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# 带有弱奇性核的多项分数阶非线性随机微分方程的改进 Euler-Maruyama 格式*

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**摘要：** 针对一类带有弱奇性核的多项分数阶非线性随机微分方程构造了改进 Euler-Maruyama（EM）格式，并证明了该格式的强收敛性．具体地，利用随机积分解的充分条件，将此多项分数阶随机微分方程等价地转化为随机 Volterra 积分方程的形式，详细推导出对应的改进 EM 格式，并对该格式进行了强收敛性分析，其强收敛阶为 $\alpha_m - \alpha_{m-1}$，其中 $\alpha_i$ 为分数阶导数的指标，且满足 $0 < \alpha_1 < \cdots < \alpha_{m-1} < \alpha_m < 1$．最后，通过数值实验验证了理论分析结果的正确性．

**关　键　词：** 多项分数阶随机微分方程；　弱奇性核；　Euler-Maruyama 格式；　强收敛性

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# A Modified Euler-Maruyama Scheme for Multi-Term Fractional Nonlinear Stochastic Differential Equations With Weakly Singular Kernels

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**Abstract**: A modified Euler-Maruyama (EM) scheme was constructed for a class of multi-term fractional nonlinear stochastic differential equations with weak singularity kernels, and the strong convergence of this modified EM scheme was proved. Specifically, according to the sufficient condition for stochastic integral decomposition, the multi-term fractional stochastic differential equation was equivalently transformed into the stochastic Volterra integral equation, and then the corresponding modified EM scheme and its strong convergence were derived and proved, respectively. The order of strong convergence is $\alpha_m - \alpha_{m-1}$, where $\alpha_i$ is the index of fractional derivative satisfying $0 < \alpha_1 < \cdots < \alpha_{m-1} < \alpha_m < 1$. Finally, numerical experiments verify the correctness of the theoretical results.

**Key words**: multi-term fractional stochastic differential equation; weakly singular kernel; Euler-Maruyama scheme; strong convergence

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# 引 言

众所周知,分数阶微分方程是常微分方程的一种推广,它可以用来建模具有非局部性的复杂物理现象和过程.近年来,随着分数阶微分方程的不断发展,其在图像处理[1]、电磁波[2-3]和人口系统[4]等领域都有着十分重要的应用.另一方面,为了“捕捉”环境中一些无法忽视的噪声扰动,随机微分方程的重要性也愈发凸显,在量化金融分析[5]和人口增长模型[6]等方面有着无法替代的作用.由于一些复杂的随机系统具有非局部性,因此分数阶随机微分方程可以更好地用来建模这类现象和过程,如数学金融[7-8]和流行病研究[9]等.但是,分数阶随机微分方程的解析解一般是难以获得的,所以其数值解的研究得到了广泛的关注[10-11].Liang 等[12]用 EM 格式求解了下列随机 Volterra 积分方程:

$$\boldsymbol{y}(t)=\boldsymbol{K}(t)+\int_0^t \boldsymbol{K}_1(t,s,\boldsymbol{y}(s))\mathrm{d}s+\int_0^t \boldsymbol{K}_2(t,s,\boldsymbol{y}(s))\mathrm{d}W(s). \tag{1}$$

在文献[13]中,Doan 等对以下非线性随机分数阶微分方程构造了 EM 格式,并证明了格式的强收敛性,

$$\mathrm{D}^{\alpha}\boldsymbol{y}(t)=\boldsymbol{k}_1(t,\boldsymbol{y}(t))+\boldsymbol{k}_2(t,\boldsymbol{y}(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t}.$$

本文将要研究的是以下带有弱奇性核的多项分数阶非线性随机微分方程的 EM 格式,及其相关的数值理论:

$$\sum_{j=1}^{m}\mathrm{D}^{\alpha_j}\boldsymbol{y}(t)=\boldsymbol{k}_0(t,\boldsymbol{y}(t))+\int_0^t (t-s)^{-\beta_1}\boldsymbol{k}_1(t,s,\boldsymbol{y}(s))\mathrm{d}s+\int_0^t (t-s)^{-\beta_2}\boldsymbol{k}_2(t,s,\boldsymbol{y}(s))\mathrm{d}W(s), \tag{2}$$

其中 $\beta_1\in(0,1)$, $\beta_2\in(0,1/2)$, $t\in\mathcal{J}:=[0,T]$ 且 $0\leqslant T\leqslant 1$, $\mathrm{D}^{\alpha_j}$ 表示 $\alpha_j$ 阶的 Caputo 分数阶导数且满足 $0<\alpha_1<\alpha_2<\cdots<\alpha_{m-1}<\alpha_m<1$, $W(t)$ 是完备概率空间$(\Omega,\mathcal{F},P)$ 中的 $m$ 维标准 Wiener 过程(Brown 运动), $\boldsymbol{y}(0)=\boldsymbol{y}_0\in\mathbb{R}^d$ 且 $E|\boldsymbol{y}_0|^2<\infty$, $E$ 表示此空间下的期望, $\boldsymbol{k}_i(i=0,1,2)$ 表示连续的非线性函数,且 $\boldsymbol{k}_0$: $\mathcal{J}\times\mathbb{R}^d\to\mathbb{R}^d$, $\boldsymbol{k}_1:(t,s)\times\mathbb{R}^d\to\mathbb{R}^d$, $\boldsymbol{k}_2:(t,s)\times\mathbb{R}^d\to\mathbb{R}^{d\times m}$.

本文主要受到文献[14-15]的启发,将采用随机微分方程 Fubini 定理,将方程(2)转化为方程(1)的形式,并给出相应的改进 EM 格式.

本文的结构和内容安排如下:第 1 节介绍了基本概念和相关假设;第 2 节给出方程(2)转化为方程(1)形式的详细过程;第 3 节详细推导出了方程(2) 的改进 EM 格式,并证明了该格式的强收敛性;第 4 节给出了数值实验;第 5 节给出了全文的总结.

# 1 预 备 知 识

令 $|\cdot|$ 表示 $\mathbb{R}^d$ 中的内积范数和 $\mathbb{R}^{d\times m}$ 中的迹范数,即若 $\boldsymbol{x}\in\mathbb{R}^d$,则 $|\boldsymbol{x}|$ 表示内积范数;若 $\boldsymbol{A}\in\mathbb{R}^{d\times m}$,则 $|\boldsymbol{A}|=\sqrt{\mathrm{trace}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})}$.给定两个实数 $a,b$,记 $a\vee b$ 表示 $\max\{a,b\}$, $a\wedge b$ 表示 $\min\{a,b\}$.为书写方便,下文中的 $C$ 都表示某个常数,该常数在不同的地方可能有不同的值,且与离散参数都无关.这里先给出分数阶 Riemann-Liouville 积分与 Caputo 导数的定义与性质[16].

**定义 1** 设 $f:[0,\infty)\to\mathbb{R}$, 则称

$$\mathrm{I}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau \tag{3}$$

为 $\alpha$ 阶的分数阶 Riemann-Liouville 积分,其中 $\alpha>0$, $\Gamma(\alpha)=\int_0^{\infty}\mathrm{e}^{-t}t^{\alpha-1}\mathrm{d}t$ 为 Gamma 函数.

