-t}\mathrm{d}t$．

定义 2（Riemann-Liouville 分数阶导数）　设 $f(t)\in H^n$ 在 $[t_0,+\infty)$ 是可微的函数，函数 $f(t)$ 的分数阶导数定义为

$$ {}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha} f(t)=\frac{1}{\Gamma(m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)\mathrm{d}s, $$

其中

$$ m-1<\alpha<m,\ m\in\mathbf{Z}_+ . $$

引理 1[5] 若 $z(t)\in\mathbf{R}$ 是连续可导的函数,且 $z(t)$ 的导数是可积的,则对 $\forall t>0,\alpha\in(0,1)$, 下面的不等式成立:

$$ \frac{1}{2}{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}(z^2(t))\leqslant z(t)\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}z(t). $$

引理 2 若 $p(t)\in H$ 是连续可微的函数,$\boldsymbol{M}$ 为正定的 Hermite 矩阵,则对 $\forall t>0,\alpha\in(0,1)$, 下面的不等式成立:

$$ {}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}(p^*(t)\boldsymbol{M}p(t))\leqslant p^*(t)\boldsymbol{M}\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}p(t)+{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}p^*(t)\boldsymbol{M}p(t). $$

证明 由于 $\boldsymbol{M}$ 为正定的 Hermite 矩阵,则存在一个酉矩阵 $\boldsymbol{W}\in H^{n\times n}$ 与正定的对角阵 $\boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n)$,使得 $\boldsymbol{M}=\boldsymbol{W\Lambda W}$,其中 $\lambda_m(m=1,2,\cdots,n)$ 是矩阵 $\boldsymbol{M}$ 的特征值.

令

$$ \boldsymbol{U}(t)=\boldsymbol{W}p(t)=(\boldsymbol{U}_1(t),\boldsymbol{U}_2(t),\cdots,\boldsymbol{U}_n(t))^{\mathrm{T}},\ \boldsymbol{U}_m(t)=x_m(t)+\mathrm{i}y_m(t)+\mathrm{j}z_m(t)+\mathrm{k}u_m(t), $$

则

$$ \begin{aligned} &\boldsymbol{U}^*(t)\boldsymbol{\Lambda}\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{U}(t)+{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{U}^*(t)\boldsymbol{\Lambda U}(t)=\\ &\quad\sum_{m=1}^n\lambda_m(\boldsymbol{U}^*(t)\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{U}(t)+{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{U}^*(t)\boldsymbol{U}(t))=\\ &\quad 2\sum_{m=1}^n\lambda_m(x_m^*(t)\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}x_m(t)+y_m^*(t)\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}y_m(t)+z_m^*(t)\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}z_m(t)+u_m^*(t)\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}u_m(t)). \end{aligned} $$

由引理 2 得

$$ \begin{aligned} &{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}(p^*(t)\boldsymbol{M}p(t))={}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}(p^*(t)\boldsymbol{W\Lambda W}p(t))=\\ &\quad{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}((\boldsymbol{W}p(t))^*\boldsymbol{\Lambda W}p(t))={}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\left(\sum_{m=1}^n\lambda_m\boldsymbol{U}^*(t)\boldsymbol{U}(t)\right)=\\ &\quad{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\left(\sum_{m=1}^n\lambda_m(x_m^2(t)+y_m^2(t)+z_m^2(t)+u_m^2(t))\right)\leqslant\\ &\quad 2\sum_{m=1}^n\lambda_m(x_m(t)\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}x_m(t)+y_m(t)\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}y_m(t)+z_m(t)\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}z_m(t)+u_m(t)\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}u_m(t))=\\ &\quad\boldsymbol{U}^*(t)\boldsymbol{\Lambda}\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{U}(t)+{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{U}^*(t)\boldsymbol{\Lambda U}(t)=\\ &\quad(\boldsymbol{W}p(t))^*\boldsymbol{\Lambda}\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}(\boldsymbol{W}p(t))+{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}(\boldsymbol{W}p(t))^*\boldsymbol{\Lambda W}p(t)=\\ &\quad p^*(t)\boldsymbol{M}\,{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}p(t)+{}_{t_0}^{\mathrm{RL}}\mathrm{D}_t^{\alpha}p^*(t)\boldsymbol{M}p(t). \end{aligned} $$

引理 3[17] $\boldsymbol{Q}\in H^{n\times n}$ 为正定的 Hermite 矩阵,向量函数 $\boldsymbol{u}(x):[a,b]\to H^n,a<b$, 有

$$ \left(\int_a^b\boldsymbol{u}^*(x)\mathrm{d}x\right)\boldsymbol{Q}\left(\int_a^b\boldsymbol{u}(x)\mathrm{d}x\right)\leqslant(b-a)\int_a^b\boldsymbol{u}^*(x)\boldsymbol{Qu}(x)\mathrm{d}x. $$

引理 4[24] $\boldsymbol{Q}\in C^{n\times n}$ 为正定的 Hermite 矩阵,向量函数 $\boldsymbol{u}(x):[a,b]\to C^n,a<b$, 有

$$ \left(\int_a^b\int_s^b\boldsymbol{u}^*(x)\mathrm{d}x\mathrm{d}\theta\right)\boldsymbol{Q}\left(\int_a^b\int_s^b\boldsymbol{u}(x)\mathrm{d}x\mathrm{d}\theta\right)\leqslant\frac{(b-a)^2}{2}\int_a^b\int_s^b\boldsymbol{u}^*(x)\boldsymbol{Qu}(x)\mathrm{d}x\mathrm{d}\theta. $$

引理 5 $\boldsymbol{Q}\in H^{n\times n}$ 为正定的 Hermite 矩阵,向量函数 $\boldsymbol{u}(x):[a,b]\to H^n,a<b$, 有

$$ \left(\int_a^b\int_s^b\boldsymbol{u}^*(x)\mathrm{d}x\mathrm{d}\theta\right)\boldsymbol{Q}\left(\int_a^b\int_s^b\boldsymbol{u}(x)\mathrm{d}x\mathrm{d}\theta\right)\leqslant\frac{(b-a)^2}{2}\int_a^b\int_s^b\boldsymbol{u}^*(x)\boldsymbol{Qu}(x)\mathrm{d}x\mathrm{d}\theta. $$

证明 假设 $\boldsymbol{Q}=\boldsymbol{Q}_1+\boldsymbol{Q}_2J,u(x)=u_1(x)+u_2(x)J$,其中 $\boldsymbol{Q}_1\in C^{n\times n},\boldsymbol{Q}_2\in C^{n\times n},\boldsymbol{u}_1(x)\in C^n,\boldsymbol{u}_2(x)\in C^n$. 因为 $\boldsymbol{Q}$ 为 Hermite 矩阵,所以 $\boldsymbol{Q}_1^*=\boldsymbol{Q}_1$, $\boldsymbol{Q}_2^*=-\bar{\boldsymbol{Q}}_2$, 由引理 4 得

$$ \left(\int_a^b\int_s^b\boldsymbol{u}^*(x)\mathrm{d}x\mathrm{d}\theta\right)\boldsymbol{Q}\left(\int_a^b\int_s^b\boldsymbol{u}(x)\mathrm{d}x\mathrm{d}\theta\right)= $$

$$\left(\int_a^b\int_s^b\begin{pmatrix}u_1(x)\\u_2(x)\end{pmatrix}\mathrm{d}x\mathrm{d}\theta\right)^*\begin{pmatrix}\boldsymbol{Q}_1 & -\boldsymbol{Q}_2\\ \bar{\boldsymbol{Q}}_2 & \bar{\boldsymbol{Q}}_1\end{pmatrix}\left(\int_a^b\int_s^b\begin{pmatrix}u_1(x)\\u_2(x)\end{pmatrix}\mathrm{d}x\mathrm{d}\theta\right)\leqslant$$

$$\frac{(b-a)^2}{2}\int_a^b\int_s^b\begin{pmatrix}u_1(x)\\u_2(x)\end{pmatrix}^*\begin{pmatrix}\boldsymbol{Q}_1 & -\boldsymbol{Q}_2\\ \bar{\boldsymbol{Q}}_2 & \bar{\boldsymbol{Q}}_1\end{pmatrix}\begin{pmatrix}u_1(x)\\u_2(x)\end{pmatrix}\mathrm{d}x\mathrm{d}\theta=$$

$$\frac{(b-a)^2}{2}\int_a^b\int_s^b\boldsymbol{u}^*(x)\boldsymbol{Q}\boldsymbol{u}(x)\mathrm{d}x\mathrm{d}\theta.$$

