应用数学和力学
APPLIED MATHEMATICS AND MECHANICS

拟凸函数的近似次微分及其在多目标优化问题中的应用

史小波，高 英

Properties of Quasiconvex Functions and Their Applications in Multiobjective Optimization Problems

SHI Xiaobo and GAO Ying

在线阅读 View online: https://doi.org/10.21656/1000-0887.420275 http://www.applmathmech.cn

拟凸函数的近似次微分及其在多目标优化问题中的应用*

史小波，　高　英

（重庆师范大学 数学科学学院，重庆 401331）

摘要： 针对拟凸函数提出一类新的近似次微分，研究其性质，并将近似次微分应用到拟凸多目标优化问题近似解的刻画中. 首先，对已有的近似次微分进行改进，得到拟凸函数新的近似次微分，并给出其与已有次微分之间的关系及一系列性质. 随后，利用新的近似次微分给出拟凸多目标优化问题近似有效解、近似真有效解的最优性条件.

关 键 词： 拟凸函数； 近似次微分； 多目标优化问题； 近似解； 最优性条件

中图分类号： O221.6　　**文献标志码：** A　　DOI：10.21656/1000-0887.420275

Properties of Quasiconvex Functions and Their Applications in Multiobjective Optimization Problems

SHI Xiaobo，　GAO Ying

（*School of Mathematical Sciences*, *Chongqing Normal University*, *Chongqing* 401331, *P.R.China*）

Abstract: A new type of approximate subdifferential was proposed for quasiconvex functions. Their properties were studied, and the approximate subdifferential was applied to the characterization of approximate solutions to quasiconvex multiobjective optimization problems. Firstly, the existing approximate subdifferentials were improved to get a new approximate subdifferential of the quasiconvex function, and their relationships and properties were given. Then, the optimality conditions for approximate efficient solutions and approximate properly efficient solutions to quasiconvex multiobjective optimization problems were obtained by means of the new approximate subdifferential.

Key words: quasiconvex function; approximate subdifferential; multiobjective optimization problem; approximate solution; optimality condition

引　言

凸性在优化问题中有着广泛的应用，但在实际问题的研究中，我们遇到的函数或者集合大多数是非凸的. 于是研究凸性的各种推广形式具有重要的意义. 20 世纪 60 年代以来，凸函数的概念已被推广到不同类型

* **收稿日期：** 2021-09-09; **修订日期：** 2021-09-23
基金项目： 国家自然科学基金 (11771064; 11991024); 重庆市科学技术研究重点项目 (KJZDK202001104); 重庆市高校创新研究群体项目 (CXQT20014); 重庆市留学人员回国创业创新支持计划 (cx2020096)
作者简介： 史小波 (1997—), 女, 硕士生 (E-mail: sxb792663@163.com);
高英 (1982—), 女, 教授, 博士, 硕士生导师 (通讯作者. E-mail: gaoying@cqnu.edu.cn).
引用格式： 史小波, 高英. 拟凸函数的近似次微分及其在多目标优化问题中的应用[J]. 应用数学和力学, 2022, **43**(3): 322-329.

的广义凸函数, 例如拟凸函数[1]、E-凸函数[2] 等. 拟凸函数作为一类特殊的广义凸函数, 在非凸优化中有着广泛的应用[3-4]. De Finetti 和 Fenchel 在文献 [5-6] 中首次给出了拟凸函数的定义. 1965年, Mangasarian[1] 首次引进伪凸的概念并给出拟凸函数和伪凸函数的若干性质. 随后, 杨新民等[7-9] 进一步给出拟凸函数的性质及其在优化理论中的应用.

对于凸函数来说, 次微分和近似次微分[10] 是给出凸优化问题最优性条件的基本工具. 因此, 如何引进拟凸函数的次微分是研究拟凸优化问题最优性条件的首要问题. 20 世纪 70 年代以来, 文献 [11-14] 先后引进了 Greenberg-Pierskalla 次微分、星型次微分、Gutiérrez 次微分、Plastria 次微分并给出了一些性质. 随后, Penot 在文献 [15-16] 中首次给出这四种次微分之间的关系并给出了一些推广性质. 有了这些次微分的概念及基本性质, 一些学者开始利用这些次微分给出拟凸数值优化问题的最优性条件[17-19]. 2019 年, 陈瑞婷等[20] 给出了拟凸意义下的近似次微分以及近似解的概念, 并给出了拟凸多目标优化问题近似解的最优性条件.

本文在文献 [10,20] 的基础上, 得到了拟凸函数新的近似次微分的概念, 研究其与已有次微分之间的关系及性质, 并将其应用到拟凸多目标优化问题中. 最后, 给出近似有效解、近似真有效解的概念, 利用新的近似次微分得到拟凸优化问题近似解的最优性条件.

1　预备知识

定义 1[10]　设$f: R^n \to \mathbf{R}$, 若对于任意的$\boldsymbol{x}_1, \boldsymbol{x}_2 \in R^n$和$\lambda \in [0,1]$, 有

$$f(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \leqslant \lambda f(\boldsymbol{x}_1) + (1-\lambda) f(\boldsymbol{x}_2),$$

则称f为R^n上的凸函数.

定义 2[9]　设$f: R^n \to \mathbf{R}$, 若对于任意的$\boldsymbol{x}_1, \boldsymbol{x}_2 \in R^n$和$\lambda \in [0,1]$, 有

$$f(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \leqslant \max\{f(\boldsymbol{x}_1), f(\boldsymbol{x}_2)\},$$

则称f为R^n上的拟凸函数.

给定$\lambda \in \mathbf{R}$, f对应于λ的水平集与严格水平集定义为

$$S_f^{\leqslant}(\lambda) = \{\boldsymbol{x} \in R^n \mid f(\boldsymbol{x}) \leqslant \lambda\},$$

$$S_f^{<}(\lambda) = \{\boldsymbol{x} \in R^n \mid f(\boldsymbol{x}) < \lambda\}.$$

针对拟凸函数, 文献 [11-14] 引进了如下次微分的概念.

