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广义算子下约束 Hamilton 系统的 Noether 定理*

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摘要： 研究了广义算子下奇异系统的 Noether 对称性与守恒量.首先,建立了广义算子下奇异系统的 Lagrange 方程,并导出该系统的初级约束,然后引入 Lagrange 乘子建立了广义算子下约束 Hamilton 方程以及相容性条件.其次,基于 Hamilton 作用量在无限小变换下的不变性,建立了广义算子下约束 Hamilton 系统的 Noether 定理,并给出了该系统的对称性及相应的守恒量.在特定条件下,广义算子下约束 Hamilton 系统的 Noether 守恒量可以退化为整数阶约束 Hamilton 系统的 Noether 守恒量.最后举例说明了结果的应用.

关　键　词： 广义算子；　奇异系统；　初级约束；　约束 Hamilton 方程；　Noether 定理；　对称性与守恒量
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Noether's Theorem for Constrained Hamiltonian System Under Generalized Operators

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Abstract: Noether's symmetry and conserved quantity of singular systems under generalized operators were studied. Firstly, the Lagrangian equation of singular systems under generalized operators was established, and the primary constraints on the system were derived. Then the Lagrangian multiplier was introduced to establish the constrained Hamilton equation and the compatibility condition under generalized operators. Secondly, based on the invariance of the Hamilton action under the infinitesimal transformation, Noether's theorem for constrained Hamiltonian systems under generalized operators was established, and the symmetry and corresponding conserved quantity of the system were given. Under certain conditions, Noether's conservation of constrained Hamiltonian systems under generalized operators can be reduced to Noether's conservation of integer-order constrained Hamiltonian systems. Finally, an example illustrates the application of the results.

Key words: generalized operator; singular system; primary constraint; constrained Hamilton equation; Noether's theorem; symmetry and conserved quantity

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引　言

用奇异 Lagrange 函数描述的系统称为奇异系统, 在过渡到相空间描述时, 其正则变量之间存在固有约束, 称为约束 Hamilton 系统. 奇异系统长期活跃在物理学的众多领域, 如杨-Mills 场、旋量场、电磁场、超引力、超弦等理论都与其息息相关. 因此奇异系统的基本理论在物理学中, 特别是在现代量子场论中有着不可或缺的作用[1-2]. Nambu[3] 率先研究了奇异 Lagrange 系统的正则形式, 此后 Bergmann 等[4] 也奠定了该系统动力学与量子化的基础.

Noether 定理是德国数学家 Noether[5] 在 1918 年提出的, 该定理首次揭示了对称性与守恒量之间的关系. 众所周知, Noether 对称性是指 Hamilton 作用量在无限小变换下的不变性. 通过 Noether 定理可以找到不同力学系统的守恒量, 而力学系统的守恒量对研究力学系统的动力学行为及稳定性都有指导意义, 因此 Noether 理论的研究一直以来是诸多学者关注的热门课题, 并且也取得了丰硕的成果[6-11]. 特别地, 李子平[12] 提出了奇异系统在相空间中的 Noether 定理.

分数阶模型相比于整数阶模型, 能够更好地描述复杂动力学及物理行为. 由于分数阶微积分具有记忆性和非局域性, 因此被广泛应用于流体力学、光学、经济学、信号图像处理以及生物医学工程等众多领域[13-15]. 分数阶算子中应用最为广泛的是 Riemann-Liouville 分数阶算子[16]、Caputo 分数阶算子[17]、Riesz-Riemann-Liouville 分数阶算子[18] 以及 Riesz-Caputo 分数阶算子[19]. 2010 年, Agrawal[20] 提出了一种新的分数阶算子, 称其为广义分数阶算子. 在特殊情形下, 广义分数阶算子可以退化为上述四种算子.

1996 年, Riewe[21-22] 首次将分数阶微积分纳入非保守力学系统, 提出并初步研究了分数阶变分问题, Frederico 等[23-24]、Agrawal[20] 也进一步研究了分数阶变分问题. 2007 年, Frederico 等[23-24] 首次研究了分数阶 Noether 对称性与守恒量并建立了 Noether 定理. 之后, 分数阶 Noether 对称性与守恒量的研究也取得了重大进展[25-31]. 特别地, Song 等[32-33] 利用 Agrawal 提出的广义分数阶算子给出了 Birkhoff 系统以及 Hamilton 系统的 Noether 对称性与守恒量, 然而在广义分数阶算子下, 对奇异系统的 Noether 对称性与守恒量的研究还未涉及. 因此本文进一步研究了广义分数阶算子下奇异系统的 Noether 对称性, 建立并证明了该系统的 Noether 定理, 同时给出了广义算子下相应的守恒量.

1　预备知识

广义分数阶算子K_M^α, A_M^α, B_M^α是由 Agrawal[20] 提出的, 这里主要列出它们的定义以及相关性质.

广义算子K_M^α, A_M^α, B_M^α定义为

$$K^\alpha_{\langle t_1,t,t_2,m,\omega\rangle} f(t) = K_M^\alpha f(t) = m\int_{t_1}^{t} \kappa_\alpha(t,\tau) f(\tau)\mathrm{d}\tau + \omega\int_t^{t_2}\kappa_\alpha(\tau,t) f(\tau)\mathrm{d}\tau, \qquad \alpha>0, \tag{1}$$

$$A_M^\alpha f(t) = \mathrm{D}^n K_M^{n-\alpha} f(t), \qquad n-1<\alpha<n, \tag{2}$$

$$B_M^\alpha f(t) = K_M^{n-\alpha}\mathrm{D}^n f(t), \qquad n-1<\alpha<n, \tag{3}$$

其中$M=\langle t_1,t,t_2,m,\omega\rangle$是参数集, m和ω是实数, $\kappa_\alpha(t,\tau)$是依赖于阶数α的核, n为整数.

算子K_M^α, A_M^α, B_M^α的分部积分分别为

$$\int_{t_1}^{t_2} g(t) K_M^\alpha f(t)\mathrm{d}t = \int_{t_1}^{t_2} f(t) K_{M^*}^\alpha g(t)\mathrm{d}t, \tag{4}$$

$$\int_{t_1}^{t_2} g(t) A_M^\alpha f(t)\mathrm{d}t = (-1)^n\int_{t_1}^{t_2} f(t) B_{M^*}^\alpha g(t)\mathrm{d}t + \sum_{j=0}^{n-1}(-\mathrm{D})^{n-1-j} g(t) A_M^{\alpha+j-n} f(t)\Big|_{t=t_1}^{t=t_2}, \tag{5}$$

$$\int_{t_1}^{t_2} g(t) B_M^\alpha f(t)\mathrm{d}t = (-1)^n\int_{t_1}^{t_2} f(t) A_{M^*}^\alpha g(t)\mathrm{d}t + \sum_{j=0}^{n-1}(-1)^j A_{M^*}^{\alpha+j-n} g(t)\mathrm{D}^{n-1-j} f(t)\Big|_{t=t_1}^{t=t_2}, \tag{6}$$

其中$M=\langle t_1,t,t_2,m,\omega\rangle$, $M^*=\langle t_1,t,t_2,\omega,m\rangle$, n为整数, $n-1<\alpha<n$.