2　模型描述

本文考虑如下一类具有时变时滞分数阶四元数神经网络：

$$\begin{cases}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{p}(t)=-\boldsymbol{E}\boldsymbol{p}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{p}(t))+\boldsymbol{B}\boldsymbol{g}(\boldsymbol{p}(t-\sigma(t)))+\boldsymbol{L},\\ {}_0^{\mathrm{RL}}\mathrm{D}_t^{-(1-\alpha)}\boldsymbol{p}(t)=\boldsymbol{\phi}(t),\end{cases}\quad t\in[-\sigma,0],\tag{1}$$

其中 $0<\alpha<1$，$\boldsymbol{p}(t)=(p_1(t),p_2(t),\cdots,p_n(t))^{\mathrm{T}}\in H^n$ 表示系统的状态向量；$\boldsymbol{E}=\mathrm{diag}(e_1,e_2,\cdots,e_n)$ 表示自反馈矩阵，$e_i>0,i\in 1,2,\cdots,n$；$\sigma(t)$ 表示时变时滞；$\boldsymbol{A},\boldsymbol{B}$ 分别表示 t 和 $t-\sigma(t)$ 时刻的连接权矩阵；向量激活函数 $\boldsymbol{f}(\boldsymbol{p}(t))=(f_1(p_1(t)),f_2(p_2(t)),\cdots,f_n(p_n(t)))^{\mathrm{T}}\in H^n$，$\boldsymbol{g}(\boldsymbol{p}(t-\sigma(t)))=(g_1(p_1(t-\sigma(t))),g_2(p_2(t-\sigma(t))),\cdots,g_n(p_n(t-\sigma(t))))^{\mathrm{T}}\in H^n$；$\boldsymbol{L}$ 表示相应的外部输入.

本文做假设如下.

假设 1　对任意的 $x,y\in H$，存在两个正常数 $\lambda_i,\gamma_i(i=1,2,\cdots,n)$，使得

$$|f_i(x)-f_i(y)|\leqslant\lambda_i|x-y|,\ |g_i(x)-g_i(y)|\leqslant\gamma_i|x-y|.\tag{2}$$

令

$$\boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n),\ \boldsymbol{\Gamma}=\mathrm{diag}(\gamma_1,\gamma_2,\cdots,\gamma_n).$$

假设 2　时滞 $\sigma(t)$ 是连续可导的，且满足如下条件：

$$0<\sigma(t)\leqslant\sigma,\ \dot{\sigma}(t)\leqslant\tilde{\sigma}.\tag{3}$$

考虑响应系统如下：

$$\begin{cases}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{q}(t)=-\boldsymbol{E}\boldsymbol{q}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{q}(t))+\boldsymbol{B}\boldsymbol{g}(\boldsymbol{q}(t-\sigma(t)))+\boldsymbol{L}+\boldsymbol{u}(t),\\ {}_0\mathrm{D}_t^{-(1-\alpha)}\boldsymbol{q}(t)=\boldsymbol{\psi}(t),\end{cases}\quad t\in[-\sigma,0],\tag{4}$$

其中 $\boldsymbol{u}(t)$ 为控制器.控制器设计如下：

$$\begin{aligned}\boldsymbol{u}(t)=&-\boldsymbol{K}(\boldsymbol{q}(t)-\boldsymbol{F}\boldsymbol{p}(t))+\boldsymbol{E}\boldsymbol{F}\boldsymbol{p}(t)-\boldsymbol{F}\boldsymbol{E}\boldsymbol{p}(t)-\boldsymbol{A}\boldsymbol{f}(\boldsymbol{F}\boldsymbol{p}(t))+\boldsymbol{F}\boldsymbol{A}\boldsymbol{f}(\boldsymbol{p}(t))-\\&\boldsymbol{B}\boldsymbol{g}(\boldsymbol{F}\boldsymbol{p}(t-\sigma(t)))+\boldsymbol{F}\boldsymbol{B}\boldsymbol{g}(\boldsymbol{p}(t-\sigma(t)))-(\boldsymbol{I}-\boldsymbol{F})\boldsymbol{L},\end{aligned}\tag{5}$$

其中 $\boldsymbol{K}\in R^{n\times n}$ 是控制器 $\boldsymbol{u}(t)$ 的系数矩阵，$\boldsymbol{I}$ 为单位矩阵.

定义误差为

$$\boldsymbol{\theta}(t)=\boldsymbol{q}(t)-\boldsymbol{F}\boldsymbol{p}(t),\tag{6}$$

其中

$$\boldsymbol{\theta}(t)=(\theta_1(t),\theta_2(t),\cdots,\theta_n(t))^*\in H^n,\ \boldsymbol{F}=\mathrm{diag}(F_1,F_2,\cdots,F_n)>\boldsymbol{0}.$$

则由系统(1)和系统(4)得到误差系统如下：

$${}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t)=-(\boldsymbol{E}+\boldsymbol{K})\boldsymbol{\theta}(t)+\boldsymbol{A}\tilde{\boldsymbol{f}}(\boldsymbol{\theta}(t))+\boldsymbol{B}\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(t-\sigma(t))),\tag{7}$$

其中

$$\begin{aligned}&\tilde{\boldsymbol{f}}(\boldsymbol{\theta}(t))=\boldsymbol{f}(\boldsymbol{\theta}(t)+\boldsymbol{F}\boldsymbol{p}(t))-\boldsymbol{f}(\boldsymbol{F}\boldsymbol{p}(t)),\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(t-\sigma(t)))=\\&\quad\boldsymbol{g}(\boldsymbol{\theta}(t-\sigma(t)))+\boldsymbol{F}\boldsymbol{p}(t-\sigma(t)))-\boldsymbol{g}(\boldsymbol{F}\boldsymbol{p}(t-\sigma(t))).\end{aligned}$$

3　主要结论

定理 1　假设 1 和假设 2 成立，如果存在正定的 Hermite 矩阵 $\boldsymbol{P}_i(i=1,2,\cdots,6)$，两个正定的对角矩阵 $\boldsymbol{K}_1,\boldsymbol{W}$，以及常数 $0<\tilde{\sigma}\leqslant 1$，满足如下线性矩阵不等式(LMI)：

$$\boldsymbol{\Omega}=\begin{bmatrix}\boldsymbol{\Omega}_{11} & \boldsymbol{\Omega}_{12} & \boldsymbol{\Omega}_{13} & \boldsymbol{\Omega}_{14} & \boldsymbol{\Omega}_{15} & \mathbf{0}\\ * & \boldsymbol{\Omega}_{22} & \boldsymbol{\Omega}_{23} & \boldsymbol{A}^*\boldsymbol{P}_3 & \sigma^3\boldsymbol{A}^*\boldsymbol{P}_5 & \mathbf{0}\\ * & * & \boldsymbol{\Omega}_{33} & \boldsymbol{B}^*\boldsymbol{P}_3 & \sigma^3\boldsymbol{B}^*\boldsymbol{P}_5 & \mathbf{0}\\ * & * & * & -\boldsymbol{P}_4 & -\sigma\boldsymbol{P}_5 & -\boldsymbol{P}_3\\ * & * & * & * & -4\boldsymbol{P}_6 & \mathbf{0}\\ * & * & * & * & * & -\boldsymbol{P}_2\end{bmatrix}<\mathbf{0}, \tag{8}$$