定义 3[11-14]　设f为拟凸函数, $\boldsymbol{x}_0 \in R^n$, f在$\boldsymbol{x}_0$处的 Greenberg-Pierskalla 次微分定义为

$$\boldsymbol{x}_0^* \in \partial^* f(\boldsymbol{x}_0) \Longleftrightarrow \langle \boldsymbol{x}_0^*, \boldsymbol{x} - \boldsymbol{x}_0 \rangle < 0, \qquad \forall \boldsymbol{x} \in [f < f(\boldsymbol{x}_0)];$$

星型次微分定义为

$$\boldsymbol{x}_0^* \in \partial^{\circledast} f(\boldsymbol{x}_0) \Longleftrightarrow \langle \boldsymbol{x}_0^*, \boldsymbol{x} - \boldsymbol{x}_0 \rangle \leqslant 0, \qquad \forall \boldsymbol{x} \in [f < f(\boldsymbol{x}_0)];$$

Gutiérrez 次微分定义为

$$\boldsymbol{x}_0^* \in \partial^{\leqslant} f(\boldsymbol{x}_0) \Longleftrightarrow \langle \boldsymbol{x}_0^*, \boldsymbol{x} - \boldsymbol{x}_0 \rangle \leqslant f(\boldsymbol{x}) - f(\boldsymbol{x}_0), \qquad \forall \boldsymbol{x} \in [f \leqslant f(\boldsymbol{x}_0)];$$

Plastria 下次微分定义为

$$\boldsymbol{x}_0^* \in \partial^{<} f(\boldsymbol{x}_0) \Longleftrightarrow \langle \boldsymbol{x}_0^*, \boldsymbol{x} - \boldsymbol{x}_0 \rangle \leqslant f(\boldsymbol{x}) - f(\boldsymbol{x}_0), \qquad \forall \boldsymbol{x} \in [f < f(\boldsymbol{x}_0)].$$

注 1　2000年, Penot 在文献 [15] 中给出了四种次微分之间的关系:

$$\partial^{\leqslant} f(\boldsymbol{x}_0) \subseteq \partial^{<} f(\boldsymbol{x}_0) \subseteq \partial^* f(\boldsymbol{x}_0) \subseteq \partial^{\circledast} f(\boldsymbol{x}_0).$$

2019 年, 文献 [20] 针对拟凸函数给出了近似水平集和近似次微分的概念.

设$\varepsilon \geqslant 0$, 拟凸函数f的近似水平集与严格近似水平集可以表示为

$$[f \leqslant f(\boldsymbol{x}_0) - \varepsilon] = \{\boldsymbol{x} \in R^n : f(\boldsymbol{x}) \leqslant f(\boldsymbol{x}_0) - \varepsilon\},$$

$$[f < f(\boldsymbol{x}_0) - \varepsilon] = \{\boldsymbol{x} \in R^n : f(\boldsymbol{x}) < f(\boldsymbol{x}_0) - \varepsilon\}.$$

定义 4[20]　设f为拟凸函数, $\varepsilon \geqslant 0$, $\boldsymbol{x}_0 \in R^n$, f在$\boldsymbol{x}_0$处的几种近似次微分定义为

$$\boldsymbol{x}_0^* \in \partial_\varepsilon^* f(\boldsymbol{x}_0) \Longleftrightarrow \langle \boldsymbol{x}_0^*, \boldsymbol{x}-\boldsymbol{x}_0\rangle < \varepsilon, \qquad \forall \boldsymbol{x} \in [f < f(\boldsymbol{x}_0)-\varepsilon],$$
$$\boldsymbol{x}_0^* \in \partial_\varepsilon^{\circledast} f(\boldsymbol{x}_0) \Longleftrightarrow \langle \boldsymbol{x}_0^*, \boldsymbol{x}-\boldsymbol{x}_0\rangle \leqslant \varepsilon, \qquad \forall \boldsymbol{x} \in [f < f(\boldsymbol{x}_0)-\varepsilon],$$
$$\boldsymbol{x}_0^* \in \partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0) \Longleftrightarrow \langle \boldsymbol{x}_0^*, \boldsymbol{x}-\boldsymbol{x}_0\rangle \leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\varepsilon, \qquad \forall \boldsymbol{x} \in [f \leqslant f(\boldsymbol{x}_0)-\varepsilon],$$
$$\boldsymbol{x}_0^* \in \partial_\varepsilon^{<} f(\boldsymbol{x}_0) \Longleftrightarrow \langle \boldsymbol{x}_0^*, \boldsymbol{x}-\boldsymbol{x}_0\rangle \leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\varepsilon, \qquad \forall \boldsymbol{x} \in [f < f(\boldsymbol{x}_0)-\varepsilon].$$

注 2 文献 [20] 指出,当$\varepsilon \geqslant 0$时,以上四种近似次微分有如下关系:

$$\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0) \subseteq \partial_\varepsilon^{<} f(\boldsymbol{x}_0) \subseteq \partial_\varepsilon^{*} f(\boldsymbol{x}_0) \subseteq \partial_\varepsilon^{\circledast} f(\boldsymbol{x}_0).$$

次微分与对应的近似次微分的关系为

$$\partial^{\leqslant} f(\boldsymbol{x}_0) \subseteq \partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0),\ \partial^{<} f(\boldsymbol{x}_0) \subseteq \partial_\varepsilon^{<} f(\boldsymbol{x}_0),$$
$$\partial^{*} f(\boldsymbol{x}_0) \subseteq \partial_\varepsilon^{*} f(\boldsymbol{x}_0),\ \partial^{\circledast} f(\boldsymbol{x}_0) \subseteq \partial_\varepsilon^{\circledast} f(\boldsymbol{x}_0),$$

且$\varepsilon = 0$时,近似次微分退化为精确的次微分.