当$\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, $M=M_1=\langle t_1,t,t_2,1,0\rangle$时, 得

$$A_{M_1}^\alpha f(t) = \mathrm{D}^n K_{M_1}^{n-\alpha} f(t) = {}_{t_1}^{\mathrm{RL}}\mathrm{D}_t^\alpha f(t), \tag{7}$$

$$B_{M_1}^{\alpha} f(t) = K_{M_1}^{n-\alpha} \mathrm{D}^n f(t) = {}_{t_1}^{\mathrm{C}}\mathrm{D}_t^{\alpha} f(t). \tag{8}$$

当$\kappa_\alpha(t,\tau) = (t-\tau)^{\alpha-1}/\Gamma(\alpha)$, $M = M_2 = \langle t_1, t, t_2, 0, 1\rangle$时, 得

$$A_{M_2}^{\alpha} f(t) = \mathrm{D}^n K_{M_2}^{n-\alpha} f(t) = (-1)^n {}_{t}^{\mathrm{RL}}\mathrm{D}_{t_2}^{\alpha} f(t), \tag{9}$$

$$B_{M_2}^{\alpha} f(t) = K_{M_2}^{n-\alpha} \mathrm{D}^n f(t) = (-1)^n {}_{t}^{\mathrm{C}}\mathrm{D}_{t_2}^{\alpha} f(t). \tag{10}$$

当$\kappa_\alpha(t,\tau) = (t-\tau)^{\alpha-1}/\Gamma(\alpha)$, $M = M_3 = \left\langle t_1, t, t_2, \dfrac{1}{2}, \dfrac{1}{2}\right\rangle$时, 得

$$A_{M_3}^{\alpha} f(t) = \mathrm{D}^n K_{M_3}^{n-\alpha} f(t) = {}_{t_1}^{\mathrm{R}}\mathrm{D}_{t_2}^{\alpha} f(t), \tag{11}$$

$$B_{M_3}^{\alpha} f(t) = K_{M_3}^{n-\alpha} \mathrm{D}^n f(t) = {}_{t_1}^{\mathrm{RC}}\mathrm{D}_{t_2}^{\alpha} f(t). \tag{12}$$

式(7)～(12)分别为左 Riemann-Liouville 分数阶导数、左 Caputo 分数阶导数、右 Riemann-Liouville 分数阶导数、右 Caputo 分数阶导数、Riesz-Riemann-Liouville 分数阶导数、Riesz-Caputo 分数阶导数, 以上都是算子A_M^{α}, B_M^{α}的特例.

2 广义算子下奇异 Lagrange 系统和初级约束

2.1 算子A_M^{α}下奇异 Lagrange 系统和初级约束

算子A_M^{α}下的 Hamilton 作用量定义为

$$S_A = \int_{t_1}^{t_2} L_A(t, \boldsymbol{q}_A, A_M^{\alpha}\boldsymbol{q}_A)\mathrm{d}t, \tag{13}$$

其中

$$\boldsymbol{q}_A = (q_{A1}, q_{A2}, \cdots, q_{An}),\ A_M^{\alpha}\boldsymbol{q}_A = (A_M^{\alpha}q_{A1}, A_M^{\alpha}q_{A2}, \cdots, A_M^{\alpha}q_{An}), \qquad 0<\alpha<1.$$

等时变分

$$\delta S_A = 0 \tag{14}$$

满足交换关系

$$\delta A_M^{\alpha} q_{Ai} = A_M^{\alpha}\delta q_{Ai}, \qquad i = 1,2,\cdots,n \tag{15}$$

和端点条件

$$\boldsymbol{q}_{A1} = \boldsymbol{q}_A(t_1),\ \boldsymbol{q}_{A2} = \boldsymbol{q}_A(t_2), \tag{16}$$

其中$\boldsymbol{q}_{A1} = (q_{A11}, q_{A21}, \cdots, q_{An1})$, $\boldsymbol{q}_{A2} = (q_{A12}, q_{A22}, \cdots, q_{An2})$, 称为算子$A_M^{\alpha}$下的分数阶 Hamilton 原理.

由式(5)、(14)～(16)可得

$$\frac{\partial L_A}{\partial q_{Ai}} - B_{M^*}^{\alpha}\frac{\partial L_A}{\partial A_M^{\alpha}q_{Ai}} + m\kappa_{1-\alpha}(t_2,t)\frac{\partial L_A(t_2)}{\partial A_M^{\alpha}q_{Ai}} - \omega\kappa_{1-\alpha}(t,t_1)\frac{\partial L_A(t_1)}{\partial A_M^{\alpha}q_{Ai}} = 0, \qquad i = 1,2,\cdots,n, \tag{17}$$

其中$L_A(t_1) = L_A(t_1, \boldsymbol{q}_A(t_1), A_M^{\alpha}\boldsymbol{q}_A(t_1))$, $L_A(t_2) = L_A(t_2, \boldsymbol{q}_A(t_2), A_M^{\alpha}\boldsymbol{q}_A(t_2))$. 式(17)为算子$A_M^{\alpha}$下的分数阶 Lagrange 方程.

定义广义动量和 Hamilton 量为

$$p_{Ai} = \frac{\partial L_A(t, \boldsymbol{q}_A, A_M^{\alpha}\boldsymbol{q}_A)}{\partial A_M^{\alpha}q_{Ai}}, \tag{18}$$

$$H_A = p_{Ai}A_M^{\alpha}q_{Ai} - L_A(t, \boldsymbol{q}_A, A_M^{\alpha}\boldsymbol{q}_A), \qquad i = 1,2,\cdots,n. \tag{19}$$

这里考虑$L_A(t, \boldsymbol{q}_A, A_M^{\alpha}\boldsymbol{q}_A)$是奇异的, 即$A_M^{\alpha}q_{Ai}$只有一部分能解出来, 假设可以解出 R 个$A_M^{\alpha}q_{Ai}$, $0 \leqslant R < n$, 接下来讨论两种情况分别为$1 \leqslant R < n$和$R = 0$.

当$1 \leqslant R < n$时, 有

$$A_M^{\alpha}q_{A\sigma} = f_A^{\sigma}(t, \boldsymbol{q}_A, \boldsymbol{p}_{AE}, A_M^{\alpha}\boldsymbol{q}_{A\rho}), \qquad \sigma, E = 1,2,\cdots,R;\ \rho = R+1, R+2, \cdots, n, \tag{20}$$

其中

$$\boldsymbol{p}_{AE} = (p_{A1}, p_{A2}, \cdots, p_{AR}),\ A_M^{\alpha}\boldsymbol{q}_{A\rho} = (A_M^{\alpha}q_{A(R+1)}, A_M^{\alpha}q_{A(R+2)}, \cdots, A_M^{\alpha}q_{An}), \qquad 1 \leqslant R < n.$$

将式(20)代入式(18)得

$$p_{Ai} = g_{Ai}(t, \boldsymbol{q}_A, f_A^{\sigma}(t, \boldsymbol{q}_A, \boldsymbol{p}_{AE}, A_M^{\alpha}\boldsymbol{q}_{A\rho}), A_M^{\alpha}\boldsymbol{q}_{A\rho}) = g_{Ai}(t, \boldsymbol{q}_A, \boldsymbol{p}_{AE}, A_M^{\alpha}\boldsymbol{q}_{A\rho}), \qquad i = 1,2,\cdots,n. \tag{21}$$

当$i=1,2,\cdots,R$时, 式(21)是成立的, 若$i=R+1,R+2,\cdots,n$, 有

$$p_{A\rho}=g_{A\rho}(t,\boldsymbol{q}_A,\boldsymbol{p}_{AE}),\quad \text{or}\quad \boldsymbol{p}_{AF}=\boldsymbol{g}_{AF}(t,\boldsymbol{q}_A,\boldsymbol{p}_{AE}),\qquad 1\leqslant R<n,\tag{22}$$

其中

$$\boldsymbol{p}_{AF}=(p_{A(R+1)},p_{A(R+2)},\cdots,p_{An}),\ \boldsymbol{g}_{AF}=(g_{A(R+1)},g_{A(R+2)},\cdots,g_{An}),\ \rho=R+1,R+2,\cdots,n,$$