其中，$*$表示矩阵块的共轭转置，$\boldsymbol{\Omega}_{11}=-(\boldsymbol{E}+\boldsymbol{K})^*\boldsymbol{P}_1-\boldsymbol{P}_1(\boldsymbol{E}+\boldsymbol{K})+(\boldsymbol{E}+\boldsymbol{K})^*(\boldsymbol{P}_2+\sigma^2\boldsymbol{P}_4+\sigma^4\boldsymbol{P}_6)(\boldsymbol{E}+\boldsymbol{K})+\boldsymbol{\Lambda K\Lambda}+\boldsymbol{\Gamma W\Gamma}$，$\boldsymbol{\Omega}_{12}=\boldsymbol{P}_1\boldsymbol{A}-(\boldsymbol{E}+\boldsymbol{K})^*(\boldsymbol{P}_2+\sigma^2\boldsymbol{P}_4+\sigma^4\boldsymbol{P}_6)\boldsymbol{A}$，$\boldsymbol{\Omega}_{13}=\boldsymbol{P}_1\boldsymbol{B}-(\boldsymbol{E}+\boldsymbol{K})^*(\boldsymbol{P}_2+\sigma^2\boldsymbol{P}_4+\sigma^4\boldsymbol{P}_6)\boldsymbol{B}$，$\boldsymbol{\Omega}_{14}=-(\boldsymbol{E}+\boldsymbol{K})^*\boldsymbol{P}_3$，$\boldsymbol{\Omega}_{15}=-\sigma^3(\boldsymbol{E}+\boldsymbol{K})^*\boldsymbol{P}_5$，$\boldsymbol{\Omega}_{22}=\boldsymbol{A}^*(\boldsymbol{P}_2+\sigma^2\boldsymbol{P}_4+\sigma^4\boldsymbol{P}_6)\boldsymbol{A}-\boldsymbol{K}_1$，$\boldsymbol{\Omega}_{23}=\boldsymbol{A}^*(\boldsymbol{P}_2+\sigma^2\boldsymbol{P}_4+\sigma^4\boldsymbol{P}_6)\boldsymbol{B}$，$\boldsymbol{\Omega}_{33}=\boldsymbol{B}^*(\boldsymbol{P}_2+\sigma^2\boldsymbol{P}_4+\sigma^4\boldsymbol{P}_6)\boldsymbol{B}-(1-\tilde{\sigma})\boldsymbol{W}$. 则系统(7)的零点是稳定的，即系统(4)和系统(1)可以实现投影同步.

证明 选择如下 Lyapunov 泛函：

$$V(t)=V_1(t)+V_2(t)+V_3(t), \tag{9}$$

其中

$$\begin{aligned}V_1(t)={}&{}^{\mathrm{RL}}_{0}\mathrm{D}_t^{-(1-\alpha)}(\boldsymbol{\theta}^*(t)\boldsymbol{P}_1\boldsymbol{\theta}(t))+\int_{t-\sigma(t)}^{t}\tilde{\boldsymbol{g}}^*(\boldsymbol{\theta}(s))\boldsymbol{W}\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(s))\mathrm{d}s+\\&\int_{t-\sigma}^{t}({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))^*\boldsymbol{P}_2({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))\mathrm{d}s,\end{aligned} \tag{10}$$

$$\begin{aligned}V_2(t)={}&\left(\int_{t-\sigma}^{t}{}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)^*\boldsymbol{P}_3\left(\int_{t-\sigma}^{t}{}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)+\\&\sigma\int_{-\sigma}^{0}\int_{t+\xi}^{t}({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))^*\boldsymbol{P}_4({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))\mathrm{d}\xi\mathrm{d}s,\end{aligned} \tag{11}$$

$$\begin{aligned}V_3(t)={}&\left(\sigma\int_{-\sigma}^{0}\int_{t+\xi}^{t}{}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}\xi\mathrm{d}s\right)^*\boldsymbol{P}_5\left(\sigma\int_{-\sigma}^{0}\int_{t+\xi}^{t}{}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}\xi\mathrm{d}s\right)+\\&2\sigma^2\int_{-\sigma}^{0}\int_{\xi}^{0}\int_{t+\tau}^{t}({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))^*\boldsymbol{P}_6({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))\mathrm{d}s\mathrm{d}\xi\mathrm{d}\tau.\end{aligned} \tag{12}$$

对 $V_1(t)$，$V_2(t)$，$V_3(t)$ 求导，得

$$\begin{aligned}\dot{V}_1(t)={}&{}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}(\boldsymbol{\theta}^*(t)\boldsymbol{P}_1\boldsymbol{\theta}(t))+\tilde{\boldsymbol{g}}^*(\boldsymbol{\theta}(t))\boldsymbol{W}\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(t))-\\&(1-\dot{\sigma}(t))\tilde{\boldsymbol{g}}^*(\boldsymbol{\theta}(t-\sigma(t)))\boldsymbol{W}\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(t-\sigma(t)))+\\&({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))^*\boldsymbol{P}_2({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))-({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t-\sigma))^*\boldsymbol{P}_2({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t-\sigma))\leqslant\\&\boldsymbol{\theta}^*(t)\boldsymbol{P}_1({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))+({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))^*\boldsymbol{P}_1\boldsymbol{\theta}^*(t)+\tilde{\boldsymbol{g}}^*(\boldsymbol{\theta}(t))\boldsymbol{W}\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(t))-\\&(1-\tilde{\sigma})\tilde{\boldsymbol{g}}^*(\boldsymbol{\theta}(t-\sigma(t)))\boldsymbol{W}\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(t-\sigma(t)))+({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))^*\boldsymbol{P}_2({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))-\\&({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t-\sigma))^*\boldsymbol{P}_2({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t-\sigma))\leqslant\\&\boldsymbol{\theta}^*(t)(-\boldsymbol{P}_1(\boldsymbol{E}+\boldsymbol{K})-(\boldsymbol{E}+\boldsymbol{K})^*\boldsymbol{P}_1+(\boldsymbol{E}+\boldsymbol{K})^*\boldsymbol{P}_2(\boldsymbol{E}+\boldsymbol{K}))\boldsymbol{\theta}(t)+\\&\boldsymbol{\theta}^*(t)(\boldsymbol{P}_1\boldsymbol{A}-(\boldsymbol{E}+\boldsymbol{K})^*\boldsymbol{P}_2\boldsymbol{A})\tilde{\boldsymbol{f}}(\boldsymbol{\theta}(t))+\tilde{\boldsymbol{f}}^*(\boldsymbol{\theta}(t))(\boldsymbol{A}^*\boldsymbol{P}_1-\boldsymbol{A}^*\boldsymbol{P}_2(\boldsymbol{E}+\boldsymbol{K}))\boldsymbol{\theta}(t)+\\&\boldsymbol{\theta}^*(t)(\boldsymbol{P}_1\boldsymbol{B}-(\boldsymbol{E}+\boldsymbol{K})^*\boldsymbol{P}_2\boldsymbol{B})\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(t-\sigma(t)))+\\&\tilde{\boldsymbol{g}}^*(\boldsymbol{\theta}(t-\sigma(t)))(\boldsymbol{B}^*\boldsymbol{P}_1-\boldsymbol{B}^*\boldsymbol{P}_2(\boldsymbol{E}+\boldsymbol{K}))\boldsymbol{\theta}(t)+\\&\tilde{\boldsymbol{f}}^*(\boldsymbol{\theta}(t))\boldsymbol{A}^*\boldsymbol{P}_2\boldsymbol{A}\tilde{\boldsymbol{f}}(\boldsymbol{\theta}(t))+\tilde{\boldsymbol{f}}^*(\boldsymbol{\theta}(t))\boldsymbol{A}^*\boldsymbol{P}_2\boldsymbol{B}\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(t-\sigma(t)))+\\&\tilde{\boldsymbol{g}}^*(\boldsymbol{\theta}(t))\boldsymbol{W}\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(t))+\tilde{\boldsymbol{g}}^*(\boldsymbol{\theta}(t-\sigma(t)))(\boldsymbol{B}^*\boldsymbol{P}_2\boldsymbol{B}-(1-\tilde{\sigma})\boldsymbol{W})\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(t-\sigma(t)))+\\&\tilde{\boldsymbol{g}}^*(\boldsymbol{\theta}(t-\sigma(t)))\boldsymbol{B}^*\boldsymbol{P}_2\boldsymbol{A}\tilde{\boldsymbol{f}}(\boldsymbol{\theta}(t))-({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t-\sigma))^*\boldsymbol{P}_2({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t-\sigma)),\end{aligned} \tag{13}$$