定义 5[10] 设函数$f: S \subseteq R^n \to \mathbf{R}$,对任意的$\boldsymbol{x} \in S$,存在一个正数$L,\delta$,使得

$$|f(\boldsymbol{x}_1)-f(\boldsymbol{x}_2)| \leqslant L\|\boldsymbol{x}_1-\boldsymbol{x}_2\|, \qquad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in B(\boldsymbol{x},\delta),$$

则称f为S上的局部 Lipschitz 函数.

定义 6[10] 设f为R^n上的局部 Lipschitz 函数,$f(\boldsymbol{x})$在点$\boldsymbol{x}$处关于方向$\boldsymbol{d} \in R^n$的广义方向导数定义为

$$f^o(\boldsymbol{x};\boldsymbol{d}) = \lim_{t \downarrow 0} \sup_{\boldsymbol{y} \to \boldsymbol{x}} \frac{f(\boldsymbol{y}+t\boldsymbol{d})-f(\boldsymbol{y})}{t},$$

f在点$\boldsymbol{x}$处的 Clarke 次微分定义为

$$\partial_{\mathrm{C}} f(\boldsymbol{x}) = \{\boldsymbol{\xi} \in R^n : f^o(\boldsymbol{x};\boldsymbol{v}) \geqslant \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{v}, \forall \boldsymbol{v} \in R^n\}.$$

若f为R^n上的凸函数,则 Clarke 次微分退化为凸意义下的次微分:

$$\partial f(\boldsymbol{x}_0) = \{\boldsymbol{\xi} \in R^n | \boldsymbol{\xi}^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{x}_0) \leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0), \forall \boldsymbol{x} \in R^n\}.$$

文献 [10] 针对凸函数给出了如下近似次微分的概念.

定义 7[10] 设f为R^n上的凸函数,$\varepsilon \geqslant 0$,f在点$\boldsymbol{x}_0$点的ε次微分定义为

$$\partial_\varepsilon f(\boldsymbol{x}_0) = \{\boldsymbol{\xi} \in R^n | \langle \boldsymbol{\xi}, \boldsymbol{x}-\boldsymbol{x}_0\rangle \leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\varepsilon, \forall \boldsymbol{x} \in R^n\}.$$

Clarke 次微分是针对局部 Lipschitz 连续函数给出的.显然,拟凸不一定连续.从而,也不一定是局部 Lipschitz 连续.下面给出一个例子说明拟凸连续函数也不一定是局部 Lipschitz 连续的.

例 1 设$f(x) = \sqrt{x}$, $x \in [0,+\infty)$,则$f(x)$在$[0,+\infty)$上为拟凸连续函数.

由定义显然有$f(x)$在$[0,+\infty)$上为拟凸连续函数.下面说明$f(x)$在$x=0$处不是局部 Lipschitz 连续的.事实上,对任意的$L>0,\delta>0$,特别取$x_n = \dfrac{1}{L^4 n^4}$, $y_n = \dfrac{1}{L^2 n^2}$,$n \in \mathbf{Z}_+$,当n充分大时,x_n, $y_n \in O(0,\delta) \bigcap [0,+\infty)$且

$$|f(x_n)-f(y_n)| > L|x_n-y_n|,$$

即

$$\left|\frac{f(x_n)-f(y_n)}{x_n-y_n}\right| = \left|\frac{1}{\sqrt{x_n}+\sqrt{y_n}}\right| = \frac{L^2 n^2}{Ln+1} \geqslant \left|\frac{Ln}{2}\right| > L.$$

从而$f(x)$不是局部 Lipschitz 连续的.

2 拟凸函数的近似次微分

本节针对拟凸函数$f: R^n \to \mathbf{R}$,在定义 4 的基础上,提出如下近似次微分的概念,并研究其性质.

定义 8 设$\varepsilon \geqslant 0$, $\boldsymbol{x}_0 \in R^n$, f在$\boldsymbol{x}_0$处的ε-下次微分定义为

$$\boldsymbol{x}_0^* \in \partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0) \Longleftrightarrow \langle \boldsymbol{x}_0^*, \boldsymbol{x}-\boldsymbol{x}_0\rangle \leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\varepsilon, \qquad \forall \boldsymbol{x} \in [f \leqslant f(\boldsymbol{x}_0)].$$

由定义可知,定义 4 中的近似次微分包含于定义 8 中的近似次微分,但反包含不一定成立,见例 2.

例 2 考虑拟凸函数

$$f(x)=\begin{cases}0, & x<0,\\ 1, & x=0,\\ -1, & x>0.\end{cases}$$

取$x_0=0,\ \varepsilon=2$, 则定义 4 中的近似次微分$\partial_\varepsilon^{\leqslant} f(x_0)=(-\infty,0]$, 而定义8中的近似次微分$\partial_\varepsilon^{\leqslant} f(x_0)=\{0\}$.

在实际应用中, 存在一些拟凸函数, 其精确的次微分可能为空集, 而改进后的近似次微分不一定非空, 见例 3.

例 3　考虑拟凸函数

$$f(x)=\begin{cases}x, & x\leqslant -2,\\ 1, & -2<x<0,\\ 2, & x\geqslant 0,\end{cases}$$

取$x_0=0,\ \varepsilon=2$,

$$\partial^{<} f(x_0)=\partial^{\leqslant} f(x_0)=\varnothing,\ \partial_\varepsilon^{\leqslant} f(x_0)=[1,+\infty).$$

而对于某些拟凸函数来说, 定义 4 中的近似水平集有可能为空集, 而本文定义的水平集不一定非空, 见例 4.

例 4　考虑拟凸函数

$$f(x)=\begin{cases}-x, & x\leqslant -2,\\ 1, & -2<x<0,\\ \dfrac{3}{2}, & x\geqslant 0.\end{cases}$$

取$x_0=0,\ \varepsilon=2$, 则定义 4 中的近似水平集$x\in\varnothing$,而定义8中的水平集$x\in(-2,+\infty)$, 此时, $\partial_\varepsilon^{\leqslant} f(x_0)=\left[-\dfrac{3}{4},0\right]$.