或记为

$$\boldsymbol{\phi}_A(t,\boldsymbol{q}_A,\boldsymbol{p}_A)=\boldsymbol{p}_{AF}-\boldsymbol{g}_{AF}(t,\boldsymbol{q}_A,\boldsymbol{p}_{AE})=\boldsymbol{0},\tag{23}$$

其中

$$\boldsymbol{\phi}_A=(\phi_{A1},\phi_{A2},\cdots,\phi_{A(n-R)}),\ \boldsymbol{p}_A=(p_{A1},p_{A2},\cdots,p_{An}),\qquad 1\leqslant R<n.$$

当$R=0$时, 则没有$A_M^\alpha q_{Ai}$能被解出, 由式(18)得

$$p_{Ai}=g_{Ai}(t,\boldsymbol{q}_A),\qquad i=1,2,\cdots,n,\tag{24}$$

$$\phi_{Aa}(t,\boldsymbol{q}_A,\boldsymbol{p}_A)=p_{Aa}-g_{Aa}(t,\boldsymbol{q}_A)=0,\qquad a=1,2,\cdots,n.\tag{25}$$

由式(23)、(25)得

$$\phi_{Aa}(t,\boldsymbol{q}_A,\boldsymbol{p}_A)=0,\qquad a=1,2,\cdots,n-R,\ 0\leqslant R<n.\tag{26}$$

式(26)称为算子A_M^α下的初级约束.

注 1　令$\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, 当$M=M_1$, $M=M_2$以及$M=M_3$时, 由式(17)、(26)分别得到左 Riemann-Liouville 分数阶导数、右 Riemann-Liouville 分数阶导数以及 Riesz-Riemann-Liouville 分数阶导数下的 Lagrange 方程和初级约束.

2.2 算子B_M^α下奇异 Lagrange 系统和初级约束

算子B_M^α下的 Hamilton 作用量定义为

$$S_B=\int_{t_1}^{t_2}L_B(t,\boldsymbol{q}_B,B_M^\alpha\boldsymbol{q}_B)\mathrm{d}t,\tag{27}$$

其中

$$\boldsymbol{q}_B=(q_{B1},q_{B2},\cdots,q_{Bn}),\ B_M^\alpha\boldsymbol{q}_B=(B_M^\alpha q_{B1},B_M^\alpha q_{B2},\cdots,B_M^\alpha q_{Bn}),\qquad 0<\alpha<1.$$

等时变分

$$\delta S_B=0\tag{28}$$

满足交换关系

$$\delta B_M^\alpha q_{Bi}=B_M^\alpha\delta q_{Bi},\qquad i=1,2,\cdots,n\tag{29}$$

和端点条件

$$\boldsymbol{q}_{B1}=\boldsymbol{q}_B(t_1),\ \boldsymbol{q}_{B2}=\boldsymbol{q}_B(t_2),\tag{30}$$

其中$\boldsymbol{q}_{B1}=(q_{B11},q_{B21},\cdots,q_{Bn1})$, $\boldsymbol{q}_{B2}=(q_{B12},q_{B22},\cdots,q_{Bn2})$, 称为算子$B_M^\alpha$下的分数阶 Hamilton 原理.

由式(6)、(28)～(30)可得

$$\frac{\partial L_B}{\partial q_{Bi}}-A_{M^*}^\alpha\frac{\partial L_B}{\partial B_M^\alpha q_{Bi}}=0,\qquad i=1,2,\cdots,n,\tag{31}$$

其中$L_B(t_1)=L_B(t_1,\boldsymbol{q}_B(t_1),B_M^\alpha\boldsymbol{q}_B(t_1))$, $L_B(t_2)=L_B(t_2,\boldsymbol{q}_B(t_2),B_M^\alpha\boldsymbol{q}_B(t_2))$. 式(31)为算子$B_M^\alpha$下的分数阶 Lagrange 方程.

定义广义动量和 Hamilton 量为

$$p_{Bi}=\frac{\partial L_B(t,\boldsymbol{q}_B,B_M^\alpha\boldsymbol{q}_B)}{\partial B_M^\alpha q_{Bi}},\qquad i=1,2,\cdots,n,\tag{32}$$

$$H_B=p_{Bi}B_M^\alpha q_{Bi}-L_B(t,\boldsymbol{q}_B,B_M^\alpha\boldsymbol{q}_B),\qquad i=1,2,\cdots,n,\tag{33}$$

这里考虑$L_B(t,\boldsymbol{q}_B,B_M^\alpha\boldsymbol{q}_B)$是奇异的, 即$B_M^\alpha q_{Bi}$只有一部分能解出来, 假设可以解出 R 个$B_M^\alpha q_{Bi}$, $0\leqslant R<n$, 接下来讨论两种情况分别为$1\leqslant R<n$和$R=0$.

当$1\leqslant R<n$时, 有

$$B_M^\alpha q_{B\sigma}=f_B^\sigma(t,\boldsymbol{q}_B,\boldsymbol{p}_{BE},B_M^\alpha q_{B\rho}),\qquad \sigma,E=1,2,\cdots,R;\ \rho=R+1,R+2,\cdots,n,\tag{34}$$

其中

$$\boldsymbol{p}_{BE}=(p_{B1},p_{B2},\cdots,p_{BR}),\ B_M^\alpha\boldsymbol{q}_{B\rho}=(B_M^\alpha q_{BR+1},B_M^\alpha\boldsymbol{q}_{BR+2},\cdots,B_M^\alpha q_{Bn}),\qquad 1\leqslant R<n.$$

将式(34)代入式(32)得

$$p_{Bi}=g_{Bi}(t,\boldsymbol{q}_B,f_B^\sigma(t,\boldsymbol{q}_B,\boldsymbol{p}_{BE},B_M^\alpha q_{B\rho}),B_M^\alpha q_{B\rho})=g_{Bi}(t,\boldsymbol{q}_B,\boldsymbol{p}_{BE},B_M^\alpha q_{B\rho}),\qquad i=1,2,\cdots,n.\tag{35}$$

当$i=1,2,\cdots,R$时，式(35)是成立的，若$i=R+1,R+2,\cdots,n$，有

$$p_{B\rho}=g_{B\rho}(t,\boldsymbol{q}_B,\boldsymbol{p}_{BE}),\quad \text{or}\quad \boldsymbol{p}_{BF}=\boldsymbol{g}_{BF}(t,\boldsymbol{q}_B,\boldsymbol{p}_{BE}),\qquad 1\leqslant R<n,\tag{36}$$

其中

$$\boldsymbol{p}_{BF}=(p_{BR+1},p_{BR+2},\cdots,p_{Bn}),\ \boldsymbol{g}_{BF}=(g_{BR+1},g_{BR+2},\cdots,g_{Bn}),\ \rho=R+1,R+2,\cdots,n,$$

或记为

$$\boldsymbol{\phi}_B(t,\boldsymbol{q}_B,\boldsymbol{p}_B)=\boldsymbol{p}_{BF}-\boldsymbol{g}_{BF}(t,\boldsymbol{q}_B,\boldsymbol{p}_{BE})=\boldsymbol{0},\tag{37}$$

其中

$$\boldsymbol{\phi}_B=(\phi_{B1},\phi_{B2},\cdots,\phi_{Bn-R}),\ \boldsymbol{p}_B=(p_{B1},p_{B2},\cdots,p_{Bn}),\qquad 1\leqslant R<n.$$

当$R=0$时，则没有$B_M^\alpha q_{Bi}$能被解出，由式(32)得

$$p_{Bi}=g_{Bi}(t,\boldsymbol{q}_B),\qquad i=1,2,\cdots,n,\tag{38}$$

$$\phi_{Ba}(t,\boldsymbol{q}_B,\boldsymbol{p}_B)=p_{Ba}-g_{Ba}(t,\boldsymbol{q}_B)=0,\qquad a=1,2,\cdots,n,\tag{39}$$

其中

$$\boldsymbol{q}_B=(q_{B1},q_{B2},\cdots,q_{Bn}),\ \boldsymbol{p}_B=(p_{B1},p_{B2},\cdots,p_{Bn}).$$

由式(37)、(39)得

$$\phi_{Ba}(t,\boldsymbol{q}_B,\boldsymbol{p}_B)=0,\qquad a=1,2,\cdots,n-R;\ 0\leqslant R<n.\tag{40}$$

式(40)称为算子B_M^α下的初级约束.