$$\dot{V}_2(t)=({}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t)-{}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t-\sigma))^*\boldsymbol{P}_3\left(\int_{t-\sigma}^{t}{}^{\mathrm{RL}}_{0}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)+$$

$$\begin{aligned}&\left(\int_{t-\sigma}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)^{*}\boldsymbol{P}_3({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t)-{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t-\sigma))+\\&\sigma\left[\sigma({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))^{*}\boldsymbol{P}_4({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))-\int_{t-\sigma}^{t}({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))^{*}\boldsymbol{P}_4({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))\mathrm{d}s\right]\leqslant\\&({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))^{*}\boldsymbol{P}_3\left(\int_{t-\sigma}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)-({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t-\sigma))^{*}\boldsymbol{P}_3\left(\int_{t-\sigma}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)+\\&\left(\int_{t-\sigma}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)^{*}\boldsymbol{P}_3({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))-\left(\int_{t-\sigma}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)^{*}\boldsymbol{P}_3({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t-\sigma))+\\&\sigma^2({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))^{*}\boldsymbol{P}_4({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))-\left(\int_{t-\sigma}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)^{*}\boldsymbol{P}_4\left(\int_{t-\sigma}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right),\end{aligned}\tag{14}$$

$$\begin{aligned}\dot{V}_3(t)=&\left[\sigma^2({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))-\left(\int_{t-\sigma}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)\right]^{*}\boldsymbol{P}_5\left(\sigma\int_{-\sigma}^{0}\int_{t+\xi}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}\xi\mathrm{d}s\right)+\\&\left(\sigma\int_{-\sigma}^{0}\int_{t+\xi}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}\xi\mathrm{d}s\right)^{*}\boldsymbol{P}_5\left[\sigma^2({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))-\left(\int_{t-\sigma}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)\right]+\\&2\sigma^2\left[\frac{\sigma^2}{2}({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))^{*}\boldsymbol{P}_6({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))-\int_{-\sigma}^{0}\int_{t+\xi}^{t}({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))^{*}\boldsymbol{P}_6({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))\mathrm{d}\xi\mathrm{d}s\right]\leqslant\\&\sigma^3({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))^{*}\boldsymbol{P}_5\left(\int_{-\sigma}^{0}\int_{t+\xi}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}\xi\mathrm{d}s\right)+\sigma^4({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))^{*}\boldsymbol{P}_6({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))-\\&\left(\int_{t-\sigma}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)^{*}\boldsymbol{P}_5\left(\sigma\int_{-\sigma}^{0}\int_{t+\xi}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}\xi\mathrm{d}s\right)+\\&\left(\int_{-\sigma}^{0}\int_{t+\xi}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}\xi\mathrm{d}s\right)^{*}\sigma^3\boldsymbol{P}_5({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t))-\\&\left(\int_{-\sigma}^{0}\int_{t+\xi}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}\xi\mathrm{d}s\right)^{*}\sigma\boldsymbol{P}_5\left(\int_{t-\sigma}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}s\right)-\\&\left(\int_{-\sigma}^{0}\int_{t+\xi}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}\xi\mathrm{d}s\right)^{*}4\boldsymbol{P}_6\left(\int_{-\sigma}^{0}\int_{t+\xi}^{t}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s)\mathrm{d}\xi\mathrm{d}s\right).\end{aligned}\tag{15}$$

在假设 1 条件下,存在正定对角矩阵 $\boldsymbol{K}_1,\boldsymbol{W}$, 有

$$\boldsymbol{\theta}^{*}(t)\boldsymbol{\Lambda}\boldsymbol{K}_1\boldsymbol{\Lambda}\boldsymbol{\theta}(t)-\tilde{\boldsymbol{f}}^{*}(\boldsymbol{\theta}(t))\boldsymbol{K}_1\tilde{\boldsymbol{f}}(\boldsymbol{\theta}(t))\geqslant0,\tag{16}$$

$$\boldsymbol{\theta}^{*}(t)\boldsymbol{\Gamma}\boldsymbol{W}\boldsymbol{\Gamma}\boldsymbol{\theta}(t)-\tilde{\boldsymbol{g}}^{*}(\boldsymbol{\theta}(t))\boldsymbol{W}\tilde{\boldsymbol{g}}(\boldsymbol{\theta}(t))\geqslant0.\tag{17}$$

由式(13)—(17)可知

$$\dot{V}(t)\leqslant\boldsymbol{\xi}^{*}(t)\boldsymbol{\Omega}\boldsymbol{\xi}(t)<0,\tag{18}$$

其中

$$\begin{aligned}\boldsymbol{\xi}(t)=&\left[\boldsymbol{\theta}^{*}(t),\tilde{\boldsymbol{f}}^{*}(\boldsymbol{\theta}(t)),\tilde{\boldsymbol{g}}^{*}(\boldsymbol{\theta}(t-\sigma(t))),\int_{t-\sigma}^{t}({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))^{*}\mathrm{d}s,\right.\\&\left.\int_{-\sigma}^{0}\int_{t+\xi}^{t}({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(s))^{*}\mathrm{d}\xi\mathrm{d}s,({}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{\theta}(t-\sigma))^{*}\right]^{*},\end{aligned}$$

根据 Lyapunov 理论,可知系统(1)和系统(4)是投影同步的.