注 3　当$f(x)$为局部 Lipschitz 连续时, Clarke 次微分不一定包含在$\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0)$, 见例 5.

例 5　设$f(x)=x^3,\ x\in\mathbf{R}$, 则$f(x)$为拟凸局部 Lipschitz 连续函数. 特别取$x_0=0,\ \varepsilon=1$, 则$\partial_{\mathrm{C}} f(x_0)=\{0\}$, $\partial_\varepsilon^{\leqslant} f(x_0)=\varnothing$.

下面, 我们研究了近似次微分的一些性质.

定理 1　设$\varepsilon\geqslant 0,\ \boldsymbol{x}_0\in R^n$, 则$\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0)$为闭凸集.

证明　首先证明闭性, 任取$\{\boldsymbol{x}_n^*\}\subset\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0)$, 若$\lim_{n\to\infty}\boldsymbol{x}_n^*=\boldsymbol{x}^*$, 下面证明$\boldsymbol{x}^*\in\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0)$.

由$\boldsymbol{x}_n^*\in\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0)$可知, 对任意的$\boldsymbol{x}\in[f\leqslant f(\boldsymbol{x}_0)]$, 有

$$\langle \boldsymbol{x}_n^*,\boldsymbol{x}-\boldsymbol{x}_0\rangle\leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\varepsilon.$$

又由$\lim_{n\to\infty}\boldsymbol{x}_n^*=\boldsymbol{x}^*$可知

$$\langle \boldsymbol{x}^*,\boldsymbol{x}-\boldsymbol{x}_0\rangle=\langle \lim_{n\to\infty}\boldsymbol{x}_n^*,\boldsymbol{x}-\boldsymbol{x}_0\rangle\leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\varepsilon.$$

这表明$\boldsymbol{x}^*\in\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0)$, 从而$\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0)$是闭集.

下面再证明凸性. 任取$\boldsymbol{\xi}_1,\ \boldsymbol{\xi}_2\in\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0)$, 由定义可知

$$\langle \boldsymbol{\xi}_1,\boldsymbol{x}-\boldsymbol{x}_0\rangle\leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\varepsilon,$$
$$\langle \boldsymbol{\xi}_2,\boldsymbol{x}-\boldsymbol{x}_0\rangle\leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\varepsilon.$$

任取$\lambda\in[0,1]$, 有

$$\lambda\langle \boldsymbol{\xi}_1,\boldsymbol{x}-\boldsymbol{x}_0\rangle\leqslant \lambda(f(\boldsymbol{x})-f(\boldsymbol{x}_0))+\lambda\varepsilon,$$
$$(1-\lambda)\langle \boldsymbol{\xi}_2,\boldsymbol{x}-\boldsymbol{x}_0\rangle\leqslant (1-\lambda)(f(\boldsymbol{x})-f(\boldsymbol{x}_0))+(1-\lambda)\varepsilon.$$

将上式相加得

$$\langle \lambda\boldsymbol{\xi}_1+(1-\lambda)\boldsymbol{\xi}_2,\boldsymbol{x}-\boldsymbol{x}_0\rangle\leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\varepsilon.$$

即

$$\lambda\boldsymbol{\xi}_1+(1-\lambda)\boldsymbol{\xi}_2\in\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}),$$

故$\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0)$为凸集.

综上所述, $\partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0)$为闭凸集.

定理 2　设$\boldsymbol{x}_0\in R^n,\ 0<\varepsilon_1<\varepsilon_2$, 则$\partial_{\varepsilon_1}^{\leqslant} f(\boldsymbol{x}_0)\subset\partial_{\varepsilon_2}^{\leqslant} f(\boldsymbol{x}_0)$.

证明 任取$\boldsymbol{\xi} \in \partial_{\varepsilon_1}^{\leqslant} f(\boldsymbol{x}_0)$, 对于任意的$\boldsymbol{x} \in [f \leqslant f(\boldsymbol{x})]$, 有

$$\langle \boldsymbol{\xi}, \boldsymbol{x}-\boldsymbol{x}_0 \rangle \leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\varepsilon_1.$$

由$\varepsilon_1 < \varepsilon_2$可知

$$\langle \boldsymbol{\xi}, \boldsymbol{x}-\boldsymbol{x}_0 \rangle \leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\varepsilon_2.$$

这表明$\boldsymbol{\xi} \in \partial_{\varepsilon_2}^{\leqslant} f(\boldsymbol{x}_0)$. 再由$\boldsymbol{\xi}$的任意性可知

$$\partial_{\varepsilon_1}^{\leqslant} f(\boldsymbol{x}_0) \subset \partial_{\varepsilon_2}^{\leqslant} f(\boldsymbol{x}_0).$$

定理 3 设$\boldsymbol{x}_0 \in R^n$, 对任意的$\varepsilon > 0$, 有

$$\partial^{\leqslant} f(\boldsymbol{x}_0) = \bigcap_{\varepsilon>0} \partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0).$$

证明 首先证明$\bigcap_{\varepsilon>0} \partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0) = \bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0)$, 其中$n \in \mathbf{Z}_+$. 由定义显然有$\bigcap_{\varepsilon>0} \partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0) \subset \bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0)$.

下面证明$\bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0) \subset \bigcap_{\varepsilon>0} \partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0)$. 任取$\varepsilon > 0$和$\boldsymbol{\xi} \in \bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0)$, 则存在$N \in \mathbf{Z}_+$, 当$n > N$时有$\frac{1}{n} < \varepsilon$. 由定理 2 可知$\partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0) \subset \partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0)$, 从而$\bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0) \subset \bigcap_{\varepsilon>0} \partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0)$. 这表明

$$\bigcap_{\varepsilon>0} \partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0) = \bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0).$$

现只需证$\partial^{\leqslant} f(\boldsymbol{x}_0) = \bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0)$. 由定义显然有$\partial^{\leqslant} f(\boldsymbol{x}_0) \subset \bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0)$.