注 2 令$\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$，当$M=M_1$, $M=M_2$以及$M=M_3$时，由式(31)、(40)分别得到左 Caputo 分数阶导数、右 Caputo 分数阶导数以及 Riesz-Caputo 分数阶导数下的 Lagrange 方程和初级约束.

3 广义算子下约束 Hamilton 方程及相容性条件

3.1 算子A_M^α下约束 Hamilton 方程

由式(18)和(19)得

$$\begin{aligned}\delta H_A=&P_{Ai}\cdot\delta A_M^\alpha q_{Ai}+A_M^\alpha q_{Ai}\delta p_{Ai}-\frac{\partial L_A}{\partial q_{Ai}}\delta q_{Ai}-\frac{\partial L_A}{\partial A_M^\alpha q_{Ai}}\delta A_M^\alpha q_{Ai}=\\&A_M^\alpha q_{Ai}\delta p_{Ai}-\frac{\partial L_A}{\partial q_{Ai}}\delta q_{Ai},\qquad i=1,2,\cdots,n.\end{aligned}\tag{41}$$

由式(17)得

$$\frac{\partial L_A}{\partial q_{Ai}}=B_{M^*}^\alpha\frac{\partial L_A}{\partial A_M^\alpha q_{Ai}}-m\kappa_{1-\alpha}(t_2,t)\frac{\partial L_A(t_2)}{\partial A_M^\alpha q_{Ai}}+\omega\kappa_{1-\alpha}(t,t_1)\frac{\partial L_A(t_1)}{\partial A_M^\alpha q_{Ai}}.\tag{42}$$

由 Hamilton 量$H_A(t,\boldsymbol{q}_A,\boldsymbol{p}_A)$可得

$$\delta H_A=\frac{\partial H_A}{\partial q_{Ai}}\delta q_{Ai}+\frac{\partial H_A}{\partial p_{Ai}}\delta p_{Ai},\qquad i=1,2,\cdots,n.\tag{43}$$

由式(26)的等时变分得

$$\delta\phi_{Aa}=\frac{\partial\phi_{Aa}}{\partial q_{Ai}}\delta q_{Ai}+\frac{\partial\phi_{Aa}}{\partial p_{Ai}}\delta p_{Ai}=0,\qquad i=1,2,\cdots,n;\ a=1,2,\cdots,n-R;\ 0\leqslant R<n.\tag{44}$$

引入 Lagrange 乘子$\lambda_{Aa}(t)$, $a=1,2,\cdots,n-R$, $0\leqslant R<n$，由式(41)～(44)得

$$\begin{aligned}&B_{M^*}^\alpha p_{Ai}=-\frac{\partial H_A}{\partial q_{Ai}}-\lambda_{Aa}\frac{\partial\phi_{Aa}}{\partial q_{Ai}}+m\kappa_{1-\alpha}(t_2,t)p_{Ai}(t_2)-\omega\kappa_{1-\alpha}(t,t_1)p_{Ai}(t_1),\ A_M^\alpha q_{Ai}=\frac{\partial H_A}{\partial p_{Ai}}+\lambda_{Aa}\frac{\partial\phi_{Aa}}{\partial p_{Ai}},\\&\qquad\qquad i=1,2,\cdots,n;\ a=1,2,\cdots,n-R;\ 0\leqslant R<n.\end{aligned}\tag{45}$$

方程(45)为算子A_M^α下的约束 Hamilton 方程.

引入总 Hamilton 量$H_{AT}=H_A+\lambda_{Aa}\phi_{Aa}$, 式(45)也可表示为

$$A_M^\alpha q_{Ai}=\frac{\partial H_{AT}}{\partial p_{Ai}},\ B_{M^*}^\alpha p_{Ai}=-\frac{\partial H_{AT}}{\partial q_{Ai}}+m\kappa_{1-\alpha}(t_2,t)p_{Ai}(t_2)-\omega\kappa_{1-\alpha}(t,t_1)p_{Ai}(t_1),$$
$$i=1,2,\cdots,n;\ a=1,2,\cdots,n-R;\ 0\leqslant R<n. \tag{46}$$

注 3　令$\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, 当$M=M_1$, $M=M_2$以及$M=M_3$时, 由式(45)分别得到左 Riemann-Liouville 分数阶导数、右 Riemann-Liouville 分数阶导数以及 Riesz-Riemann-Liouville 分数阶导数下的约束 Hamilton 方程.

3.2 算子B_M^α下约束 Hamilton 方程

由式(32)和(33)得

$$\delta H_B=P_{Bi}\cdot\delta B_M^\alpha q_{Bi}+B_M^\alpha q_{Bi}\delta p_{Bi}-\frac{\partial L_B}{\partial q_{Bi}}\delta q_{Bi}-\frac{\partial L_B}{\partial B_M^\alpha q_{Bi}}\delta B_M^\alpha q_{Bi}=$$
$$B_M^\alpha q_{Bi}\delta\, p_{Bi}-\frac{\partial L_B}{\partial q_{Bi}}\delta\, q_{Bi},\qquad i=1,2,\cdots,n. \tag{47}$$

由式(31)得

$$\frac{\partial L_B}{\partial q_{Bi}}=A_{M^*}^\alpha\frac{\partial L_B}{\partial B_M^\alpha q_{Bi}}=A_{M^*}^\alpha p_{Bi}. \tag{48}$$

由 Hamilton 量$H_B(t,\boldsymbol{q}_B,\boldsymbol{p}_B)$可得

$$\delta H_B=\frac{\partial H_B}{\partial q_{Bi}}\delta q_{Bi}+\frac{\partial H_B}{\partial p_{Bi}}\delta p_{Bi},\qquad i=1,2,\cdots,n. \tag{49}$$

由式(40)的等时变分得

$$\delta\phi_{Ba}=\frac{\partial\phi_{Ba}}{\partial q_{Bi}}\cdot\delta q_{Bi}+\frac{\partial\phi_{Ba}}{\partial p_{Bi}}\cdot\delta p_{Bi}=0,\qquad i=1,2,\cdots,n;\ a=1,2,\cdots,n-R;\ 0\leqslant R<n. \tag{50}$$

引入 Lagrange 乘子$\lambda_{Ba}(t)$, $a=1,2,\cdots,n-R$, $0\leqslant R<n$, 由式(47)～(50)得

$$B_M^\alpha q_{Bi}=\frac{\partial H_B}{\partial p_{Bi}}+\lambda_{Ba}\frac{\partial\phi_{Ba}}{\partial p_{Bi}},\ A_{M^*}^\alpha p_{Bi}=-\frac{\partial H_B}{\partial q_{Bi}}-\lambda_{Ba}\frac{\partial\phi_{Ba}}{\partial q_{Bi}},$$
$$i=1,2,\cdots,n;\ a=1,2,\cdots,n-R;\ 0\leqslant R<n. \tag{51}$$

方程(51)为算子B_M^α下的约束 Hamilton 方程.