**定义 2** 设函数 $f\in C[0,+\infty)$, $0<\alpha\leqslant 1$, 则称

$$\mathrm{D}^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau \tag{4}$$

为 $\alpha$ 阶的分数阶 Caputo 导数.

**性质 1** 分数阶 Riemann-Liouville 积分与 Caputo 导数之间有如下性质:

① $\mathrm{I}^{\alpha}(\mathrm{D}^{\alpha}f(t))=f(t)-f(0)$, $\alpha\in(0,1)$;

② $\mathrm{D}^{\alpha}C=0$;

③ $D^{\alpha}(I^{\alpha}f(t))=f(t)$.

为了确保方程(2)解的存在唯一性[14],我们给出以下 4 个假设.

**假设 1**　存在常数 $L>0$, 对所有的 $t_1,t_2,s\in\mathcal{J},\boldsymbol{y}\in\mathbb{R}^d$, 使得 $\boldsymbol{k}_1$ 和 $\boldsymbol{k}_2$ 满足

$$|\boldsymbol{k}_i(t_1,s,\boldsymbol{y})-\boldsymbol{k}_i(t_2,s,\boldsymbol{y})|\leqslant L(1+|\boldsymbol{y}|)|t_1-t_2|,\qquad i=1,2. \tag{5}$$

**假设 2**(Lipschitz 条件)　存在常数 $L>0$, 对所有的 $t,s\in\mathcal{J},\boldsymbol{y}_1,\boldsymbol{y}_2\in\mathbb{R}^d$, 使得 $\boldsymbol{k}_0,\boldsymbol{k}_1$ 和 $\boldsymbol{k}_2$ 满足

$$|\boldsymbol{k}_0(s,\boldsymbol{y}_1)-\boldsymbol{k}_0(s,\boldsymbol{y}_2)|\vee|\boldsymbol{k}_i(t,s,\boldsymbol{y}_1)-\boldsymbol{k}_i(t,s,\boldsymbol{y}_2)|\leqslant L|\boldsymbol{y}_1-\boldsymbol{y}_2|,\qquad i=1,2. \tag{6}$$

**假设 3**(线性增长条件)　存在常数 $L>0$, 对所有的 $t,s\in\mathcal{J},\boldsymbol{y}\in\mathbb{R}^d$, 使得 $\boldsymbol{k}_0,\boldsymbol{k}_1$ 和 $\boldsymbol{k}_2$ 满足

$$|\boldsymbol{k}_0(s,\boldsymbol{y})|\vee|\boldsymbol{k}_i(t,s,\boldsymbol{y})|\leqslant L(1+|\boldsymbol{y}|),\qquad i=1,2. \tag{7}$$

**假设 4**　存在常数 $L>0$, 使得对所有的 $s_1,s_2,t\in\mathcal{J},\boldsymbol{y}\in\mathbb{R}^d,\boldsymbol{k}_1$ 和 $\boldsymbol{k}_2$ 满足

$$|\boldsymbol{k}_0(s_1,\boldsymbol{y})-\boldsymbol{k}_0(s_2,\boldsymbol{y})|\vee|\boldsymbol{k}_i(t,s_1,\boldsymbol{y})-\boldsymbol{k}_i(t,s_2,\boldsymbol{y})|\leqslant L(1+|\boldsymbol{y}|)|s_1-s_2|,\qquad i=1,2. \tag{8}$$

## 2　方程的转化

在本节中,我们将把方程(2)转化为方程(1)的形式.首先,利用性质 1 中的①,在方程(2)的两边同时作用 Riemann-Liouville 积分算子 $I^{\alpha_m}$, 可以得到

$$\begin{aligned}\boldsymbol{y}(t)=\boldsymbol{y}_0&+\frac{1}{\Gamma(\alpha_m)}\int_0^t(t-\tau)^{\alpha_m-1}\boldsymbol{k}_0(\tau,\boldsymbol{y}(\tau))\mathrm{d}\tau+\\&\frac{1}{\Gamma(\alpha_m)}\int_0^t(t-\tau)^{\alpha_m-1}\left(\int_0^\tau(\tau-s)^{-\beta_1}\boldsymbol{k}_1(\tau,s,\boldsymbol{y}(s))\mathrm{d}s\right)\mathrm{d}\tau+\\&\frac{1}{\Gamma(\alpha_m)}\int_0^t(t-\tau)^{\alpha_m-1}\left(\int_0^\tau(\tau-s)^{-\beta_2}\boldsymbol{k}_2(\tau,s,\boldsymbol{y}(s))\mathrm{d}W(s)\right)\mathrm{d}\tau+\\&\sum_{j=1}^{m-1}\frac{t^{\alpha_m-\alpha_j}}{\Gamma(\alpha_m-\alpha_j+1)}\boldsymbol{y}_0-\sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\int_0^t(t-\tau)^{\alpha_m-\alpha_j-1}\boldsymbol{y}(\tau)\mathrm{d}\tau.\end{aligned} \tag{9}$$

根据文献[17]中的定义 2.1,可知 $(\tau-s)^{-\beta_2}\boldsymbol{k}_2(\tau,s,\boldsymbol{y}(s))\in\mathcal{L}^2(\{(t,s):0\leqslant s\leqslant\tau\leqslant T\}\times\mathbb{R}^d;\mathbb{R}^{d\times m})$,从而对 $\alpha\in(0,1)$ 存在常数 $C$, 使得

$$\int_0^t(t-\tau)^{\alpha-1}\left(\int_0^\tau E(|(t-s)^{-\beta_2}\boldsymbol{k}_2(\tau,s,\boldsymbol{y}(s))|^2)\mathrm{d}s\right)^{1/2}\mathrm{d}\tau<C\int_0^t(t-\tau)^{\alpha-1}\mathrm{d}\tau<+\infty.$$

故根据文献[18]中的定理 4.33,可知式(9)满足 Fubini 定理的充分条件,经过积分运算后可以得到

$$\begin{aligned}\boldsymbol{y}(t)=\boldsymbol{y}_0&+\frac{1}{\Gamma(\alpha_m)}\int_0^t(t-s)^{\alpha_m-1}\boldsymbol{k}_0(s,\boldsymbol{y}(s))\mathrm{d}s+\\&\int_0^t\frac{(t-s)^{\alpha_m-\beta_1}}{\Gamma(\alpha_m)}\int_0^1(1-u)^{\alpha_m-1}u^{-\beta_1}\boldsymbol{k}_1((t-s)u+s,s,\boldsymbol{y}(s))\mathrm{d}u\mathrm{d}s+\\&\int_0^t\frac{(t-s)^{\alpha_m-\beta_2}}{\Gamma(\alpha_m)}\int_0^1(1-u)^{\alpha_m-1}u^{-\beta_2}\boldsymbol{k}_2((t-s)u+s,s,\boldsymbol{y}(s))\mathrm{d}u\mathrm{d}W(s)+\\&\sum_{j=1}^{m-1}\frac{t^{\alpha_m-\alpha_j}}{\Gamma(\alpha_m-\alpha_j+1)}\boldsymbol{y}_0-\sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\int_0^t(t-s)^{\alpha_m-\alpha_j-1}\boldsymbol{y}(s)\mathrm{d}s.\end{aligned} \tag{10}$$