下面证明$\bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0) \subset \partial^{\leqslant} f(\boldsymbol{x}_0)$. 由$\partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0)$的单调性可知, $\left\{\partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x})\right\}$是单调递减的, 故

$$\lim_{n\to\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}) = \bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}).$$

对任意的$\boldsymbol{\xi} \in \bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0)$, 有

$$\langle \boldsymbol{\xi}, \boldsymbol{x}-\boldsymbol{x}_0 \rangle \leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0)+\frac{1}{n}, \qquad \forall n \in \mathbf{N}_+,\ \boldsymbol{x} \in [f \leqslant f(\boldsymbol{x}_0)].$$

令$n \to +\infty$, 则有

$$\langle \boldsymbol{\xi}, \boldsymbol{x}-\boldsymbol{x}_0 \rangle \leqslant f(\boldsymbol{x})-f(\boldsymbol{x}_0), \qquad \forall\, \boldsymbol{x} \in [f \leqslant f(x_0)].$$

从而$\boldsymbol{\xi} \in \partial^{\leqslant} f(\boldsymbol{x}_0)$. 再由$\boldsymbol{\xi}$的任意性可知$\bigcap_{n=1}^{\infty} \partial_{\frac{1}{n}}^{\leqslant} f(\boldsymbol{x}_0) \subset \partial^{\leqslant} f(\boldsymbol{x}_0)$, 故

$$\partial^{\leqslant} f(\boldsymbol{x}_0) = \bigcap_{\varepsilon>0} \partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0).$$ □

与凸意义下的近似次微分类似, 下述结论成立.

定理 4 设$\boldsymbol{x}_0 \in R^n$,则有

(ⅰ) $\partial_{\varepsilon}^{\leqslant} g(\boldsymbol{x}_0) = \partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0)$, 其中$g(\boldsymbol{x}) = f(\boldsymbol{x})+C$, C为常数;

(ⅱ) $\partial_{\varepsilon}^{\leqslant} g(\boldsymbol{x}_0) = a\partial_{\frac{\varepsilon}{a}}^{\leqslant} f(\boldsymbol{x}_0)$, 其中$g(\boldsymbol{x}) = af(\boldsymbol{x})$, $a > 0$;

(ⅲ) $\partial_{\varepsilon}^{\leqslant} g(\boldsymbol{x}+\boldsymbol{x}_0) = \partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0)$, 其中$g(\boldsymbol{x}) = f(\boldsymbol{x}-\boldsymbol{x}_0)$, $f(\boldsymbol{x})$为单调函数;

(ⅳ) $\partial_{\varepsilon}^{\leqslant} f(\boldsymbol{x}_0)+\{\alpha\} \subset \partial_{\varepsilon}^{\leqslant} g(\boldsymbol{x}_0)$, 其中$g(\boldsymbol{x}) = f(\boldsymbol{x})+\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{x}$, $\boldsymbol{\alpha} \in R_+^n$, $g(\boldsymbol{x})$为单调递增函数.

定理 5 设$\varepsilon_1, \varepsilon_2 \geqslant 0$, $\varepsilon_1+\varepsilon_2 \leqslant \varepsilon$, 对任意$\boldsymbol{x}_0 \in R^n$, 若$L_{f_1(\boldsymbol{x}_0)} = L_{f_2(\boldsymbol{x}_0)}$, 有

$$\left\{\partial_{\varepsilon_1}^{\leqslant} f_1(\boldsymbol{x}_0)+\partial_{\varepsilon_2}^{\leqslant} f_2(\boldsymbol{x}_0)\right\} \subset \partial_{\varepsilon}^{\leqslant}(f_1(\boldsymbol{x}_0)+f_2(\boldsymbol{x}_0)).$$

证明 任取$\boldsymbol{\xi}_1 \in \partial_{\varepsilon_1}^{\leqslant} f_1(\boldsymbol{x}_0)$, $\boldsymbol{\xi}_2 \in \partial_{\varepsilon_2}^{\leqslant} f_2(\boldsymbol{x}_0)$, 由定义可知, 对任意的$\boldsymbol{x} \in L_{f_1(\boldsymbol{x}_0)}$, 有

$$\langle \boldsymbol{\xi}_1, \boldsymbol{x}-\boldsymbol{x}_0 \rangle \leqslant f_1(\boldsymbol{x})-f_1(\boldsymbol{x}_0)+\varepsilon_1,$$
$$\langle \boldsymbol{\xi}_2, \boldsymbol{x}-\boldsymbol{x}_0 \rangle \leqslant f_2(\boldsymbol{x})-f_2(\boldsymbol{x}_0)+\varepsilon_2.$$

两式相加得

$$\langle \boldsymbol{\xi}_1+\boldsymbol{\xi}_2\,,\boldsymbol{x}-\boldsymbol{x}_0\rangle \leqslant f_1(\boldsymbol{x})+f_2(\boldsymbol{x})-(f_1(\boldsymbol{x}_0)+f_2(\boldsymbol{x}_0))+\varepsilon_1+\varepsilon_2.$$

令$\boldsymbol{\xi}=\boldsymbol{\xi}_1+\boldsymbol{\xi}_2$, 有

$$\begin{aligned}\langle \boldsymbol{\xi},\boldsymbol{x}-\boldsymbol{x}_0\rangle &= f_1(\boldsymbol{x})+f_2(\boldsymbol{x})-(f_1(\boldsymbol{x}_0)+f_2(\boldsymbol{x}_0))+\varepsilon_1+\varepsilon_2 \leqslant \\ &\quad f_1(\boldsymbol{x})+f_2(\boldsymbol{x})-(f_1(\boldsymbol{x}_0)+f_2(\boldsymbol{x}_0))+\varepsilon.\end{aligned}$$

从而$\boldsymbol{\xi}\in\partial_\varepsilon^{\leqslant}(f_1(\boldsymbol{x}_0)+f_2(\boldsymbol{x}_0))$, 即包含关系成立.