注 2　当系统(1)的时变时滞退化成常时滞时,相应的驱动系统和响应系统为

$$\begin{cases}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{p}(t)=-\boldsymbol{E}\boldsymbol{p}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{p}(t))+\boldsymbol{B}\boldsymbol{g}(\boldsymbol{p}(t-\sigma))+\boldsymbol{L},\\{}_0\mathrm{D}_t^{-(1-\alpha)}\boldsymbol{p}(t)=\boldsymbol{\phi}(t),\end{cases}\quad t\in[-\sigma,0],\tag{19}$$

$$\begin{cases}{}_0^{\mathrm{RL}}\mathrm{D}_t^{\alpha}\boldsymbol{q}(t)=-\boldsymbol{E}\boldsymbol{q}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{q}(t))+\boldsymbol{B}\boldsymbol{g}(\boldsymbol{q}(t-\sigma))+\boldsymbol{L}+\boldsymbol{u}(t),\\{}_0\mathrm{D}_t^{-(1-\alpha)}\boldsymbol{q}(t)=\boldsymbol{\psi}(t),\end{cases}\quad t\in[-\sigma,0],\tag{20}$$

其中控制器 $\boldsymbol{u}(t)=-\boldsymbol{K}(\boldsymbol{q}(t)-\boldsymbol{F}\boldsymbol{p}(t))+\boldsymbol{E}\boldsymbol{F}\boldsymbol{p}(t)-\boldsymbol{F}\boldsymbol{E}\boldsymbol{p}(t)-\boldsymbol{A}\boldsymbol{f}(\boldsymbol{F}\boldsymbol{p}(t))+\boldsymbol{F}\boldsymbol{A}\boldsymbol{f}(\boldsymbol{p}(t))-\boldsymbol{B}\boldsymbol{g}(\boldsymbol{F}\boldsymbol{p}(t-\sigma))+\boldsymbol{F}\boldsymbol{B}\boldsymbol{g}(\boldsymbol{p}(t-\sigma))-(\boldsymbol{I}-\boldsymbol{F})\boldsymbol{L}$.

推论 1　若假设 1 成立,如果存在正定的 Hermite 矩阵 $\boldsymbol{P}_i(i=1,2,\cdots,6)$,两个正定的对角矩阵 $\boldsymbol{K}_1,\boldsymbol{W}$,满足如下线性矩阵不等式:

$$
\boldsymbol{\Pi}=\begin{bmatrix}
\boldsymbol{\Pi}_{11} & \boldsymbol{\Pi}_{12} & \boldsymbol{\Pi}_{13} & \boldsymbol{\Pi}_{14} & \boldsymbol{\Pi}_{15} & \mathbf{0} \\
* & \boldsymbol{\Pi}_{22} & \boldsymbol{\Pi}_{23} & \boldsymbol{A}^{*}\boldsymbol{P}_{3} & \sigma^{3}\boldsymbol{A}^{*}\boldsymbol{P}_{5} & \mathbf{0} \\
* & * & \boldsymbol{\Pi}_{33} & \boldsymbol{B}^{*}\boldsymbol{P}_{3} & \sigma^{3}\boldsymbol{B}^{*}\boldsymbol{P}_{5} & \mathbf{0} \\
* & * & * & -\boldsymbol{P}_{4} & -\sigma\boldsymbol{P}_{5} & -\boldsymbol{P}_{3} \\
* & * & * & * & -4\boldsymbol{P}_{6} & \mathbf{0} \\
* & * & * & * & * & -\boldsymbol{P}_{2}
\end{bmatrix}<\mathbf{0}, \tag{21}
$$

其中 $\boldsymbol{\Pi}_{11}=-(\boldsymbol{E}+\boldsymbol{K})^{*}\boldsymbol{P}_{1}-\boldsymbol{P}_{1}(\boldsymbol{E}+\boldsymbol{K})+(\boldsymbol{E}+\boldsymbol{K})^{*}(\boldsymbol{P}_{2}+\sigma^{2}\boldsymbol{P}_{4}+\sigma^{4}\boldsymbol{P}_{6})(\boldsymbol{E}+\boldsymbol{K})+\boldsymbol{\Lambda}\boldsymbol{K}_{1}\boldsymbol{\Lambda}+\boldsymbol{\Gamma}\boldsymbol{W}\boldsymbol{\Gamma}$，$\boldsymbol{\Pi}_{12}=\boldsymbol{P}_{1}\boldsymbol{A}-(\boldsymbol{E}+\boldsymbol{K})^{*}(\boldsymbol{P}_{2}+\sigma^{2}\boldsymbol{P}_{4}+\sigma^{4}\boldsymbol{P}_{6})\boldsymbol{A}$，$\boldsymbol{\Pi}_{13}=\boldsymbol{P}_{1}\boldsymbol{B}-(\boldsymbol{E}+\boldsymbol{K})^{*}(\boldsymbol{P}_{2}+\sigma^{2}\boldsymbol{P}_{4}+\sigma^{4}\boldsymbol{P}_{6})\boldsymbol{B}$，$\boldsymbol{\Pi}_{14}=-(\boldsymbol{E}+\boldsymbol{K})^{*}\boldsymbol{P}_{3}$，$\boldsymbol{\Pi}_{15}=-\sigma^{3}(\boldsymbol{E}+\boldsymbol{K})^{*}\boldsymbol{P}_{5}$，$\boldsymbol{\Pi}_{22}=\boldsymbol{A}^{*}(\boldsymbol{P}_{2}+\sigma^{2}\boldsymbol{P}_{4}+\sigma^{4}\boldsymbol{P}_{6})\boldsymbol{A}-\boldsymbol{K}_{1}$，$\boldsymbol{\Pi}_{23}=\boldsymbol{A}^{*}(\boldsymbol{P}_{2}+\sigma^{2}\boldsymbol{P}_{4}+\sigma^{4}\boldsymbol{P}_{6})\boldsymbol{B}$，$\boldsymbol{\Pi}_{33}=\boldsymbol{B}^{*}(\boldsymbol{P}_{2}+\sigma^{2}\boldsymbol{P}_{4}+\sigma^{4}\boldsymbol{P}_{6})\boldsymbol{B}-\boldsymbol{W}$.

根据 Lyapunov 理论，可知系统(1)和系统(4)是投影同步的.

注 3 文献[26]研究了分数阶四元数值神经网络的同步性，但没有考虑时滞.该模型是本文研究模型的特例，本文研究的模型更符合实际，研究结果也更具有普遍性.

4 数值例子

例 1 考虑以下二维的时变时滞分数阶四元数神经网络作为驱动系统：

$$
\begin{cases}
{}^{\mathrm{RL}}_{0}\mathrm{D}_{t}^{\alpha}\boldsymbol{p}(t)=-\boldsymbol{E}\boldsymbol{p}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{p}(t))+\boldsymbol{B}\boldsymbol{g}(\boldsymbol{p}(t-\sigma(t)))+\boldsymbol{L}, \\
{}_{0}\mathrm{D}_{t}^{-(1-\alpha)}\boldsymbol{p}(t)=\boldsymbol{\phi}(t),
\end{cases} \quad t\in[-\sigma,0], \tag{22}
$$

其中 $\alpha=0.98$，$\sigma(t)=|\sin(2t)|$，$\boldsymbol{p}(t)=(p_1(t),p_2(t))^{\mathrm{T}}\in H^2$，$f_1(p(t))=f_2(p(t))=g_1(p(t))=g_2(p(t))=0.25\tanh(p(t))$，

$$
\boldsymbol{E}=\begin{pmatrix}15 & 0\\ 0 & 15\end{pmatrix},\ \boldsymbol{L}=\begin{pmatrix}0\\0\end{pmatrix},
$$

$$
\boldsymbol{A}=\begin{pmatrix}5-2\mathrm{i}+3\mathrm{j}-2\mathrm{k} & 3+4\mathrm{i}-3\mathrm{j}+5\mathrm{k}\\ 4-2\mathrm{i}+5\mathrm{j}-3\mathrm{k} & 1+6\mathrm{i}+\mathrm{j}+4\mathrm{k}\end{pmatrix},
$$

$$
\boldsymbol{B}=\begin{pmatrix}5+3\mathrm{i}-\mathrm{j}+3\mathrm{k} & 1-2\mathrm{i}+\mathrm{j}-4\mathrm{k}\\ -4+3\mathrm{i}-4\mathrm{j}+3\mathrm{k} & 5-7\mathrm{i}-2\mathrm{j}-3\mathrm{k}\end{pmatrix}.
$$