从而,我们可以将方程(2)等价地改写成方程(1)的形式, 即有

$$\begin{aligned}\boldsymbol{y}(t)=\boldsymbol{y}_0&+\int_0^t\boldsymbol{K}_0(t,s,\boldsymbol{y}(s))\mathrm{d}s+\int_0^t\boldsymbol{K}_1(t,s,\boldsymbol{y}(s))\mathrm{d}s+\int_0^t\boldsymbol{K}_2(t,s,\boldsymbol{y}(s))\mathrm{d}W(s)+\\&\sum_{j=1}^{m-1}\frac{t^{\alpha_m-\alpha_j}}{\Gamma(\alpha_m-\alpha_j+1)}\boldsymbol{y}_0-\left(\sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\int_0^t(t-s)^{\alpha_m-\alpha_j-1}\boldsymbol{y}(s)\mathrm{d}s\right),\end{aligned} \tag{11}$$

其中

$$\begin{cases}\boldsymbol{K}_0(t,s,\boldsymbol{y}(s)):=\dfrac{1}{\Gamma(\alpha_m)}(t-s)^{\alpha_m-1}\boldsymbol{k}_0(s,\boldsymbol{y}(s)),\\[2ex]\boldsymbol{K}_i(t,s,\boldsymbol{y}(s)):=\dfrac{(t-s)^{\alpha_m-\beta_i}}{\Gamma(\alpha_m)}\displaystyle\int_0^1(1-u)^{\alpha_m-1}u^{-\beta_i}\boldsymbol{k}_i((t-s)u+s,s,\boldsymbol{y}(s))\mathrm{d}u,\qquad i=1,2.\end{cases} \tag{12}$$

## 3 改进 EM 格式的构造及强收敛性分析

本节将首先在上节结论的基础上详细推导出方程(2)的 EM 格式以及改进 EM 格式，然后证明改进 EM 格式的强收敛性.

### 3.1 EM 格式和改进 EM 格式

为了构造数值格式，对区间 $[0,T]$ 进行等距剖分，记 $t_n = nh$ 为网格点，其中 $n = 0,1,\cdots,N$，$h = T/N$. 于是，由式(11)可得

$$\begin{aligned}
\boldsymbol{y}(t_n) = {} & \boldsymbol{y}_0 + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_0(t_n,s,\boldsymbol{y}(s))\mathrm{d}s + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_1(t_n,s,\boldsymbol{y}(s))\mathrm{d}s + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_2(t_n,s,\boldsymbol{y}(s))\mathrm{d}W(s) + {} \\
& \sum_{j=1}^{m-1}\frac{t_n^{\alpha_m-\alpha_j}}{\Gamma(\alpha_m-\alpha_j+1)}\boldsymbol{y}_0 - \sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-s)^{\alpha_m-\alpha_j-1}\boldsymbol{y}(s)\mathrm{d}s \approx {} \\
& \boldsymbol{y}_0 + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_0(t_n,s,\boldsymbol{y}(t_i))\mathrm{d}s + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_1(t_n,s,\boldsymbol{y}(t_i))\mathrm{d}s + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_2(t_n,s,\boldsymbol{y}(t_i))\mathrm{d}W(s) + {} \\
& \sum_{j=1}^{m-1}\frac{t_n^{\alpha_m-\alpha_j}}{\Gamma(\alpha_m-\alpha_j+1)}\boldsymbol{y}_0 - \sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-s)^{\alpha_m-\alpha_j-1}\boldsymbol{y}(t_i)\mathrm{d}s.
\end{aligned} \tag{13}$$

令

$$\begin{aligned}
\boldsymbol{y}^N(t_n) = {} & \boldsymbol{y}_0 + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_0(t_n,s,\boldsymbol{y}^N(t_i))\mathrm{d}s + {} \\
& \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_1(t_n,s,\boldsymbol{y}^N(t_i))\mathrm{d}s + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_2(t_n,s,\boldsymbol{y}^N(t_i))\mathrm{d}W(s) + {} \\
& \sum_{j=1}^{m-1}\frac{t_n^{\alpha_m-\alpha_j}}{\Gamma(\alpha_m-\alpha_j+1)}\boldsymbol{y}_0 - \sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-s)^{\alpha_m-\alpha_j-1}\boldsymbol{y}^N(t_i)\mathrm{d}s.
\end{aligned} \tag{14}$$

于是式(2)的 EM 格式可写为

$$\begin{aligned}
\boldsymbol{y}^N(t) = {} & \boldsymbol{y}_0 + \int_0^t\boldsymbol{K}_0(t,s,\hat{\boldsymbol{y}}^N(s))\mathrm{d}s + \int_0^t\boldsymbol{K}_1(t,s,\hat{\boldsymbol{y}}^N(s))\mathrm{d}s + \int_0^t\boldsymbol{K}_2(t,s,\hat{\boldsymbol{y}}^N(s))\mathrm{d}W(s) + {} \\
& \sum_{j=1}^{m-1}\frac{t^{\alpha_m-\alpha_j}}{\Gamma(\alpha_m-\alpha_j+1)}\boldsymbol{y}_0 - \sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\int_0^t(t-s)^{\alpha_m-\alpha_j-1}\hat{\boldsymbol{y}}^N(s)\mathrm{d}s,
\end{aligned} \tag{15}$$

其中 $\hat{\boldsymbol{y}}^N(s) = \sum_{n=0}^{N-1}\boldsymbol{y}^N(t_n)\mathcal{X}_{[t_n,t_{n+1})}(t)$. 函数 $\mathcal{X}_{(U)}(t)$ 为指示函数，即当 $t\in U$ 时，$\mathcal{X}_{(U)}(t)=1$；当 $t\notin U$ 时，$\mathcal{X}_{(U)}(t)=0$. 为了避免计算随机积分，下面推导出改进 EM 格式. 令

$$\begin{aligned}
\boldsymbol{Y}(t_n) = {} & \boldsymbol{y}_0 + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_0(t_n,t_i,\boldsymbol{Y}(t_i))\mathrm{d}s + {} \\
& \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_1(t_n,t_i,\boldsymbol{Y}(t_i))\mathrm{d}s + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\boldsymbol{K}_2(t_n,t_i,\boldsymbol{Y}(t_i))\mathrm{d}W(t_i) + {} \\
& \sum_{j=1}^{m-1}\frac{t_n^{\alpha_m-\alpha_j}}{\Gamma(\alpha_m-\alpha_j+1)}\boldsymbol{Y}_0 - \sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-t_i)^{\alpha_m-\alpha_j-1}\boldsymbol{Y}(t_i)\mathrm{d}s = {} \\
& \boldsymbol{Y}_0 + h\sum_{i=1}^{n-1}\boldsymbol{K}_0(t_n,t_i,\boldsymbol{Y}(t_i)) + h\sum_{i=1}^{n-1}\boldsymbol{K}_1(t_n,t_i,\boldsymbol{Y}(t_i)) + \sum_{i=1}^{n-1}\boldsymbol{K}_2(t_n,t_i,\boldsymbol{Y}(t_i))\Delta W_i + {} \\
& \sum_{j=1}^{m-1}\frac{t_n^{\alpha_m-\alpha_j}}{\Gamma(\alpha_m-\alpha_j+1)}\boldsymbol{y}_0 - \sum_{j=1}^{m-1}\frac{h}{\Gamma(\alpha_m-\alpha_j)}\sum_{i=1}^{n-1}(t_n-t_i)^{\alpha_m-\alpha_j-1}\boldsymbol{Y}(t_i),
\end{aligned} \tag{16}$$