注 4　定理中$L_{f_1(\boldsymbol{x}_0)}=L_{f_2(\boldsymbol{x}_0)}$条件必不可少.

例 6　考虑函数

$$f_1(x)=x,\ f_2(x)=x^2,$$

取$x_0=0,\ \varepsilon=1$, 则$L_{f_1(x_0)}=(-\infty,0],\ L_{f_2(x_0)}=\{0\},\ L_{f_1+f_2}(x_0)=[-1,0]$.

由定义有

$$\partial_\varepsilon^{\leqslant} f_1(x_0)=[1,+\infty),\ \partial_\varepsilon^{\leqslant} f_2(x_0)=\mathbf{R},$$
$$\{\partial_\varepsilon^{\leqslant} f_1(x_0)+\partial_\varepsilon^{\leqslant} f_2(x_0)\}=\mathbf{R},$$

然而

$$\partial_\varepsilon^{\leqslant}(f_1(x_0)+f_2(x_0))=[-1,+\infty).$$

由此可见$\mathbf{R}\not\subset[-1,+\infty)$. 因此该条件必不可少.

3　拟凸多目标优化问题近似解的最优性条件

本节考虑如下的多目标优化问题 (MOP):

$$\min\ \boldsymbol{f}(\boldsymbol{x}),\qquad \text{s.t.}\quad \boldsymbol{x}\in S,$$

其中, $S\subset R^n$为凸集, $\boldsymbol{f}(\boldsymbol{x})=(f_1(\boldsymbol{x}),f_2(\boldsymbol{x}),\cdots,f_m(\boldsymbol{x}))^{\mathrm T}$, $f_i:S\to\mathbf{R}, i=1,2,\cdots,m$为拟凸函数, 利用第 2 节中的近似次微分给出 MOP 的近似解的最优性条件.

定义 9[18]　设$\varepsilon\geqslant 0$, $\boldsymbol{x}_0\in S$, 则S在$\boldsymbol{x}_0$处的法锥定义为

$$N(S,\boldsymbol{x}_0)=\{\boldsymbol{x}_0^*\in R^n:\langle \boldsymbol{x}_0^*,\boldsymbol{x}-\boldsymbol{x}_0\rangle\leqslant 0,\forall \boldsymbol{x}\in S\}.$$

S在$\boldsymbol{x}_0$处的ε-法锥定义为

$$N_\varepsilon(S,\boldsymbol{x}_0)=\{\boldsymbol{x}_0^*\in R^n:\langle \boldsymbol{x}_0^*,\boldsymbol{x}-\boldsymbol{x}_0\rangle\leqslant \varepsilon,\forall \boldsymbol{x}\in S\}.$$

定义 10[20]　设$\boldsymbol{\varepsilon}=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m)^{\mathrm T}\geqslant 0$, $\boldsymbol{x}_0\in S$,

(ⅰ) 若不存在$\boldsymbol{x}\in S$, 使得$f_i(\boldsymbol{x})<f_i(\boldsymbol{x}_0)-\varepsilon_i,\forall i\in\{1,2,\cdots,m\}$, 则称$\boldsymbol{x}_0$为 MOP 的$\boldsymbol{\varepsilon}$-弱有效解.

(ⅱ) 若不存在$\boldsymbol{x}\in S$, 使得

$$\begin{aligned}&f_i(\boldsymbol{x})\leqslant f_i(\boldsymbol{x}_0)-\varepsilon_i, &&\forall i\in\{1,2,\cdots,m\},\\ &f_j(\boldsymbol{x})< f_j(\boldsymbol{x}_0)-\varepsilon_j, &&\exists j\in\{1,2,\cdots,m\},\end{aligned}$$

则称$\boldsymbol{x}_0$为 MOP 的$\boldsymbol{\varepsilon}$-有效解.

定义 11[21]　设$\boldsymbol{\varepsilon}=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m)^{\mathrm T}\geqslant \mathbf{0}$, $\boldsymbol{x}_0\in S$, 若

(ⅰ) $\boldsymbol{x}_0$是 MOP 的$\boldsymbol{\varepsilon}$-有效解,

(ⅱ) 存在常数$M>0$, 使得对任意的$i\in\{1,2,\cdots,m\}$和$\boldsymbol{x}\in S$, 满足$f_i(\boldsymbol{x})<f_i(\boldsymbol{x}_0)-\varepsilon_i$, 存在$j\in\{1,2,\cdots,m\}\backslash\{i\}$满足$f_j(\boldsymbol{x}_0)<f_j(\boldsymbol{x})+\varepsilon_j$, 且

$$\frac{f_i(\boldsymbol{x}_0)-f_i(\boldsymbol{x})-\varepsilon_i}{f_j(\boldsymbol{x})-f_j(\boldsymbol{x}_0)+\varepsilon_j}\leqslant M,$$

则称$\boldsymbol{x}_0$为 MOP 的$\boldsymbol{\varepsilon}$-真有效解.

与文献 [20] 中定理2.2类似, 容易得出以下结论.

定理 6　设$\boldsymbol{x}_0\in S$, $\boldsymbol{\varepsilon}=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m)^{\mathrm T}\geqslant 0$. $\lambda_i\geqslant 0$, $i\in\{1,2,\cdots,m\}$, $\sum_{i=1}^m\lambda_i\varepsilon_i=\epsilon$, 若存在$\boldsymbol{x}_0^*\in\sum_{i=1}^m\lambda_i\partial_{\varepsilon_i}^{\leqslant}f_i(\boldsymbol{x}_0)\bigcap(-N_\epsilon(S,\boldsymbol{x}_0))$, 满足$\langle \boldsymbol{x}_0^*,\boldsymbol{x}-\boldsymbol{x}_0\rangle\notin[-\epsilon,\epsilon]$, $\forall \boldsymbol{x}\in S\backslash\{\boldsymbol{x}_0\}$, 则$\boldsymbol{x}_0$是 MOP 的$\boldsymbol{\varepsilon}$-有效解.