引入总 Hamilton 量$H_{BT}=H_B+\lambda_{Ba}\phi_{Ba}$, 式(51)也可表示为

$$B_M^\alpha q_{Bi}=\frac{\partial H_{BT}}{\partial p_{Bi}},\ A_{M^*}^\alpha p_{Bi}=-\frac{\partial H_{BT}}{\partial q_{Bi}},\qquad i=1,2,\cdots,n;\ a=1,2,\cdots,n-R;\ 0\leqslant R<n. \tag{52}$$

注 4　令$\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, 当$M=M_1$, $M=M_2$以及$M=M_3$时, 由式(51)分别得到左 Caputo 分数阶导数、右 Caputo 分数阶导数以及 Riesz-Caputo 分数阶导数下的约束 Hamilton 方程.

3.3 广义算子下约束Hamilton 系统的相容性条件

令$F=F(t,\boldsymbol{q},\boldsymbol{p})$, $G=G(t,\boldsymbol{q},\boldsymbol{p})$, 定义 Poisson 括号:

$$\{F,G\}=\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i},\qquad i=1,2,\cdots,n, \tag{53}$$

其中$\boldsymbol{q}=(q_1,q_2,\cdots,q_n)$, $\boldsymbol{p}=(p_1,p_2,\cdots,p_n)$. 利用 Poisson 括号和式(46)可得

$$(\{\phi_{Aa},H_A\}+\lambda_{Ab}\{\phi_{Aa},\phi_{Ab}\})\frac{\partial\phi_{Aa}}{\partial q_{Aj}}\dot q_{Aj}+\left(\frac{\partial\phi_{Aa}}{\partial t}+\frac{\partial\phi_{Aa}}{\partial p_{Aj}}\dot p_{Aj}\right)\frac{\partial\phi_{Aa}}{\partial q_{Ai}}A_M^\alpha q_{Ai}-$$
$$\left(\frac{\partial\phi_{Aa}}{\partial p_{Ai}}\cdot\frac{\partial\phi_{Aa}}{\partial q_{Aj}}\dot q_{Aj}\right)[B_{M^*}^\alpha p_{Ai}-m\kappa_{1-\alpha}(t_2,t)p_{Ai}(t_2)+\omega\kappa_{1-\alpha}(t,t_1)p_{Ai}(t_1)]=0,$$
$$i,j=1,2,\cdots,n;\ a=1,2,\cdots,n-R;\ 0\leqslant R<n. \tag{54}$$

式(54)称为算子A_M^α下约束 Hamilton 系统的相容性条件.

同理, 算子B_M^α下约束 Hamilton 系统的相容性条件为

$$(\{\phi_{Ba},H_B\}+\lambda_{Bb}\{\phi_{Ba},\phi_{Bb}\})\frac{\partial\phi_{Ba}}{\partial q_{Bj}}\dot{q}_{Bj}+\left(\frac{\partial\phi_{Ba}}{\partial t}+\frac{\partial\phi_{Ba}}{\partial p_{Bj}}\dot{p}_{Bj}\right)\frac{\partial\phi_{Ba}}{\partial q_{Bi}}B^{\alpha}_{M}q_{Bi}-$$

$$\frac{\partial\phi_{Ba}}{\partial p_{Bi}}\cdot\frac{\partial\phi_{Ba}}{\partial q_{Bj}}\dot{q}_{Bj}\cdot A^{\alpha}_{M^*}p_{Bi}=0,\qquad i,j=1,2,\cdots,n;\ a=1,2,\cdots,n-R;\ 0\leqslant R<n. \tag{55}$$

如果$\det[\{\phi_{Aa},\phi_{Ab}\}]\neq 0(\det[\{\phi_{Ba},\phi_{Bb}\}]\neq 0)$,则所有的 Lagrange 乘子$\lambda_{Aa}(\lambda_{Ba})$可由式(54)(式(55))全部解出.

如果$\det[\{\phi_{Aa},\phi_{Ab}\}]=0(\det[\{\phi_{Ba},\phi_{Bb}\}]=0)$,则矩阵$[\{\phi_{Aa},\phi_{Ab}\}]([\{\phi_{Ba},\phi_{Bb}\}])$是奇异的,此时$\lambda_{Aa}(\lambda_{Ba})$不能全部解出,则产生新的约束称为次级约束,次级约束是由初级约束的相容性条件所导出.同理,若次级约束仍然不能解出所有的 Lagrange 乘子,就会又导出新的次级约束.对有限自由度系统,这种次级约束相容性条件经过有限次步骤后,就不再产生新的次级约束.

注 5 令$\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$,当$M=M_1$, $M=M_2$以及$M=M_3$时,由式(54)和(55)分别得到左 Riemann-Liouville 分数阶导数、左 Caputo 分数阶导数、右 Riemann-Liouville 分数阶导数、右 Caputo 分数阶导数、Riesz-Riemann-Liouville 分数阶导数以及 Riesz-Caputo 分数阶导数下初级约束的相容性条件.

4 广义算子下约束 Hamilton 系统的 Noether 定理

Noether 对称性是 Hamilton 作用量在无限小变换下的不变性.Noether 对称性导致 Noether 守恒量.这里我们将分别研究算子A^{α}_{M}和B^{α}_{M}下约束 Hamilton 系统的 Noether 定理.

4.1 算子A^{α}_{M}下约束 Hamilton 系统的 Noether 定理

考虑算子A^{α}_{M}的 Hamilton 作用量:

$$S_A=\int_{t_1}^{t_2}[p_{Ai}\cdot A^{\alpha}_{M}q_{Ai}-H_A(t,\boldsymbol{q}_A,\boldsymbol{p}_A)]\mathrm{d}t,\qquad i=1,2,\cdots,n. \tag{56}$$

引入无限小变换:

$$\bar{t}=t+\Delta t,\ \bar{q}_{Ai}=q_{Ai}(t)+\Delta q_{Ai},\ \bar{p}_{Ai}(\bar{t})=p_{Ai}(t)+\Delta p_{Ai}, \tag{57}$$

其展开式为

$$\begin{cases}\bar{t}=t+\theta_A\xi_{A0}(t,\boldsymbol{q}_A,\boldsymbol{p}_A)+o(\theta_A),\ \bar{q}_{Ai}=q_{Ai}(t)+\theta_A\xi_{Ai}(t,\boldsymbol{q}_A,\boldsymbol{p}_A)+o(\theta_A),\\ \bar{p}_{Ai}(\bar{t})=p_{Ai}(t)+\theta_A\eta_{Ai}(t,\boldsymbol{q}_A,\boldsymbol{p}_A)+o(\theta_A),\end{cases} \tag{58}$$

其中θ_A是无限小参数, $\xi_{A0},\xi_{Ai},\eta_{Ai}$为无限小生成元, $i=1,2,\cdots,n$.

$$\Delta S_A=\bar{S}_A-S_A=\int_{\bar{t}_1}^{\bar{t}_2}\left[\bar{p}_{Ai}A^{\alpha}_{\bar{M}}\bar{q}_{Ai}-H_A(\bar{t},\bar{\boldsymbol{q}}_A,\bar{\boldsymbol{p}}_A)\right]\mathrm{d}\bar{t}-\int_{t_1}^{t_2}[p_{Ai}A^{\alpha}_{M}q_{Ai}-H_A(t,\boldsymbol{q}_A,\boldsymbol{p}_A)]\mathrm{d}t=$$

$$\theta_A\int_{t_1}^{t_2}\Bigg[p_{Ai}A^{\alpha}_{M}(\xi_{Ai}-\dot{q}_{Ai}\xi_{A0})+\left(p_{Ai}\frac{\mathrm{d}}{\mathrm{d}t}A^{\alpha}_{M}q_{Ai}-\frac{\partial H_A}{\partial t}\right)\xi_{A0}-\frac{\partial H_A}{\partial q_{Ai}}\xi_{Ai}+(p_{Ai}A^{\alpha}_{M}q_{Ai}-H_A)\dot{\xi}_{A0}+$$

$$\lambda_{Aa}\frac{\partial\phi_{Aa}}{\partial p_{Ai}}\eta_{Ai}+\omega p_{Ai}q_{Ai}(t_2)\xi_{A0}(t_2)\frac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(t_2,t)-mp_{Ai}q_{Ai}(t_1)\xi_{A0}(t_1)\frac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(t,t_1)\Bigg]\mathrm{d}t, \tag{59}$$