其中 $\Delta W_i = W(t_{i+1}) - W(t_i)$. 令 $\underline{s} = t_n$，$s\in[t_n,t_{n+1})$，$\hat{\boldsymbol{Y}}(s) = \sum_{n=0}^{N-1}\boldsymbol{Y}(t_n)\mathcal{X}_{[t_n,t_{n+1})}(t)$，于是，可以得到以下改进 EM 格式：

$$\begin{aligned}
\boldsymbol{Y}(t) = {} & \boldsymbol{y}_0 + \int_0^t\boldsymbol{K}_0(t,\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}s + \int_0^t\boldsymbol{K}_1(t,\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}s + \int_0^t\boldsymbol{K}_2(t,\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}W(s) + {} \\
& \sum_{j=1}^{m-1}\frac{t^{\alpha_m-\alpha_j}}{\Gamma(\alpha_m-\alpha_j+1)}\boldsymbol{y}_0 - \sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\int_0^t(t-\underline{s})^{\alpha_m-\alpha_j-1}\hat{\boldsymbol{Y}}(s)\mathrm{d}s.
\end{aligned} \tag{17}$$

## 3.2 强收敛性

根据文献[14]中的引理 4.1 与引理 4.2, 可得如下引理.

**引理 1**　若 $a,b \in (0,1]$, 对任意$t \in [t_n,t_{n+1})$, $n = 0,1,\cdots,N-1$, 则有

$$\int_0^{t_n} |(t-\underline{s})^{a-b}-(t_n-\underline{s})^{a-b}|\mathrm{d}s \leqslant Ch^{1\wedge(1+a-b)},$$

$$\int_0^{t} |(t-\underline{s})^{a-b}-(t-s)^{a-b}|\mathrm{d}s \leqslant Ch^{1\wedge(1+a-b)}.$$

**引理 2**　若 $a\in(0,1]$, $b\in(0,1/2)$, 则 $a-b\in(-1/2,1)$, 对任意 $t\in[t_n,t_{n+1})$, $n=0,1,\cdots,N-1$, 有

$$\int_0^{t_n} |(t-s)^{a-b}-(t_n-s)^{a-b}|^2\mathrm{d}s = \begin{cases} Ch^2, & a-b\in(1/2,1), \\ C|\ln h|h^2, & a-b=1/2, \\ Ch^{1+2(a-b)}, & a-b\in(-1/2,1/2), \end{cases}$$

$$\int_0^{t} |(t-\underline{s})^{a-b}-(t_n-s)^{a-b}|^2\mathrm{d}s = \begin{cases} Ch^2, & a-b\in(1/2,1), \\ C|\ln h|h^2, & a-b=1/2, \\ Ch^{1+2(a-b)}, & a-b\in(-1/2,1/2). \end{cases}$$

在证明接下来的引理之前,我们先给出一个初等不等式:设 $x_1,x_2,\cdots,x_n \geqslant 0$, 则

$$\left(\sum_{i=1}^n x_i\right)^2 \leqslant n\sum_{i=1}^n x_i^2. \tag{18}$$

**引理 3**　在假设 1 与假设 3 成立的条件下,有

$$E|\hat{\boldsymbol{Y}}(t)|^2 \leqslant C,\ E|\boldsymbol{Y}(t)|^2 \leqslant C.$$

**证明**　对 $\forall t \in \mathcal{J}$,存在正整数 $n$ 使得当 $t \in [t_n,t_{n+1})$ 时,成立等式 $\hat{\boldsymbol{Y}}(t)=\boldsymbol{Y}(t_n)$.因此,由式(17)可知

$$\begin{aligned} &E|\hat{\boldsymbol{Y}}(t)|^2 = E|\boldsymbol{Y}(t_n)|^2 = \\ &E\Bigg|\boldsymbol{y}_0 + \int_0^{t_n}\boldsymbol{K}_0(t_n,\underline{s},\hat{\boldsymbol{Y}}(\underline{s}))\mathrm{d}s + \int_0^{t_n}\boldsymbol{K}_1(t_n,\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}s + \\ &\int_0^{t_n}\boldsymbol{K}_2(t_n,\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}W(s) + \sum_{j=1}^{m-1}\frac{t_n^{\alpha_m-\alpha_k}}{\Gamma(\alpha_m-\alpha_k+1)}\boldsymbol{y}_0 - \\ &\sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_k)}\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-\alpha_k-1}\hat{\boldsymbol{Y}}(s)\mathrm{d}s\Bigg|^2. \end{aligned}$$

利用不等式(18),可得

$$\begin{aligned} &E|\hat{\boldsymbol{Y}}(t)|^2 \leqslant 6\Bigg(E|\boldsymbol{y}_0|^2 + E\left|\int_0^{t_n}\boldsymbol{K}_0(t_n,\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}s\right|^2 + E\left|\int_0^{t_n}\boldsymbol{K}_1(t_n,\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}s\right|^2 + \\ &E\left|\int_0^{t_n}\boldsymbol{K}_2(t_n,\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}W(s)\right|^2 + E\left|\sum_{j=1}^{m-1}\frac{t_n^{\alpha_m-\alpha_k}}{\Gamma(\alpha_m-\alpha_k+1)}\boldsymbol{y}_0\right|^2 + \\ &E\left|\sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_k)}\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-\alpha_k-1}\hat{\boldsymbol{Y}}(s)\mathrm{d}s\right|^2\Bigg) := \\ &6(M_1+M_2+M_3+M_4+M_5+M_6). \end{aligned}$$

下面,我们分别给出 $M_2,M_3,M_4,M_5$ 和 $M_6$ 的估计.首先根据 Hölder 不等式和假设 3,有

$$\begin{aligned} &M_2 = \frac{1}{\Gamma^2(\alpha_m)}E\left|\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-1}\boldsymbol{k}_0(\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}s\right|^2 \leqslant \\ &\frac{1}{\Gamma^2(\alpha_m)}E\left|\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-1}\mathrm{d}s\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-1}\boldsymbol{k}_0{}^2(s,\hat{\boldsymbol{Y}}(s))\mathrm{d}s\right| \leqslant \\ &\frac{C}{\Gamma^2(\alpha_m)}\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-1}\mathrm{d}s\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-1}(1+E|\hat{\boldsymbol{Y}}(s)|^2)\mathrm{d}s \leqslant \\ &C\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-1}(1+E|\hat{\boldsymbol{Y}}(s)|^2)\mathrm{d}s. \end{aligned}$$