注意到, 文献 [20] 并没有给出近似真有效解的最优性条件, 下面利用近似次微分给出 MOP 近似真有效解的最优性条件.

定理 7 设$\boldsymbol{x}_0 \in S$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m)^{\mathrm{T}} \geqslant \boldsymbol{0}$. $\lambda_i > 0, i \in \{1,2,\cdots,m\}$, $\sum_{i=1}^m \lambda_i \varepsilon_i = \epsilon$, 若存在$\boldsymbol{x}_0^* \in \sum_{i=1}^m \lambda_i \partial_{\varepsilon_i}^{\leqslant} f_i(\boldsymbol{x}_0) \bigcap (-N_\epsilon(S, \boldsymbol{x}_0) \backslash \{\boldsymbol{0}\})$, 满足$\langle \boldsymbol{x}_0^*, \boldsymbol{x} - \boldsymbol{x}_0 \rangle \notin (-\epsilon, \epsilon), \forall \boldsymbol{x} \in S \backslash \{\boldsymbol{x}_0\}$, 则$\boldsymbol{x}_0$是 MOP 的$\boldsymbol{\varepsilon}$-真有效解.

证明 首先, 由定理6得出$\boldsymbol{x}_0$为 MOP 的$\boldsymbol{\varepsilon}$-有效解, 下面证明$\boldsymbol{x}_0$是 MOP 的$\boldsymbol{\varepsilon}$-真有效解.

反证, 假设$\boldsymbol{x}_0$不是$\boldsymbol{\varepsilon}$-真有效解, 则对任意的$M > 0$, 存在满足$f_i(\boldsymbol{x}) < f_i(\boldsymbol{x}_0) - \varepsilon_i$的$\boldsymbol{x} \in S \backslash \{\boldsymbol{x}_0\}, i \in \{1,2,\cdots,m\}$使得对任意满足$f_j(\boldsymbol{x}_0) < f_j(\boldsymbol{x}) + \varepsilon_j$的$j \in \{1,2,\cdots,m\} \backslash \{i\}$有

$$\frac{f_i(\boldsymbol{x}_0) - f_i(\boldsymbol{x}) - \varepsilon_i}{f_j(\boldsymbol{x}) - f_j(\boldsymbol{x}_0) + \varepsilon_j} > M.$$

令$M = (m-1)\dfrac{\lambda_j}{\lambda_i}$, 有

$$f_i(\boldsymbol{x}_0) - f_i(\boldsymbol{x}) - \varepsilon_i > (m-1)\frac{\lambda_j}{\lambda_i}(f_j(\boldsymbol{x}_0) - f_j(\boldsymbol{x}) + \varepsilon_j).$$

由$\lambda_i > 0$得

$$\lambda_i(f_i(\boldsymbol{x}_0) - f_i(\boldsymbol{x}) - \varepsilon_i) > (m-1)\lambda_j(f_j(\boldsymbol{x}_0) - f_j(\boldsymbol{x}) + \varepsilon_j).$$

整理得

$$\lambda_i f_i(\boldsymbol{x}_0) + (m-1)\lambda_j f_j(\boldsymbol{x}_0) > \lambda_i f_i(\boldsymbol{x}) + (m-1)\lambda_j f_j(\boldsymbol{x}) + \lambda_i \varepsilon_i + (m-1)\lambda_j \varepsilon_j.$$

即

$$\sum_{i=1}^m \lambda_i f_i(\boldsymbol{x}_0) > \sum_{i=1}^m \lambda_i f_i(\boldsymbol{x}) + \sum_{i=1}^m \lambda_i \varepsilon_i.$$

从而

$$\sum_{i=1}^m \lambda_i (f_i(\boldsymbol{x}) - f_i(\boldsymbol{x}_0)) + \epsilon < 0.$$

取$\boldsymbol{x}_0^* = \sum_{i=1}^m \lambda_i z_i \in (-N_\epsilon(S, \boldsymbol{x}_0) \backslash \{\boldsymbol{0}\})$. 其中, $z_i \in \partial_{\varepsilon_i}^{\leqslant} f_i(\boldsymbol{x}_0), i \in \{1,2,\cdots,m\}$, 则

$$\langle \boldsymbol{x}_0^*, \boldsymbol{x} - \boldsymbol{x}_0 \rangle = \left\langle \sum_{i=1}^m \lambda_i z_i, \boldsymbol{x} - \boldsymbol{x}_0 \right\rangle = \sum_{i=1}^m \lambda_i \langle z_i, \boldsymbol{x} - \boldsymbol{x}_0 \rangle \leqslant \sum_{i=1}^m \lambda_i (f_i(\boldsymbol{x}) - f_i(\boldsymbol{x}_0) + \varepsilon_i) < 0 < \epsilon,$$

即

$$\langle \boldsymbol{x}_0^*, \boldsymbol{x} - \boldsymbol{x}_0 \rangle < \epsilon,$$

又由于$\boldsymbol{x}_0^* \in -N_\epsilon(S, \boldsymbol{x}_0)$, 从而有

$$\langle \boldsymbol{x}_0^*, \boldsymbol{x} - \boldsymbol{x}_0 \rangle > -\epsilon.$$

这与$\langle \boldsymbol{x}_0^*, \boldsymbol{x} - \boldsymbol{x}_0 \rangle \notin (-\epsilon, \epsilon)$, $\forall \boldsymbol{x} \in S \backslash \{\boldsymbol{x}_0\}$相矛盾, 故$\boldsymbol{x}_0$是 MOP 的$\varepsilon$-真有效解. □

推论 1 设 MOP 中$m = 1$, $S = R^n$, $\varepsilon \geqslant 0$, 若$\boldsymbol{0} \in \partial_\varepsilon^{\leqslant} f(\boldsymbol{x}_0)$, 则$\boldsymbol{x}_0$为$\min_{\boldsymbol{x} \in R^n} f(\boldsymbol{x})$的$\varepsilon$-最优解.