其中

$$\xi_{A0}(t_1)=\xi_{A0}(t_1,\boldsymbol{q}_A(t_1),\boldsymbol{p}_A(t_1)),\ \xi_{A0}(t_2)=\xi_{A0}(t_2,\boldsymbol{q}_A(t_2),\boldsymbol{p}_A(t_2)),\ \bar{M}=\langle\bar{t}_1,\bar{t},\bar{t}_2,m,\omega\rangle.$$

算子A^{α}_{M}下约束 Hamilton 系统的 Noether 对称性是指 Hamilton 作用量在无限小变换(57)下的不变性,即要满足$\Delta S_A=0$,由式(59)得

$$p_{Ai}\cdot A^{\alpha}_{M}(\xi_{Ai}-\dot{q}_{Ai}\xi_{A0})+\left(p_{Ai}\frac{\mathrm{d}}{\mathrm{d}t}A^{\alpha}_{M}q_{Ai}-\frac{\partial H_A}{\partial t}\right)\xi_{A0}-\frac{\partial H_A}{\partial q_{Ai}}\xi_{Ai}+(p_{Ai}A^{\alpha}_{M}q_{Ai}-H_A)\dot{\xi}_{A0}+$$

$$\lambda_{Aa}\frac{\partial\phi_{Aa}}{\partial p_{Ai}}\eta_{Ai}+\omega p_{Ai}q_{Ai}(t_2)\xi_{A0}(t_2)\frac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(t_2,t)-mp_{Ai}q_{Ai}(t_1)\xi_{A0}(t_1)\frac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(t,t_1)=0. \tag{60}$$

式(60)称为算子A^{α}_{M}下约束 Hamilton 系统的 Noether 等式.

对于无限小变换(57),如果满足$\Delta S_A=-\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G_A)\mathrm{d}t$成立,其中$\Delta G_A=\theta_A G_A$, $G_A(t,\boldsymbol{q}_A,\boldsymbol{p}_A)$为算子$A^{\alpha}_{M}$下的规范函数,即

$$p_{Ai}A_M^\alpha(\xi_{Ai}-\dot q_{Ai}\xi_{A0})+\left(p_{Ai}\frac{\mathrm d}{\mathrm dt}A_M^\alpha q_{Ai}-\frac{\partial H_A}{\partial t}\right)\xi_{A0}-\frac{\partial H_A}{\partial q_{Ai}}\xi_{Ai}+(p_{Ai}A_M^\alpha q_{Ai}-H_A)\dot\xi_{A0}+$$

$$\lambda_{Aa}\frac{\partial\phi_{Aa}}{\partial p_{Ai}}\eta_{Ai}+\omega p_{Ai}q_{Ai}(t_2)\xi_{A0}(t_2)\frac{\mathrm d}{\mathrm dt}\kappa_{1-\alpha}(t_2,t)-mp_{Ai}q_{Ai}(t_1)\xi_{A0}(t_1)\frac{\mathrm d}{\mathrm dt}\kappa_{1-\alpha}(t,t_1)+\dot G_A=0. \quad (61)$$

则变换(57)称为算子A_M^α下约束 Hamilton 系统的 Noether 准对称变换.

定理 1　对于算子A_M^α下约束 Hamilton 系统(45), 若无限小生成元$\xi_{A0},\xi_{Ai},\eta_{Ai}$满足式(60), 则存在如下的守恒量:

$$I_A=(p_{Ai}A_M^\alpha q_{Ai}-H_A)\xi_{A0}+\int_{t_1}^t\{p_{Ai}A_M^\alpha(\xi_{Ai}-\dot q_{Ai}\xi_{A0})+$$

$$(\xi_{Ai}-\dot q_{Ai}\xi_{A0})[B_{M^*}^\alpha p_{Ai}-m\kappa_{1-\alpha}(t_2,\tau)p_{Ai}(t_2)+\omega\kappa_{1-\alpha}(\tau,t_1)p_{Ai}(t_1)]\}\mathrm d\tau+$$

$$\omega q_{Ai}(t_2)\xi_{A0}(t_2)\int_{t_1}^t p_{Ai}(\tau)\frac{\mathrm d}{\mathrm d\tau}\kappa_{1-\alpha}(t_2,\tau)\mathrm d\tau-mq_{Ai}(t_1)\xi_{A0}(t_1)\int_{t_1}^t p_{Ai}(\tau)\frac{\mathrm d}{\mathrm d\tau}\kappa_{1-\alpha}(\tau,t_1)\mathrm d\tau=\mathrm{const}. \quad (62)$$

证明　由式(26)、(45)、(60)可得$\mathrm dI_A/\mathrm dt=0$.

定理 2　对于算子A_M^α下约束 Hamilton 系统(45), 若存在一个规范函数G_A, 使无限小生成元$\xi_{A0},\xi_{Ai},\eta_{Ai}$满足式(61), 则存在如下守恒量:

$$I_{AG}=(p_{Ai}A_M^\alpha q_{Ai}-H_A)\xi_{A0}+\int_{t_1}^t\{p_{Ai}A_M^\alpha(\xi_{Ai}-\dot q_{Ai}\xi_{A0})+$$

$$(\xi_{Ai}-\dot q_{Ai}\xi_{A0})[B_{M^*}^\alpha p_{Ai}-m\kappa_{1-\alpha}(t_2,\tau)p_{Ai}(t_2)+\omega\kappa_{1-\alpha}(\tau,t_1)p_{Ai}(t_1)]\}\mathrm d\tau+$$

$$\omega q_{Ai}(t_2)\xi_{A0}(t_2)\int_{t_1}^t p_{Ai}(\tau)\frac{\mathrm d}{\mathrm d\tau}\kappa_{1-\alpha}(t_2,\tau)\mathrm d\tau-mq_{Ai}(t_1)\xi_{A0}(t_1)\int_{t_1}^t p_{Ai}(\tau)\frac{\mathrm d}{\mathrm d\tau}\kappa_{1-\alpha}(\tau,t_1)\mathrm d\tau+G_A=\mathrm{const}. \quad (63)$$

证明　由式(26)、(45)、(61)可得$\mathrm dI_{AG}/\mathrm dt=0$.

注 6　令$\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, 当$M=M_1$, $M=M_2$以及$M=M_3$时, 由式(60)、(61)和定理 1、定理 2 分别得到左 Riemann-Liouville 分数阶导数、右 Riemann-Liouville 分数阶导数、Riesz-Riemann-Liouville 分数阶导数下分数阶约束 Hamilton 系统的 Noether 对称性与 Noether 准对称性以及导致的守恒量.