与上述推导类似,可以推出

$$M_3 = \frac{1}{\Gamma^2(\alpha_m)} E \left| \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \beta_1} \int_0^1 (1-u)^{\alpha_m - 1} u^{-\beta_1} \boldsymbol{k}_1((t_n - \underline{s})u + \underline{s}, \underline{s}, \hat{\boldsymbol{Y}}(s)) \mathrm{d}u\mathrm{d}s \right|^2 \leqslant$$

$$\frac{1}{\Gamma^2(\alpha_m)} E \left| \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \beta_1} \max_{0<u<1} \boldsymbol{k}_1((t_n - \underline{s})u + \underline{s}, \underline{s}, \hat{\boldsymbol{Y}}(s)) \int_0^1 (1-u)^{\alpha_m - 1} u^{-\beta_1} \mathrm{d}u\mathrm{d}s \right|^2 \leqslant$$

$$C \frac{B^2(\alpha_m, 1-\beta_1)}{\Gamma^2(\alpha_m)} \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \beta_1} \mathrm{d}s \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \beta_1} (1 + E|\hat{\boldsymbol{Y}}(s)|^2) \mathrm{d}s \leqslant$$

$$C \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \beta_1} (1 + E|\hat{\boldsymbol{Y}}(s)|^2) \mathrm{d}s.$$

利用 Hölder 不等式、Itô 等距性以及假设 3,可知

$$M_4 = \frac{1}{\Gamma^2(\alpha_m)} E \left| \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \beta_2} \int_0^1 (1-u)^{\alpha_m - 1} u^{-\beta_2} \boldsymbol{k}_2((t_n - s)u + \underline{s}, \underline{s}, \hat{\boldsymbol{Y}}(s)) \mathrm{d}u\mathrm{d}W(s) \right|^2 \leqslant$$

$$\frac{B^2(\alpha_m, 1-\beta_2)}{\Gamma^2(\alpha_m)} \left| \int_0^{t_n} (t_n - \underline{s})^{2(\alpha_m - \beta_2)} E(\max_{0<u<1} |\boldsymbol{k}_2((t_n - \underline{s})u + \underline{s}, \underline{s}, \hat{\boldsymbol{Y}}(s))|^2) \mathrm{d}s \right| \leqslant$$

$$C \int_0^{t_n} (t_n - \underline{s})^{2(\alpha_m - \beta_2)} (1 + E|\hat{\boldsymbol{Y}}(s)|^2) \mathrm{d}s.$$

由于 $T \leqslant 1$, 则 $t_n^{\alpha_m - \alpha_j} \leqslant t_n^{\alpha_m - \alpha_{m-1}}$, $j = 1, 2, \cdots, m-2$, 且利用不等式(18),可得

$$M_5 = E \left| \sum_{j=1}^{m-1} \frac{t_n^{\alpha_m - \alpha_j}}{\Gamma(\alpha_m - \alpha_j + 1)} \boldsymbol{y}_0 \right|^2 \leqslant C(m-1)^2 E|t_n^{\alpha_m - \alpha_{m-1}} \boldsymbol{y}_0|^2 \leqslant CE|\boldsymbol{y}_0|^2.$$

再利用 Hölder 不等式与不等式(18),则有

$$M_6 = E \left| \sum_{j=1}^{m-1} \frac{1}{\Gamma(\alpha_m - \alpha_j)} \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \alpha_j - 1} \hat{\boldsymbol{Y}}(s) \mathrm{d}s \right|^2 \leqslant$$

$$CE \left| \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \alpha_1 - 1} \hat{\boldsymbol{Y}}(s) \mathrm{d}s + \cdots + \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \alpha_{m-1} - 1} \hat{\boldsymbol{Y}}(s) \mathrm{d}s \right|^2 \leqslant$$

$$C(m-1)^2 \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \alpha_{m-1} - 1} \mathrm{d}s \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \alpha_{m-1} - 1} (E|\hat{\boldsymbol{Y}}(s)|^2) \mathrm{d}s \leqslant$$

$$C \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \alpha_{m-1} - 1} (E|\hat{\boldsymbol{Y}}(s)|^2) \mathrm{d}s.$$

综合上述分析,可知

$$E|\hat{\boldsymbol{Y}}(t)|^2 = E|\boldsymbol{Y}(t_n)|^2 \leqslant 6(M_1 + M_2 + M_3 + M_4 + M_5 + M_6) \leqslant$$

$$C\left(1 + \int_0^{t_n} (t_n - \underline{s})^{\alpha_m - \alpha_{m-1} - 1} E|\hat{\boldsymbol{Y}}(s)|^2 \mathrm{d}s\right) =$$

$$C\left(1 + \int_0^{t} (t_n - \underline{s})^{\alpha_m - \alpha_{m-1} - 1} \mathcal{X}_{(0, t_n)}(s) E|\hat{\boldsymbol{Y}}(s)|^2 \mathrm{d}s\right).$$

从而根据 Gronwall 不等式,可知 $E|\hat{\boldsymbol{Y}}(t)|^2 \leqslant C$,以及 $E|\boldsymbol{Y}(t)|^2 \leqslant C$. □

**引理 4** 若假设 1 与假设 3 成立,则有

$$E|\boldsymbol{Y}(t) - \hat{\boldsymbol{Y}}(t)|^2 = \begin{cases} C(|\ln h| h^2 \vee h^{2(\alpha_m - \alpha_{m-1})}), & \alpha_m - \beta_2 = 1/2, \\ Ch^{2(\alpha_m - \alpha_{m-1})}, & \alpha_m - \beta_2 \neq 1/2. \end{cases}$$

**证明** 当 $t \in [t_n, t_{n+1})$ 时, $\hat{\boldsymbol{Y}}(t) = \boldsymbol{Y}(t_n)$, 则根据式(17)与不等式(18),可推出

$$E|\boldsymbol{Y}(t) - \hat{\boldsymbol{Y}}(t)|^2 = E|\boldsymbol{Y}(t) - \boldsymbol{Y}(t_n)|^2 \leqslant$$

$$5\left\{ E \left| \int_0^t \boldsymbol{K}_0(t, \underline{s}, \hat{\boldsymbol{Y}}(s)) \mathrm{d}s - \int_0^{t_n} \boldsymbol{K}_0(t_n, \underline{s}, \hat{\boldsymbol{Y}}(s)) \mathrm{d}s \right|^2 + \right.$$

$$E \left| \int_0^t \boldsymbol{K}_1(t, \underline{s}, \hat{\boldsymbol{Y}}(s)) \mathrm{d}s - \int_0^{t_n} \boldsymbol{K}_1(t_n, \underline{s}, \hat{\boldsymbol{Y}}(s)) \mathrm{d}s \right|^2 +$$

$$E \left| \int_0^t \boldsymbol{K}_2(t, \underline{s}, \hat{\boldsymbol{Y}}(s)) \mathrm{d}W(s) - \int_0^{t_n} \boldsymbol{K}_1(t_n, \underline{s}, \hat{\boldsymbol{Y}}(s)) \mathrm{d}W(s) \right|^2 +$$

$$E \left| \sum_{j=1}^{m-1} \frac{t^{\alpha_m - \alpha_j}}{\Gamma(\alpha_m - \alpha_j + 1)} \boldsymbol{y}_0 - \sum_{j=1}^{m-1} \frac{t_n^{\alpha_m - \alpha_j}}{\Gamma(\alpha_m - \alpha_j + 1)} \boldsymbol{y}_0 \right|^2 +$$

$$E\left|\sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\int_0^t(t-\underline{s})^{\alpha_m-\alpha_j-1}\hat{\boldsymbol{Y}}(s)\mathrm{d}s-\right.$$
$$\left.\sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-\alpha_j-1}\hat{\boldsymbol{Y}}(s)\mathrm{d}s\right|^2\Big\}:=$$
$$5(H_1+H_2+H_3+H_4+H_5).$$