4 结　　论

本文在已有拟凸函数的近似次微分基础上, 对次微分定义进行了改进, 给出了其与已有次微分之间的关系及一系列性质, 如凸性闭性等, 并通过实例说明其合理性. 随后, 利用新的近似次微分给出拟凸单目标优化问题的近似最优解, 拟凸多目标优化问题的近似有效解、近似真有效解的刻画.

参考文献(References):

[1] MANGASARIAN O L. Pseudo function[J]. *Journal of the Society for Industrial and Applied Mathematics*, 1965, **3**(2): 23-32.

[2] DUCA D I, LUPA L. On the *E*-epigraph of an *E*-convex function[J]. *Journal of Optimization Theory and Ap-*

plications, 2006, **129**(2): 341-348.

[3] 王海英, 符祖峰. D-η-E-半预不变凸映射和D-η-E-半预不变真拟凸映射[J]. 应用数学和力学, 2019, **40**(3): 111-122. (WANG Haiying, FU Zufeng. D-η-E-semi-preinvex mapping and D-η-E-properly semi-prequasi-invex mapping[J]. *Applied Mathematics and Mechanics*, 2019, **40**(3): 111-122.(in Chinese))

[4] 刘娟, 龙宪军. 非光滑多目标半无限规划问题的混合型对偶[J]. 应用数学和力学, 2021, **42**(6): 595-601. (LIU Juan, LONG Xianjun. Mixed type duality for nonsmooth multiobjective semi-infinite programming problems[J]. *Applied Mathematics and Mechanics*, 2021, **42**(6): 595-601.(in Chinese))

[5] DE FINETTI B. Sulle stratificazioni convesse[J]. *Annali di Matematica Pura ed Applicata*, 1949, **30**(1): 173-183.

[6] FENCHEL W. *Convex Cones, Sets, and Functions*[M]. Princeton University, 1953.

[7] 杨新民. 拟凸函数的某些性质[J]. 工程数学学报, 1993, **10**(1): 51-56. (YANG Xinmin. Some properties of quasiconvex functions[J]. *Journal of Engineering Mathematics*, 1993, **10**(1): 51-56.(in Chinese))

[8] YANG X M, LIU S Y. Technical note three kind of generalized convexity[J]. *Journal of Optimization Theory and Applications*, 1995, **86**(2): 501-513.

[9] 杨新民, 戎卫东. 广义凸性及其应用[M]. 北京: 科学出版社, 2016. (YANG Xinmin, RONG Weidong. *Generalized Convexity and Its Application*[M]. Beijing: Science Press, 2016. (in Chinese))

[10] 高岩. 非光滑优化[M]. 北京: 科学出版社, 2008. (GAO Yan. *Nonsmooth Optimization*[M]. Beijing: Science Press, 2008. (in Chinese))

[11] GREENBERG H P, PIERSKALLA W P. Quasi-conjugate functions and surrogate duality[J]. *Cahiers du Centre d'Etude de Recherche Operationelle*, 1973, **15**: 437-448.

[12] PENOT J P, ZALINESCU C. Elements of quasiconvex subdifferential calculus[J]. *Journal of Convex Analysis*, 2000, **7**(7): 243-269.

[13] GUTIÉRREZ D J M. Infragradients and directions of decrease[J]. *Rev Real Acad Cienc Exact Fís Natur Madrid*, 1984, **78**(4): 523-532.

[14] PLASRTIA F. Lower subdifferentiable functions and their minimization by cutting planes[J]. *Journal of Optimization Theory and Applications*, 1985, **46**(1): 37-53.

[15] PENOT J P. What is quasiconvex analysis?[J]. *Optimization*, 2000, **47**(1/2): 35-110.

[16] PENOT J P. Characterization of solution sets of quasiconvex programs[J]. *Journal of Optimization Theory and Applications*, 2003, **117**(3): 627-636.

[17] NGUYEN T H L, PENOT J P. Optimality conditions for quasiconvex programs[J]. *SIAM Journal on Optimization*, 2006, **17**(2): 500-510.

[18] SUZUKI S, KUROIWA D. Optimality conditions and the basic constraint qualification for quasiconvex programming[J]. *Nonlinear Analysis Theory Methods and Applications*, 2011, **74**(4): 1279-1285.

[19] KHANH P Q, QUYEN H T, YAO J C. Optimality conditions under relaxed quasiconvexity assumptions using star and adjusted subdifferentials[J]. *European Journal of Operational Research*, 2011, **212**(2): 235-241.

[20] 陈瑞婷, 徐智会, 高英. 拟凸多目标优化问题近似解的最优性条件[J]. 运筹学学报, 2019, **23**(1): 35-44. (CHEN Ruiting, XU Zhihui, GAO Ying. The optimality conditions of approximate solutions for quasiconvex multiobjective optimization problem[J]. *Operations Research Transactions*, 2019, **23**(1): 35-44.(in Chinese))

[21] 岳瑞雪, 高英. 多目标优化问题$(\varepsilon, \bar{\varepsilon})$-拟近似真有效解的非线性标量化[J]. 数学的实践与认识, 2015, **45**(21): 282-289. (YUE Ruixue, GAO Ying. Nonlinear scalarizations for $(\varepsilon, \bar{\varepsilon})$-approximate quasi solutions of multiobjective optimization problems[J]. *Mathematics in Practice and Theory*, 2015, **45**(21): 282-289.(in Chinese))