4.2 算子B_M^α下约束 Hamilton 系统的 Noether 定理

考虑算子B_M^α的 Hamilton 作用量:

$$S_B=\int_{t_1}^{t_2}[p_{Bi}\cdot B_M^\alpha q_{Bi}-H_B(t,\boldsymbol q_B,\boldsymbol p_B)]\mathrm dt,\qquad i=1,2,\cdots,n. \quad (64)$$

引入无限小变换:

$$\bar t=t+\Delta t,\ \bar q_{Bi}=q_{Bi}(t)+\Delta q_{Bi},\ \bar p_{Bi}(\bar t)=p_{Bi}(t)+\Delta p_{Bi}, \quad (65)$$

其展开式为

$$\begin{cases}\bar t=t+\theta_B\xi_{B0}(t,\boldsymbol q_B,\boldsymbol p_B)+o(\theta_B),\ \bar q_{Bi}=q_{Bi}(t)+\theta_B\xi_{Bi}(t,\boldsymbol q_B,\boldsymbol p_B)+o(\theta_B),\\ \bar p_{Bi}(\bar t)=p_{Bi}(t)+\theta_B\eta_{Bi}(t,\boldsymbol q_B,\boldsymbol p_B)+o(\theta_B),\end{cases} \quad (66)$$

其中θ_B是无限小参数, $\xi_{B0},\xi_{Bi},\eta_{Bi}$为无限小生成元, $i=1,2,\cdots,n$.

$$\Delta S_B=\bar S_B-S_B=\int_{\bar t_1}^{\bar t_2}\left[\bar p_{Bi}B_{\bar M}^\alpha\bar q_{Bi}-H_B(\bar t,\bar{\boldsymbol q}_B,\bar{\boldsymbol p}_B)\right]\mathrm d\bar t-\int_{t_1}^{t_2}[p_{Bi}B_M^\alpha q_{Bi}-H_Bt,\boldsymbol q_B,\boldsymbol p_B]\mathrm dt=$$

$$\theta_B\int_{t_1}^{t_2}\Bigg[p_{Bi}\cdot B_M^\alpha(\xi_{Bi}-\dot q_{Bi}\xi_{B0})+\left(p_{Bi}\frac{\mathrm d}{\mathrm dt}B_M^\alpha q_{Bi}-\frac{\partial H_B}{\partial t}\right)\xi_{B0}-\frac{\partial H_B}{\partial q_{Bi}}\xi_{Bi}+(p_{Bi}B_M^\alpha q_{Bi}-H_B)\dot\xi_{B0}+$$

$$\lambda_{Ba}\frac{\partial\phi_{Ba}}{\partial p_{Bi}}\eta_{Bi}+\omega p_{Bi}\kappa_{1-\alpha}(t_2,t)\dot q_{Bi}(t_2)\xi_{B0}(t_2)-mp_{Bi}\kappa_{1-\alpha}(t,t_1)\dot q_{Bi}(t_1)\xi_{B0}(t_1)\Bigg]\mathrm dt, \quad (67)$$

其中

$$\xi_{B0}(t_1)=\xi_{B0}(t_1,\boldsymbol q_B(t_1),\boldsymbol p_B(t_1)),\ \xi_{B0}(t_2)=\xi_{B0}(t_2,\boldsymbol q_B(t_2),\boldsymbol p_B(t_2)),$$

$$\bar M=\langle\bar t_1,\bar t,\bar t_2,m,\omega\rangle.$$

算子B_M^α下约束 Hamilton 系统的 Noether 对称性是指 Hamilton 作用量在无限小变换(65)下的不变性, 即

要满足$\Delta S_B=0$, 由式(67)得

$$p_{Bi}B_M^\alpha(\xi_{Bi}-\dot q_{Bi}\xi_{B0})+\left(p_{Bi}\frac{\mathrm d}{\mathrm dt}B_M^\alpha q_{Bi}-\frac{\partial H_B}{\partial t}\right)\xi_{B0}-\frac{\partial H_B}{\partial q_{Bi}}\xi_{Bi}+(p_{Bi}B_M^\alpha q_{Bi}-H_B)\dot\xi_{B0}+ \lambda_{Ba}\frac{\partial\phi_{Ba}}{\partial p_{Bi}}\eta_{Bi}+\omega p_{Bi}\kappa_{1-\alpha}(t_2,t)\dot q_{Bi}(t_2)\xi_{B0}(t_2)-mp_{Bi}\kappa_{1-\alpha}(t,t_1)\dot q_{Bi}(t_1)\xi_{B0}(t_1)=0. \tag{68}$$

式(68)称为算子B_M^α下约束 Hamilton 系统的 Noether 等式.

对于无限小变换(65), 如果满足$\Delta S_B=-\int_{t_1}^{t_2}\frac{\mathrm d}{\mathrm dt}(\Delta G_B)\mathrm dt$成立, 其中$\Delta G_B=\theta_B G_B$, $G_B(t,\boldsymbol q_B,\boldsymbol p_B)$为算子$B_M^\alpha$下的规范函数, 即

$$p_{Bi}B_M^\alpha(\xi_{Bi}-\dot q_{Bi}\xi_{B0})+\left(p_{Bi}\frac{\mathrm d}{\mathrm dt}B_M^\alpha q_{Bi}-\frac{\partial H_B}{\partial t}\right)\xi_{B0}-\frac{\partial H_B}{\partial q_{Bi}}\xi_{Bi}+(p_{Bi}B_M^\alpha q_{Bi}-H_B)\dot\xi_{B0}+ \lambda_{Ba}\frac{\partial\phi_{Ba}}{\partial p_{Bi}}\eta_{Bi}+\omega p_{Bi}\kappa_{1-\alpha}(t_2,t)\dot q_{Bi}(t_2)\xi_{B0}(t_2)-mp_{Bi}\kappa_{1-\alpha}(t,t_1)\dot q_{Bi}(t_1)\xi_{B0}(t_1)+\dot G_B=0. \tag{69}$$

则变换(65)称为算子B_M^α下约束 Hamilton 系统的 Noether 准对称变换.

定理 3　对于算子B_M^α下约束 Hamilton 系统(51), 若无限小生成元$\xi_{B0}, \xi_{Bi}, \eta_{Bi}$满足式(68), 则存在如下的守恒量:

$$I_B=(p_{Bi}B_M^\alpha q_{Bi}-H_B)\xi_{B0}+\int_{t_1}^t[p_{Bi}B_M^\alpha(\xi_{Bi}-\dot q_{Bi}\xi_{B0})+(\xi_{Bi}-\dot q_{Bi}\xi_{B0})A_{M^*}^\alpha p_{Bi}]\mathrm d\tau+ \omega\dot q_{Bi}(t_2)\xi_{B0}(t_2)\int_{t_1}^t p_{Bi}(\tau)\kappa_{1-\alpha}(t_2,\tau)\mathrm d\tau- m\dot q_{Bi}(t_1)\xi_{B0}(t_1)\int_{t_1}^t p_{Bi}(\tau)\kappa_{1-\alpha}(\tau,t_1)\mathrm d\tau=\mathrm{const}. \tag{70}$$

证明　由式(40)、(51)、(68)可得$\mathrm dI_B/\mathrm dt=0$.

定理 4　对于算子B_M^α下约束 Hamilton 系统(51), 若存在一个规范函数G_B, 使无限小生成元$\xi_{B0}, \xi_{Bi}, \eta_{Bi}$满足式(69), 则存在如下守恒量:

$$I_{BG}=(p_{Bi}B_M^\alpha q_{Bi}-H_B)\xi_{B0}+\int_{t_1}^t[p_{Bi}B_M^\alpha(\xi_{Bi}-\dot q_{Bi}\xi_{B0})+(\xi_{Bi}-\dot q_{Bi}\xi_{B0})A_{M^*}^\alpha p_{Bi}]\mathrm d\tau+ \omega\dot q_{Bi}(t_2)\xi_{B0}(t_2)\int_{t_1}^t p_{Bi}(\tau)\kappa_{1-\alpha}(t_2,\tau)\mathrm d\tau- m\dot q_{Bi}(t_1)\xi_{B0}(t_1)\int_{t_1}^t p_{Bi}(\tau)\kappa_{1-\alpha}(\tau,t_1)\mathrm d\tau+G_B=\mathrm{const}. \tag{71}$$

证明　由式(40)、(51)、(69)可得$\mathrm dI_{BG}/\mathrm dt=0$.