下面,我们分别给 $H_1,H_2,H_3,H_4,H_5$ 的估计.首先,利用 Hölder 不等式、引理 1、引理 3 以及不等式(18),可知

$$H_1=E\left|\int_0^{t_n}(\boldsymbol{K}_0(t,\underline{s},\hat{\boldsymbol{Y}}(s))-\boldsymbol{K}_0(t_n,\underline{s},\hat{\boldsymbol{Y}}(s)))\mathrm{d}s+\int_{t_n}^t\boldsymbol{K}_0(t,\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}s\right|^2\leqslant$$
$$\frac{C}{\Gamma^2(\alpha_m)}\Big\{\int_0^{t_n}|(t-\underline{s})^{\alpha_m-1}-(t_n-\underline{s})^{\alpha_m-1}|\mathrm{d}s\times$$
$$\int_0^{t_n}|(t-\underline{s})^{\alpha_m-1}-(t_n-\underline{s})^{\alpha_m-1}|(1+E|\hat{\boldsymbol{Y}}(s)|^2)\mathrm{d}s+$$
$$\int_{t_n}^t|(t-\underline{s})^{\alpha_m-1}|\mathrm{d}s\int_{t_n}^t(t-\underline{s})^{\alpha_m-1}(1+E|\hat{\boldsymbol{Y}}(s)|^2)\mathrm{d}s\Big\}\leqslant Ch^{2\alpha_m}.$$

类似于估计 $H_1$ 的推导,可知

$$H_2=E\left|\int_0^{t_n}(\boldsymbol{K}_1(t,\underline{s},\hat{\boldsymbol{Y}}(s))-\boldsymbol{K}_1(t_n,\underline{s},\hat{\boldsymbol{Y}}(s)))\mathrm{d}s+\int_{t_n}^t\boldsymbol{K}_1(t,\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}s\right|^2\leqslant$$
$$\frac{2}{\Gamma^2(\alpha_m)}\Big\{E\Big|\int_0^{t_n}\int_0^1(1-u)^{\alpha_m-1}u^{-\beta_1}[(t-\underline{s})^{\alpha_m-\beta_1}\boldsymbol{k}_1((t-\underline{s})u+\underline{s},\underline{s},\hat{\boldsymbol{Y}}(s))-$$
$$(t_n-\underline{s})^{\alpha_m-\beta_1}\boldsymbol{k}_1((t-\underline{s})u+\underline{s},\underline{s},\hat{\boldsymbol{Y}}(s))+(t_n-\underline{s})^{\alpha_m-\beta_1}\boldsymbol{k}_1((t-\underline{s})u+\underline{s},\underline{s},\hat{\boldsymbol{Y}}(s))-$$
$$(t_n-\underline{s})^{\alpha_m-\beta_1}\boldsymbol{k}_1((t_n-\underline{s})u+\underline{s},\underline{s},\hat{\boldsymbol{Y}}(s))]\mathrm{d}u\mathrm{d}s\Big|^2+$$
$$E\left|\int_{t_n}^t(t-\underline{s})^{\alpha_m-\beta_1}\int_0^1(1-u)^{\alpha_m-1}u^{-\beta_1}\boldsymbol{k}_1((t-\underline{s})u+\underline{s},\underline{s},\hat{\boldsymbol{Y}}(s))\mathrm{d}u\mathrm{d}s\right|^2\Big\}\leqslant$$
$$\frac{C}{\Gamma^2(\alpha_m)}\Big\{\int_0^{t_n}|(t-\underline{s})^{\alpha_m-\beta_1}-(t_n-\underline{s})^{\alpha_m-\beta_1}|\mathrm{d}s\times$$
$$\int_0^{t_n}|(t-\underline{s})^{\alpha_m-\beta_1}-(t_n-\underline{s})^{\alpha_m-\beta_1}|(1+E|\hat{\boldsymbol{Y}}(s)|^2)\mathrm{d}s+$$
$$\int_{t_n}^t|(t-\underline{s})^{\alpha_m-\beta_1}(t-\underline{s})^{\alpha_m-\beta_1}|\mathrm{d}s\int_{t_n}^t|(t-\underline{s})^{\alpha_m-\beta_1}(t-\underline{s})^{\alpha_m-\beta_1}|(1+E|\hat{\boldsymbol{Y}}(s)|^2)\mathrm{d}s+$$
$$Ch^2\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-\beta_1}(1+E|\hat{\boldsymbol{Y}}(s)|^2)\mathrm{d}s\Big\}\leqslant$$
$$Ch^{2\wedge2(\alpha_m-\beta_1+1)}+Ch^{2(\alpha_m-\beta_1+1)}+Ch^2.$$

结合 Hölder 不等式、引理 1~3、不等式(18)以及 Itô 等距性,则有

$$H_3\leqslant Ch^2+Ch^{2(\alpha_m-\beta_2)-1}+\begin{cases}C|\ln h|h^2, & \alpha_m-\beta_2=1/2,\\ Ch^{1+2(\alpha_m-\alpha_{m-1})}, & \alpha_m-\beta_2\in(-1/2,1/2),\\ Ch^2, & \alpha_m-\beta_2\in(1/2,1).\end{cases}$$

根据不等式(18)和 $|\boldsymbol{x}+\boldsymbol{y}|^\alpha\leqslant|\boldsymbol{x}|^\alpha+|\boldsymbol{y}|^\alpha(0<\alpha<1)$, 可知

$$H_4\leqslant CE|(t^{\alpha_m-\alpha_1}-t_n^{\alpha_m-\alpha_1})\boldsymbol{y}_0+\cdots+(t^{\alpha_m-\alpha_{m-1}}-t_n^{\alpha_m-\alpha_{m-1}})\boldsymbol{y}_0|^2\leqslant$$
$$C(m-1)\{E|(t^{\alpha_m-\alpha_1}-t_n^{\alpha_m-\alpha_1})\boldsymbol{y}_0|^2+\cdots+E|(t^{\alpha_m-\alpha_{m-1}}-t_n^{\alpha_m-\alpha_{m-1}})\boldsymbol{y}_0|^2\}\leqslant$$
$$C(m-1)^2E|(t-t_n)^{\alpha_m-\alpha_{m-1}}\boldsymbol{y}_0|^2\leqslant C(m-1)^2(t_{n+1}-t_n)^{2(\alpha_m-\alpha_{m-1})}E|\boldsymbol{y}_0|^2\leqslant$$
$$Ch^{2(\alpha_m-\alpha_{m-1})}.$$