对应的响应系统为

$$
\begin{cases}
{}^{\mathrm{RL}}_{0}\mathrm{D}_{t}^{\alpha}\boldsymbol{q}(t)=-\boldsymbol{E}\boldsymbol{q}(t)+\boldsymbol{A}\boldsymbol{f}(\boldsymbol{q}(t))+\boldsymbol{B}\boldsymbol{g}(\boldsymbol{q}(t-\sigma(t)))+\boldsymbol{L}+\boldsymbol{u}(t), \\
{}_{0}\mathrm{D}_{t}^{-(1-\alpha)}\boldsymbol{q}(t)=\boldsymbol{\psi}(t),
\end{cases} \quad t\in[-\sigma,0], \tag{23}
$$

其中 $\alpha=0.98$，$\boldsymbol{q}(t)=(q_1(t),q_2(t))^{\mathrm{T}}\in H^2$，$f_1(q(t))=f_2(q(t))=g_1(q(t))=g_2(q(t))=0.25\tanh(p(t))$，$\tilde{\sigma}=0.76$，$\sigma=0.05$，$\boldsymbol{E},\boldsymbol{A},\boldsymbol{B},\boldsymbol{L}$ 与系统(22)一样.控制器为 $\boldsymbol{u}(t)=-\boldsymbol{K}(\boldsymbol{q}(t)-\boldsymbol{F}\boldsymbol{p}(t))+\boldsymbol{E}\boldsymbol{F}\boldsymbol{p}(t)-\boldsymbol{F}\boldsymbol{E}\boldsymbol{p}(t)-\boldsymbol{A}\boldsymbol{f}(\boldsymbol{F}\boldsymbol{p}(t))+\boldsymbol{F}\boldsymbol{A}\boldsymbol{f}(\boldsymbol{p}(t))-\boldsymbol{B}\boldsymbol{g}(\boldsymbol{F}\boldsymbol{p}(t-\sigma(t)))+\boldsymbol{F}\boldsymbol{B}\boldsymbol{g}(\boldsymbol{p}(t-\sigma(t)))-(\boldsymbol{I}-\boldsymbol{F})\boldsymbol{L}$，

$$
\boldsymbol{K}=\begin{pmatrix}15 & 0\\ 0 & 15\end{pmatrix}. \tag{24}
$$

当 $\boldsymbol{\Lambda}=\mathrm{diag}(0.25,0.25)$ 时，$\boldsymbol{\Gamma}=\mathrm{diag}(0.25,0.25)$，满足假设 1 和假设 2.

利用 MATLAB 对线性矩阵不等式(8)求得可行解为

$$
\boldsymbol{P}_1=\begin{pmatrix}52.400\,1 & 3.212\,0+1.501\,2\mathrm{i}+1.535\,5\mathrm{j}-1.601\,2\mathrm{k}\\ 3.212\,0-1.501\,2\mathrm{i}-1.535\,5\mathrm{j}+1.601\,2\mathrm{k} & 34.801\,1\end{pmatrix},
$$

$$
\boldsymbol{P}_2=\begin{pmatrix}38.226\,6 & 0.201\,3-0.345\,0\mathrm{i}+0.000\,8\mathrm{j}+0.005\,2\mathrm{k}\\ 0.201\,3+0.345\,0\mathrm{i}-0.000\,8\mathrm{j}-0.005\,2\mathrm{k} & 38.226\,6\end{pmatrix},
$$

$$
\boldsymbol{P}_3=\begin{pmatrix}0.014\,1 & -0.008\,4-0.002\,8\mathrm{i}+0.000\,9\mathrm{j}-0.001\,1\mathrm{k}\\ -0.008\,4+0.002\,8\mathrm{i}-0.000\,9\mathrm{j}+0.001\,1\mathrm{k} & 0.008\,2\end{pmatrix},
$$

$$\boldsymbol{P}_4=\begin{pmatrix}1.370\,9 & -0.024\,3+0.184\,9\mathrm{i}-0.016\,9\mathrm{j}+0.014\,7\mathrm{k}\\ -0.024\,3-0.184\,9\mathrm{i}+0.016\,9\mathrm{j}-0.014\,7\mathrm{k} & 1.340\,3\end{pmatrix},$$

$$\boldsymbol{P}_5=\begin{pmatrix}64.128\,1 & -26.448\,2-0.753\,9\mathrm{i}+1.427\,4\mathrm{j}+0.931\,8\mathrm{k}\\ -26.448\,2+0.753\,9\mathrm{i}-1.427\,4\mathrm{j}-0.931\,8\mathrm{k} & 44.092\,8\end{pmatrix},$$

$$\boldsymbol{P}_6=\begin{pmatrix}181.520\,0 & 0.721\,0+0.011\,3\mathrm{i}-0.021\,1\mathrm{j}-0.023\,7\mathrm{k}\\ 0.720\,0-0.010\,0\mathrm{i}+0.021\,1\mathrm{j}+0.023\,7\mathrm{k} & 180.210\,0\end{pmatrix},$$

$$\boldsymbol{K}_1=10^5\begin{pmatrix}1.453\,0 & 0\\ 0 & 1.449\,9\end{pmatrix},\ \boldsymbol{W}=10^4\begin{pmatrix}2.858\,2 & 0\\ 0 & 2.849\,6\end{pmatrix}.$$

因此,定理 1 的条件成立,从而系统(7)的零点是稳定的,即系统(4)和系统(1)可以实现投影同步.数值仿真选取如下初始条件:

$$\boldsymbol{p}_0=[4.5+0.9\mathrm{i}-3.5\mathrm{j}-2\mathrm{k},\ 4.5-0.9\mathrm{i}-5.5\mathrm{j}+2\mathrm{k}]^{\mathrm{T}},$$

$$\boldsymbol{q}_0=[-5.5-2\mathrm{i}-3.5\mathrm{j}-5\mathrm{k},\ -1.5-3\mathrm{i}-7.5\mathrm{j}+5\mathrm{k}]^{\mathrm{T}}.$$

图 1、图 2 给出了驱动系统(1)和响应系统(4)在未施加控制时状态变量的时间响应曲线,图 3 给出了误差系统(7)在未施加控制时状态变量的时间响应曲线.

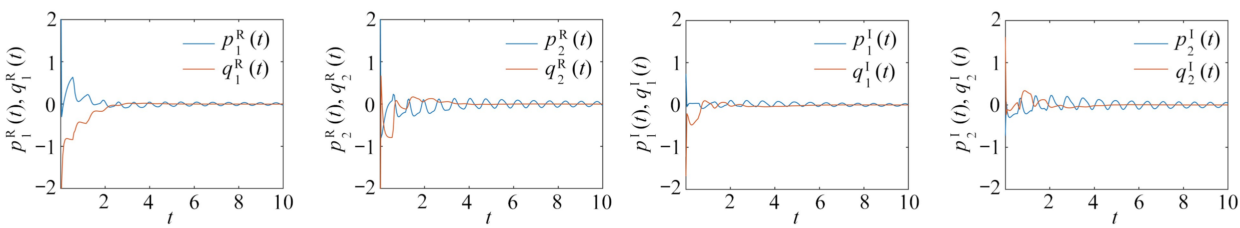

图 1　未加控制时,状态变量 $p_1^{\mathrm{R}},p_2^{\mathrm{R}},p_1^{\mathrm{I}},p_2^{\mathrm{I}},q_1^{\mathrm{R}},q_2^{\mathrm{R}},q_1^{\mathrm{I}},q_2^{\mathrm{I}}$ 的时间响应曲线