注 7　令$\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, 当$M=M_1$, $M=M_2$以及$M=M_3$时, 由式(68)、(69)和定理 3、定理 4 分别得到左 Caputo 分数阶导数、右 Caputo 分数阶导数、Riesz-Caputo 分数阶导数下分数阶约束 Hamilton 系统的 Noether 对称性与 Noether 准对称性以及导致的守恒量.

5 算　　例

算例 1　考虑算子A_M^α下的奇异系统, 其 Lagrange 函数为

$$L_A=-\frac c2(q_{A1})^2+\frac b2(q_{A2})^2+\frac12(q_{A2}A_M^\alpha q_{A1}-q_{A1}A_M^\alpha q_{A2}), \tag{72}$$

其中b, c为正常数, 研究该系统的 Noether 对称性与守恒量.

由式(18)、(19)其广义动量和 Hamilton 量分别为

$$p_{A1}=\frac{\partial L_A}{\partial A_M^\alpha q_{A1}}=\frac12 q_{A2},\ p_{A2}=\frac{\partial L_A}{\partial A_M^\alpha q_{A2}}=-\frac12 q_{A1}, \tag{73}$$

$$H_A=p_{A1}A_M^\alpha q_{A1}+p_{A2}A_M^\alpha q_{A2}-L_A=\frac c2(q_{A1})^2-\frac b2(q_{A2})^2. \tag{74}$$

由$\det[H_{Aij}]=0$, 可得 Lagrange 函数是奇异的, 则由式(26)可得两个初级约束

$$\phi_{A1}=p_{A1}-\frac{1}{2}q_{A2}=0,\ \phi_{A2}=p_{A2}+\frac{1}{2}q_{A1}=0. \tag{75}$$

由相容性可得两个 Lagrange 乘子

$$\begin{cases}b\dot{q}_{A1}q_{A2}+\lambda_{A1}\dot{q}_{A1}+\dot{p}_{A2}A_M^\alpha q_{A1}-\dot{q}_{A1}[B_{M^*}^\alpha q_{A2}-m\kappa_{1-\alpha}(t_2,t)p_{A2}(t_2)+\omega p_{A2}(t_1)\kappa_{1-\alpha}(t,t_1)]=0,\\ cq_{A1}\dot{q}_{A2}+\lambda_{A2}\dot{q}_{A2}-\dot{p}_{A1}A_M^\alpha q_{A2}+\dot{q}_{A2}[B_{M^*}^\alpha q_{A1}-m\kappa_{1-\alpha}(t_2,t)p_{A1}(t_2)+\omega p_{A1}(t_1)\kappa_{1-\alpha}(t,t_1)]=0.\end{cases} \tag{76}$$

由式(45)、(76)得

$$\begin{cases}A_M^\alpha q_{A1}=-2bq_{A2}+2[B_{M^*}^\alpha p_{A2}-m\kappa_{1-\alpha}(t_2,t)p_{A2}(t_2)+\omega p_{A2}(t_1)\kappa_{1-\alpha}(t,t_1)],\\ A_M^\alpha q_{A2}=-2cq_{A1}-2[B_{M^*}^\alpha p_{A1}-m\kappa_{1-\alpha}(t_2,t)p_{A1}(t_2)+\omega p_{A1}(t_1)\kappa_{1-\alpha}(t,t_1)].\end{cases} \tag{77}$$

式(77)为该系统的运动微分方程.

由式(61)得

$$\begin{aligned}&p_{A1}A_M^\alpha(\xi_{A1}-\dot{q}_{A1}\xi_{A0})+p_{A2}A_M^\alpha(\xi_{A2}-\dot{q}_{A2}\xi_{A0})-cq_{A1}\xi_{A1}+bq_{A2}\xi_{A2}+\\&\quad\left(p_{A1}\frac{\mathrm{d}}{\mathrm{d}t}A_M^\alpha q_{A1}+p_{A2}\frac{\mathrm{d}}{\mathrm{d}t}A_M^\alpha q_{A2}\right)\xi_{A0}+\lambda_{A1}\eta_{A1}+\lambda_{A2}\eta_{A2}+(p_{A1}A_M^\alpha q_{A1}+p_{A2}A_M^\alpha q_{A2}-H_A)\dot{\xi}_{A0}+\\&\quad\omega p_{A1}q_{A1}(t_2)\cdot\xi_{A0}(t_2)\frac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(t_2,t)-mp_{A1}q_{A1}(t_1)\xi_{A0}(t_1)\frac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(t,t_1)+\\&\quad\omega p_{A2}q_{A2}(t_2)\xi_{A0}(t_2)\frac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(t_2,t)-mp_{A2}q_{A2}(t_1)\xi_{A0}(t_1)\frac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(t,t_1)+\dot{G}_A=0.\end{aligned} \tag{78}$$

式(78)有解为

$$\xi_{A0}=-1,\ \xi_{A1}=\xi_{A2}=0,\ \eta_{A1}=\eta_{A2}=0,\ G_A^0=0. \tag{79}$$

最后由定理 2 得到守恒量

$$\begin{aligned}I_A=&-\left(p_{A1}A_M^\alpha q_{A1}+p_{A2}A_M^\alpha q_{A2}-\frac{c}{2}q_{A1}^2+\frac{b}{2}q_{A2}^2\right)+\int_{t_1}^{t}\Big\{p_{A1}\frac{\mathrm{d}}{\mathrm{d}\tau}A_M^\alpha q_{A1}+p_{A2}\frac{\mathrm{d}}{\mathrm{d}\tau}A_M^\alpha q_{A2}+\\&\quad\dot{q}_{A1}[B_{M^*}^\alpha p_{A1}-m\kappa_{1-\alpha}(t_2,\tau)p_{A1}(t_2)+\omega\kappa_{1-\alpha}(\tau,t_1)p_{A1}(t_1)]+\\&\quad\dot{q}_{A2}[B_{M^*}^\alpha p_{A2}-m\kappa_{1-\alpha}(t_2,\tau)p_{A2}(t_2)+\omega\kappa_{1-\alpha}(\tau,t_1)p_{A2}(t_1)]\Big\}\mathrm{d}\tau=\mathrm{const}.\end{aligned} \tag{80}$$

特别地, 令$\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, 当$M=M_1(M=M_2$或$M=M_3)$, $\alpha\to1$时可得$I_A=H_A=\mathrm{const}$, 此时上述算例退化为在双方没有增员和自然损失情形下的常规战斗模型的 Noether 定理, 这与文献 [7] 中的结果一致.

6 结　论

分数阶微积分作为各个领域的重要工具, 能够更好地解决一些在整数阶导数下无法解决的问题, 同时奇异系统也一直备受关注, 如相对论运动粒子, 杨-Mills 场等都是由奇异 Lagrange 量所描述. 本文提出并证明了广义算子下约束 Hamilton 系统的 Noether 定理. 主要贡献如下:

1) 给出了广义算子下奇异 Lagrange 方程以及初级约束.

2) 建立了广义算子下约束 Hamilton 系统, 并由 Poisson 括号导出该系统的相容性条件.

3) 建立并证明了广义算子下约束 Hamilton 系统的 Noether 定理.

4) 若令$\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, 且当$M=M_1$, $M=M_2$以及$M=M_3$时, 可得到基于左(右)Riemann-Liouville 分数阶导数、左(右)Caputo 分数阶导数、Riesz-Riemann-Liouville 分数阶导数和 Riesz-Caputo 分数阶导数的分数阶约束 Hamilton 系统的对称性与守恒量. 当$\alpha\to1$时, 广义算子下的约束 Hamilton 方程退化为经典整数阶情况, 这与文献 [2] 中的结果一致.

广义算子下奇异系统还有很多问题值得研究, 如 Lie 对称性、Mei 对称性等. 此外, 时间尺度微积分提供了一种可以同时研究离散系统和连续系统的有效方法, 所以时间尺度上广义算子奇异系统的对称性与守恒量也是值得研究的.

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