利用引理 1、引理 3、不等式(18)和 $|\boldsymbol{x}+\boldsymbol{y}|^\alpha\leqslant|\boldsymbol{x}|^\alpha+|\boldsymbol{y}|^\alpha(0<\alpha<1)$, 可知

$$H_5\leqslant CE\Big|\int_0^t(t-\underline{s})^{\alpha_m-\alpha_1-1}\hat{\boldsymbol{Y}}(s)\mathrm{d}s-\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-\alpha_1-1}\hat{\boldsymbol{Y}}(s)\mathrm{d}s+\cdots+$$
$$\int_0^t(t-\underline{s})^{\alpha_m-\alpha_{m-1}-1}\hat{\boldsymbol{Y}}(s)\mathrm{d}s-\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-\alpha_{m-1}-1}\hat{\boldsymbol{Y}}(s)\mathrm{d}s\Big|^2\leqslant$$

$$
\begin{aligned}
&C(m-1)\left\{E\left|\int_0^t (t-\underline{s})^{\alpha_m-\alpha_1-1}\hat{\boldsymbol{Y}}(s)\,\mathrm{d}s-\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-\alpha_1-1}\hat{\boldsymbol{Y}}(s)\,\mathrm{d}s\right|^2+\cdots+\right.\\
&\left.E\left|\int_0^t (t-\underline{s})^{\alpha_m-\alpha_{m-1}-1}\hat{\boldsymbol{Y}}(s)\,\mathrm{d}s-\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-\alpha_{m-1}-1}\hat{\boldsymbol{Y}}(s)\,\mathrm{d}s\right|^2\right\}\leqslant\\
&2C(m-1)^2\left\{E\left|\int_0^t (t-\underline{s})^{\alpha_m-\alpha_{m-1}-1}\hat{\boldsymbol{Y}}(s)\,\mathrm{d}s-\int_0^{t_n}(t_n-\underline{s})^{\alpha_m-\alpha_{m-1}-1}\hat{\boldsymbol{Y}}(s)\,\mathrm{d}s\right|^2+\right.\\
&\left.E\left|\int_t^{t_n}(t-\underline{s})^{\alpha_m-\alpha_{m-1}-1}\hat{\boldsymbol{Y}}(s)\,\mathrm{d}s\right|^2\right\}\leqslant\\
&2C(m-1)^2\int_0^{t_n}\left|(t_n-\underline{s})^{\alpha_m-\alpha_{m-1}-1}-(t-\underline{s})^{\alpha_m-\alpha_{m-1}-1}\right|\mathrm{d}s\times\\
&\int_0^{t_n}\left|(t_n-\underline{s})^{\alpha_m-\alpha_{m-1}-1}-(t-\underline{s})^{\alpha_m-\alpha_{m-1}-1}\right|E\left|\hat{\boldsymbol{Y}}(s)\right|^2\mathrm{d}s+\\
&2C(m-1)^2\int_{t_n}^t (t-\underline{s})^{\alpha_m-\alpha_{m-1}-1}\mathrm{d}s\int_{t_n}^t (t-\underline{s})^{\alpha_m-\alpha_{m-1}-1}E\left|\hat{\boldsymbol{Y}}(s)\right|^2\mathrm{d}s\leqslant\\
&Ch^{2(\alpha_m-\alpha_{m-1})}.
\end{aligned}
$$

故结合上述分析,可得

$$
E\left|\boldsymbol{Y}(t)-\hat{\boldsymbol{Y}}(t)\right|^2=\begin{cases}C(|\ln h|\,h^2\vee h^{2(\alpha_m-\alpha_{m-1})}), & \alpha_m-\beta_2=1/2,\\ Ch^{2(\alpha_m-\alpha_{m-1})}, & \alpha_m-\beta_2\neq 1/2.\end{cases}
$$

证毕. □

**定理 1**(强收敛定理) 若假设 1~4 成立,则改进 EM 格式的解 $\boldsymbol{Y}(t)$ 在均方意义下收敛到方程(2)的解 $\boldsymbol{y}(t)$, 即

$$
\lim_{h\to 0}E\left|\boldsymbol{Y}(t)-\boldsymbol{y}(t)\right|^2=0.
$$

**证明** 根据式(11)和(17)可知

$$
\begin{aligned}
E\left|\boldsymbol{Y}(t)-\boldsymbol{y}(t)\right|^2\leqslant 8&\left\{E\left|\int_0^t(\boldsymbol{K}_0(t,\underline{s},\hat{\boldsymbol{Y}}(s))-\boldsymbol{K}_0(t,s,\hat{\boldsymbol{Y}}(s)))\,\mathrm{d}s\right|^2+\right.\\
&E\left|\int_0^t(\boldsymbol{K}_0(t,s,\hat{\boldsymbol{Y}}(s))-\boldsymbol{K}_0(t,s,\boldsymbol{y}(s)))\,\mathrm{d}s\right|^2+\\
&E\left|\int_0^t(\boldsymbol{K}_1(t,\underline{s},\hat{\boldsymbol{Y}}(s))-\boldsymbol{K}_1(t,s,\hat{\boldsymbol{Y}}(s)))\,\mathrm{d}s\right|^2+\\
&E\left|\int_0^t(\boldsymbol{K}_1(t,s,\hat{\boldsymbol{Y}}(s))-\boldsymbol{K}_1(t,s,\boldsymbol{y}(s)))\,\mathrm{d}s\right|^2+\\
&E\left|\int_0^t(\boldsymbol{K}_2(t,\underline{s},\hat{\boldsymbol{Y}}(s))-\boldsymbol{K}_2(t,s,\hat{\boldsymbol{Y}}(s)))\,\mathrm{d}W(s)\right|^2+\\
&E\left|\int_0^t(\boldsymbol{K}_2(t,s,\hat{\boldsymbol{Y}}(s))-\boldsymbol{K}_2(t,s,\boldsymbol{y}(s)))\,\mathrm{d}W(s)\right|^2+\\
&E\left|\sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\int_0^t[(t-\underline{s})^{\alpha_m-\alpha_j}-(t-s)^{\alpha_m-\alpha_j}]\hat{\boldsymbol{Y}}(s)\,\mathrm{d}s\right|^2+\\
&\left.E\left|\sum_{j=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_j)}\int_0^t(t-s)^{\alpha_m-\alpha_j}[\hat{\boldsymbol{Y}}(s)-\boldsymbol{y}(s)]\,\mathrm{d}s\right|^2\right\}:=\\
&8\{L_1+L_2+L_3+L_4+L_5+L_6+L_7+L_8\}.
\end{aligned}
$$

利用假设 1~4、引理 1~3、不等式(18)和$|\boldsymbol{x}+\boldsymbol{y}|^\alpha\leqslant|\boldsymbol{x}|^\alpha+|\boldsymbol{y}|^\alpha(0<\alpha<1)$, 与引理 4 的推导过程类似, 可知

$$
L_1+L_3+L_5+L_7\leqslant\begin{cases}C(|\ln h|\,h^2\vee h^{2(\alpha_m-\alpha_{m-1})}), & \alpha_m-\beta_2=1/2,\\ Ch^{2(\alpha_m-\alpha_{m-1})}, & \alpha_m-\beta_2\neq 1/2,\end{cases}
$$

$$
\begin{aligned}
&L_2+L_4+L_6+L_8\leqslant\\
&\quad C\int_0^t(3(t-s)^{\alpha_m-\alpha_{m-1}-1}+(t-s)^{2(\alpha_m-\alpha_{m-1})})E(|\hat{\boldsymbol{Y}}(t)-\boldsymbol{Y}(t)|^2+|\boldsymbol{Y}(t)-\boldsymbol{y}(t)|^2)\,\mathrm{d}s.
\end{aligned}
$$