Fig. 1　The time response curves of state variables $p_1^{\mathrm{R}},p_2^{\mathrm{R}},p_1^{\mathrm{I}},p_2^{\mathrm{I}},q_1^{\mathrm{R}},q_2^{\mathrm{R}},q_1^{\mathrm{I}},q_2^{\mathrm{I}}$ without control

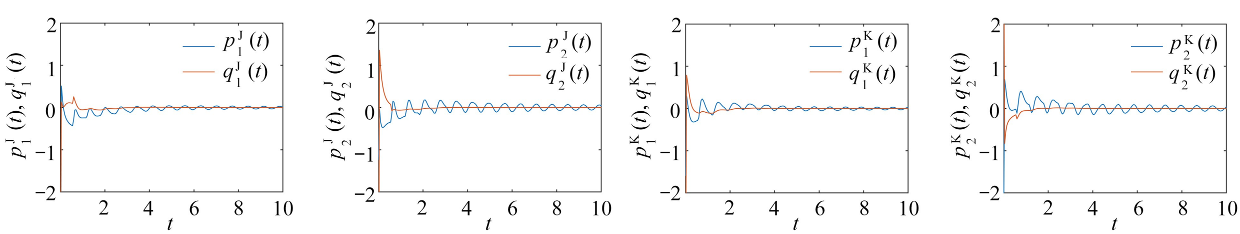

图 2　未加控制时,状态变量 $p_1^{\mathrm{J}},p_2^{\mathrm{J}},p_1^{\mathrm{K}},p_2^{\mathrm{K}},q_1^{\mathrm{J}},q_2^{\mathrm{J}},q_1^{\mathrm{K}},q_2^{\mathrm{K}}$ 的时间响应曲线

Fig. 2　The time response curves of state variables $p_1^{\mathrm{J}},p_2^{\mathrm{J}},p_1^{\mathrm{K}},p_2^{\mathrm{K}},q_1^{\mathrm{J}},q_2^{\mathrm{J}},q_1^{\mathrm{K}},q_2^{\mathrm{K}}$ without control

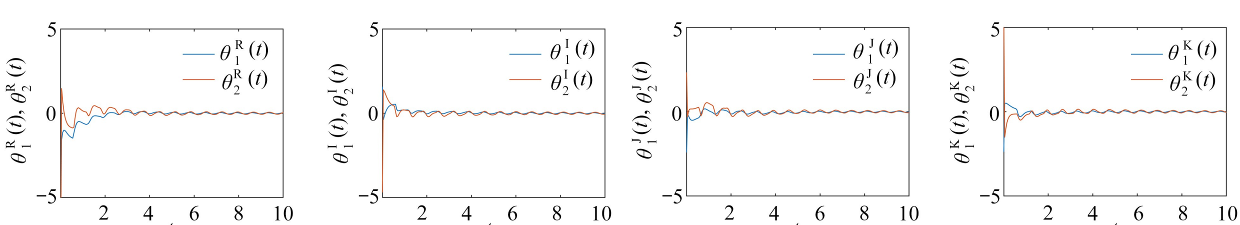

图 3　未加控制时,误差变量 $\theta_1^{\mathrm{R}}(t),\theta_2^{\mathrm{R}}(t),\theta_1^{\mathrm{I}}(t),\theta_2^{\mathrm{I}}(t),\theta_1^{\mathrm{J}}(t),\theta_2^{\mathrm{J}}(t),\theta_1^{\mathrm{K}}(t),\theta_2^{\mathrm{K}}(t)$ 的时间响应曲线

Fig. 3　Time response curves of error variables $\theta_1^{\mathrm{R}}(t),\theta_2^{\mathrm{R}}(t),\theta_1^{\mathrm{I}}(t),\theta_2^{\mathrm{I}}(t),\theta_1^{\mathrm{J}}(t),\theta_2^{\mathrm{J}}(t),\theta_1^{\mathrm{K}}(t),\theta_2^{\mathrm{K}}(t)$ without control

注　为了解释图中的颜色,读者可以参考本文的电子网页版本,后同.

当投影系数矩阵 $\boldsymbol{F}$ 取为 $\boldsymbol{F}=\mathrm{diag}(1,1)$, 驱动系统(1)和响应系统(4)完全同步.图 4、图 5 给出了驱动系统(1)和响应系统(4) 在施加控制时状态变量的时间响应曲线.图 6 给出了误差系统(7) 在施加控制时状态变量的时间响应曲线.

当投影系数矩阵 $\boldsymbol{F}$ 取为 $\boldsymbol{F}=\mathrm{diag}(-1,-1)$, 驱动系统(1)和响应系统(4)反同步 .图 7、图 8 给出了驱

动系统(1)和响应系统(4)在施加控制时状态变量的时间响应曲线.图 9 给出了误差系统(7)在施加控制时状态变量的时间响应曲线.

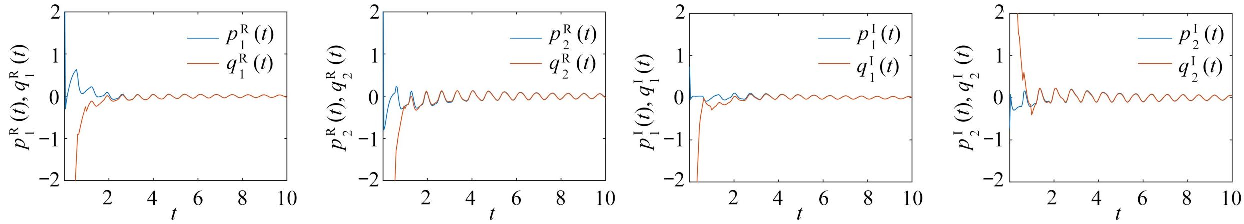

图 4 投影矩阵为 $\boldsymbol{F}$ = diag(1,1),施加控制时,状态变量 $p_1^R, p_2^R, p_1^I, p_2^I, q_1^R, q_2^R, q_1^I, q_2^I$ 的时间响应曲线

Fig. 4 Projection matrix $\boldsymbol{F}$ = diag(1,1), and the time response curves of state variables $p_1^R, p_2^R, p_1^I, p_2^I, q_1^R, q_2^R, q_1^I, q_2^I$ with control

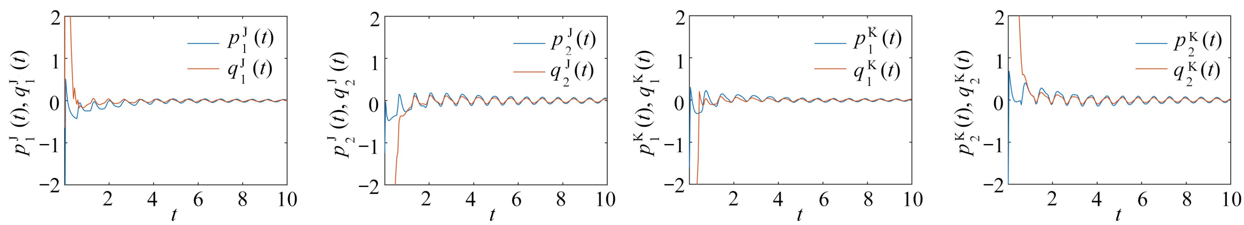

图 5 投影矩阵为 $\boldsymbol{F}$ = diag(1,1),施加控制时,状态变量 $p_1^J, p_2^J, p_1^K, p_2^K, q_1^J, q_2^J, q_1^K, q_2^K$ 的时间响应曲线

Fig. 5 Projection matrix $\boldsymbol{F}$ = diag(1,1), and the time response curves of state variables $p_1^J, p_2^J, p_1^K, p_2^K, q_1^J, q_2^J, q_1^K, q_2^K$ with control

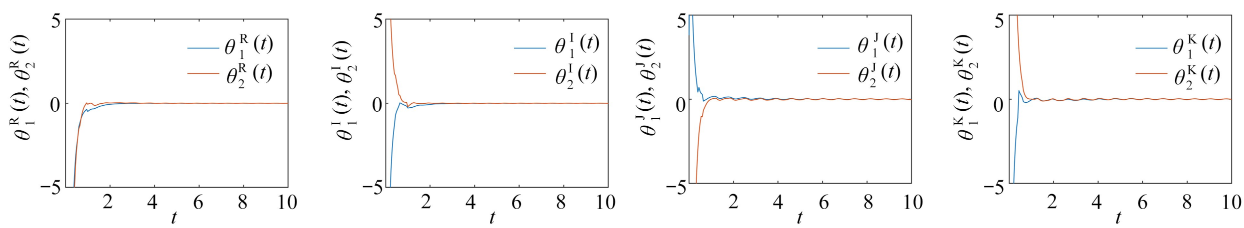

图 6 投影矩阵为 $\boldsymbol{F}$ = diag(1,1),施加控制时,误差变量 $\theta_1^R(t), \theta_2^R(t), \theta_1^I(t), \theta_2^I(t), \theta_1^J(t), \theta_2^J(t), \theta_1^K(t), \theta_2^K(t)$ 的时间响应曲线

Fig. 6 Projection matrix $\boldsymbol{F}$ = diag(1,1), and the time response curves of error variables $\theta_1^R(t), \theta_2^R(t), \theta_1^I(t), \theta_2^I(t), \theta_1^J(t), \theta_2^J(t), \theta_1^K(t), \theta_2^K(t)$ with control

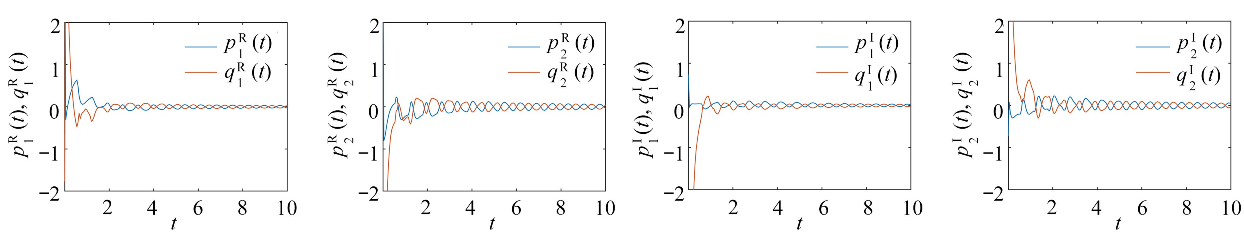

图 7 投影矩阵为 $\boldsymbol{F}$ = diag(−1, −1),施加控制时,状态变量 $p_1^R, p_2^R, p_1^I, p_2^I, q_1^R, q_2^R, q_1^I, q_2^I$ 的时间响应曲线

Fig. 7 Projection matrix $\boldsymbol{F}$ = diag(−1, −1), and the time response curves of state variables $p_1^R, p_2^R, p_1^I, p_2^I, q_1^R, q_2^R, q_1^I, q_2^I$ with control

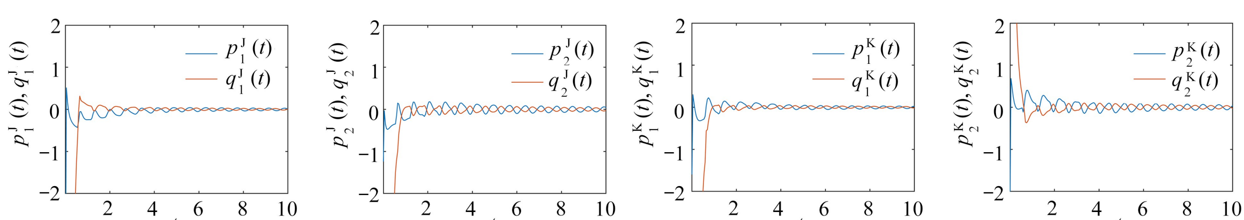

图 8 投影矩阵为 $\boldsymbol{F}$ = diag(−1, −1),施加控制时,状态变量 $p_1^J, p_2^J, p_1^K, p_2^K, q_1^J, q_2^J, q_1^K, q_2^K$ 的时间响应曲线

Fig. 8 Projection matrix $\boldsymbol{F}$ = diag(−1, −1), and the time response curves of state variables $p_1^J, p_2^J, p_1^K, p_2^K, q_1^J, q_2^J, q_1^K, q_2^K$ with control

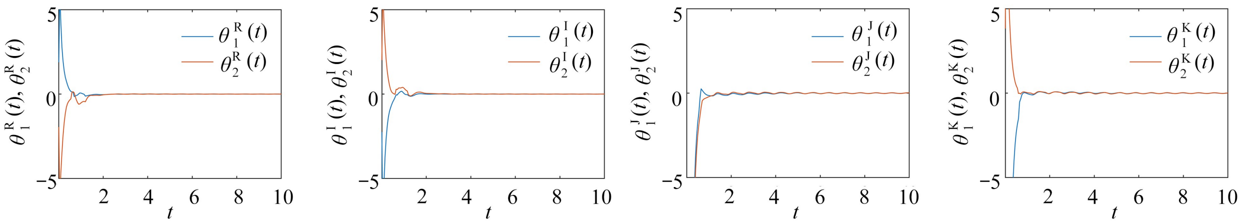

图 9　投影矩阵为 $\boldsymbol{F} = \mathrm{diag}(-1, -1)$,施加控制时,误差变量 $\theta_1^{\mathrm{R}}(t)$,$\theta_2^{\mathrm{R}}(t)$,$\theta_1^{\mathrm{I}}(t)$,$\theta_2^{\mathrm{I}}(t)$,$\theta_1^{\mathrm{J}}(t)$,$\theta_2^{\mathrm{J}}(t)$,$\theta_1^{\mathrm{K}}(t)$,$\theta_2^{\mathrm{K}}(t)$ 的时间响应曲线

Fig. 9　Projection matrix $\boldsymbol{F} = \mathrm{diag}(-1, -1)$, and the time response curves of error variables $\theta_1^{\mathrm{R}}(t)$,$\theta_2^{\mathrm{R}}(t)$,$\theta_1^{\mathrm{I}}(t)$,$\theta_2^{\mathrm{I}}(t)$,$\theta_1^{\mathrm{J}}(t)$,$\theta_2^{\mathrm{J}}(t)$,$\theta_1^{\mathrm{K}}(t)$,$\theta_2^{\mathrm{K}}(t)$ with control

如图 1、2、3 所示,状态变量的时间响应曲线验证了在不施加控制时,驱动系统(1)和响应系统(4)不同步,误差系统(7)是不稳定的.在施加控制时,图 6 和图 9 表明误差系统(7)是稳定的.当投影系数矩阵 $\boldsymbol{F}$ 取为 $\boldsymbol{F} = \mathrm{diag}(1,1)$, 图 4 和图 5 表明驱动系统(1)和响应系统(4)完全同步;当投影系数矩阵 $\boldsymbol{F}$ 取为 $\boldsymbol{F} = \mathrm{diag}(-1, -1)$, 图 7 和图 8 表明驱动系统(1)和响应系统(4)反同步.

5　结　　论

本文研究了具有时变时滞的四元数神经网络系统投影同步问题.在合适的控制器下,通过构造合适的 Lyapunov 函数,并利用一些不等式技巧,得到了具有时变时滞分数阶四元数时滞神经网络的投影同步的充分性判据.最后通过数值仿真实例验证了所得结论的有效性和可行性.

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