利用引理 4 和 Gronwall 不等式,可得

$$E|\boldsymbol{Y}(t)-\boldsymbol{y}(t)|^2 \leqslant \begin{cases} C(|\ln h|h^2 \vee h^{2(\alpha_m-\alpha_{m-1})}), & \alpha_m-\beta_2=1/2, \\ Ch^{2(\alpha_m-\alpha_{m-1})}, & \alpha_m-\beta_2 \neq 1/2. \end{cases} \tag{19}$$

证毕. □

**定理 2**(收敛阶定理)　若假设 1~4 成立,则有

$$(E|\boldsymbol{Y}(t)-\boldsymbol{y}(t)|^2)^{1/2} \leqslant \begin{cases} C(|\ln h|^{1/2}h \vee h^{(\alpha_m-\alpha_{m-1})}), & \alpha_m-\beta_2=1/2, \\ Ch^{(\alpha_m-\alpha_{m-1})}, & \alpha_m-\beta_2 \neq 1/2. \end{cases}$$

**证明**　由于假设 1~4 成立,故其结果可由式(19)直接推得. □

# 4　数值算例

在本节中,我们将引入数值算例来验证改进 EM 格式的理论收敛阶.与文献[19]中的误差计算方法类似,先预设随机轨道的数量为 2 000 条,定义如下形式的均方误差:

$$\varepsilon_T(h)=\left(\frac{1}{2\,000}\sum_{i=1}^{2\,000}|\boldsymbol{Y}_h(T,s_i)-\boldsymbol{Y}_{h/2}(T,s_i)|^2\right)^{1/2},$$

其中 $\boldsymbol{Y}_h(T,s_i)$ 表示由第 $i$ 条随机轨道驱动的, 以 $h$ 为步长的, 在 $T$ 时刻获得的数值解.

**例 1**　考虑如下二项分数阶非线性随机微分方程:

$$\sum_{j=1}^{2}\mathrm{D}^{\alpha_j}y(t)=\sin(ty(t))^2+\int_0^t\frac{t s\cos(y(s))}{(t-s)^{\beta_1}}\mathrm{d}s+\int_0^t\frac{t s\cos(y(s))}{(t-s)^{\beta_2}}\mathrm{d}W(s),$$

其中 $t\in(0,1)$, $y(0)=1$.则易知 $k_1=\sin(ty(t))^2$, $k_2=ts\cos(y(s))$, $k_3=ts\cos(y(s))$, 且满足假设 1~4.在具体计算时,固定 $\beta_1=0.8$, $\beta_2=0.4$,对 $\alpha_1$ 和 $\alpha_2$ 取三组不同的值,即 $\alpha_1=0.2$, $\alpha_2=0.5$; $\alpha_1=0.2$, $\alpha_2=0.7$; $\alpha_1=0.1$, $\alpha_2=0.8$.相关的计算结果见表 1.显然,从表 1 可以看出,随着步长 $h$ 的减小,其数值解的误差也在不断减小,且改进 EM 格式的收敛阶接近于 $\alpha_2-\alpha_1$, 这与定理 2 的结论相符.

**表 1**　二项分数阶导数时的误差与收敛阶

Table 1　The numerical errors and convergence orders for 2-term fractional derivatives

| $h$ | $\alpha_2=0.5,\alpha_1=0.2$ | | $\alpha_2=0.7,\alpha_1=0.2$ | | $\alpha_2=0.8,\alpha_1=0.1$ | |
|---|---|---|---|---|---|---|
| | error $\varepsilon_T(h)$ | order $n$ | error $\varepsilon_T(h)$ | order $n$ | error $\varepsilon_T(h)$ | order $n$ |
| 1/64 | 0.012 8 | – | 0.036 2 | – | 0.014 2 | – |
| 1/128 | 0.010 5 | 0.275 4 | 0.024 3 | 0.402 5 | 0.008 6 | 0.718 8 |
| 1/256 | 0.008 6 | 0.276 9 | 0.015 3 | 0.421 8 | 0.005 3 | 0.712 7 |
| 1/512 | 0.006 9 | 0.334 4 | 0.008 6 | 0.432 2 | 0.003 2 | 0.710 9 |

**例 2**　考虑如下具有三项分数阶导数的非线性随机微分方程:

$$\sum_{j=1}^{3}\mathrm{D}^{\alpha_j}y(t)=\sin(ty(t))^2+\int_0^t\frac{t s\cos(y(s))}{(t-s)^{\beta_1}}\mathrm{d}s+\int_0^t\frac{t s\cos(y(s))}{(t-s)^{\beta_2}}\mathrm{d}W(s),$$

其中 $t\in(0,1)$, $y(0)=1$,与例 1 相同,显然 $k_1,k_2,k_3$ 满足假设 1~4.固定 $\beta_1=0.7$, $\beta_2=0.3$,对 $\alpha_1,\alpha_2,\alpha_3$ 取三组不同的值,即 $\alpha_1=0.1,\alpha_2=0.2,\alpha_3=0.5$; $\alpha_1=0.1$, $\alpha_2=0.2,\alpha_3=0.7$; $\alpha_1=0.1,\alpha_2=0.2,\alpha_3=0.9$.经过改进 EM 格式的计算,可以得到表 2.通过表 2,可以看出改进 EM 格式的收敛阶接近于 $\alpha_3-\alpha_2$, 再次验证了定理结论的正确性.

**表 2**　三项分数阶导数时的误差与收敛阶

Table 2　The numerical errors and convergence orders for 3-term fractional derivatives

| $h$ | $\alpha_3-\alpha_2=0.3$ | | $\alpha_3-\alpha_2=0.5$ | | $\alpha_3-\alpha_2=0.7$ | |
|---|---|---|---|---|---|---|
| | error $\varepsilon_T(h)$ | order $n$ | error $\varepsilon_T(h)$ | order $n$ | error $\varepsilon_T(h)$ | order $n$ |
| 1/64 | 0.025 1 | – | 0.013 4 | – | 0.005 5 | – |
| 1/128 | 0.021 2 | 0.248 0 | 0.009 8 | 0.453 6 | 0.003 2 | 0.777 4 |
| 1/256 | 0.017 7 | 0.255 7 | 0.007 1 | 0.458 0 | 0.001 9 | 0.766 2 |
| 1/512 | 0.014 8 | 0.262 1 | 0.005 2 | 0.459 3 | 0.001 1 | 0.733 7 |

## 5 总 结

本文对一类多项分数阶非线性随机微分方程初值问题构造了一个有效的改进 EM 格式,并证明了该格式的强收敛性.数值实验与本文的理论分析结果相一致.

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