ⓒ 应用数学和力学编委会,ISSN 1000-0887 http://www.applmathmech.cn

应用多项式完全判别系统方法求解时空分数阶复 Ginzburg-Landau 方程*

胡 艳, 孙峪怀

(四川师范大学 数学科学学院, 成都 610066)

摘要： 研究了时空分数阶复 Ginzburg-Landau 方程.首先通过分数阶复变换将时空分数阶复 Ginzburg-Landau 方程转化为一个常微分方程.然后将常微分方程化为初等积分形式.最后用多项式完全判别系统法求得一系列精确解,其中包含有孤立波解、有理函数解、三角函数周期解、Jacobi 椭圆函数双周期解.

关 键 词： 时空分数阶复 Ginzburg-Landau 方程； 多项式完全判别系统方法； 精确解

中图分类号： O175 **文献标志码：** A DOI：10.21656/1000-0887.410392

Solutions to Space-Time Fractional Complex Ginzburg-Landau Equations With the Complete Discrimination System for Polynomial Method

HU Yan, SUN Yuhuai

(*School of Mathematical Sciences*, *Sichuan Normal University*, *Chengdu* 610066, *P.R.China*)

Abstract: The space-time fractional complex Ginzburg-Landau equation was studied. Firstly, the space-time fractional complex Ginzburg-Landau equation was transformed into the ordinary differential equation through the fractional complex transform. Secondly, the ordinary differential equation was reduced to an elementary integral form. Finally, a series of exact solutions including solitary wave solutions, rational function type solutions, triangle function type periodic solutions, and Jacobian elliptic function doubly-periodic solutions, were constructed with the complete discrimination system for polynomial method.

Key words: space-time fractional complex Ginzburg-Landau equation; complete discrimination system for polynomial method; exact solution

引 言

近年来,分数阶非线性偏微分方程的研究已成为流体力学、控制理论、工程应用、医学研究等多个领域的热点.在描述某些系统的动力学行为时,相对于整数阶微积分构造的模型,分数阶微积分构造的模型更符合

* **收稿日期：** 2020-12-24；**修订日期：** 2021-02-24
基金项目： 国家自然科学基金(12071323)
作者简介： 胡艳(1995—),女,硕士生(E-mail：1183353856@qq.com)；
孙峪怀(1963—),男,教授,博士(通讯作者. E-mail：sunyuhuai63@163.com).
引用格式： 胡艳, 孙峪怀. 应用多项式完全判别系统方法求解时空分数阶复 Ginzburg-Landau 方程[J]. 应用数学和力学, 2021, **42**(8)：874-880.

实际情况，因此研究分数阶偏微分方程的解有重要的理论和应用价值。目前，众多研究者在研究分析中得到了许多可行的求分数阶非线性偏微分方程精确解的方法，如 (G'/G)- 展开法[1]、变分迭代法[2]、不变子空间法[3]、首次积分法[4]、指数函数展开法[5]、同伦分析法[6]、Adomian 分析法[7]、动力系统分支方法[8-9]。但是数学研究的结果，在目前还未能提供一种普遍有效的求精确解的方法，所以发现并应用更多的方法来探究非线性偏微分方程的精确解以更好地解释非线性物理现象，是当前广大数学和物理工作者们的重要课题。

基于此，本文讨论了如下的时空分数阶复 Ginzburg-Landau 方程[10-12]：

$$\mathrm{i}D_z^{\delta}u + D_x^{2\beta}u/2 + (\beta_1 - \mathrm{i})D_{\tau}^{2\alpha}u/2 + \mathrm{i}u + (1 - \mathrm{i}r_1)u|u|^2 + \mathrm{i}r_2u|u|^4 = 0, \tag{1}$$

其中 $D_z^{\delta}u, D_x^{2\beta}u, D_{\tau}^{2\alpha}u$ 是 u 关于 z, x, τ 的整合分数阶导数，$0 < \delta, \beta, \alpha \leqslant 1$，$\mathrm{i} = \sqrt{-1}$，$\beta_1 < 0$ 是色散项系数，r_1 是三次增益系数，r_2 是五次耗散系数，β_1, r_1, r_2 是实常数，复函数 $u = u(x, z, \tau)$ 用于描述缓慢变化的电场包络，$\tau = t - z/V_0$ 是减少时间，t 为物理时间，z 和 x 是传播坐标，V_0 为载波的群速度。时空分数阶复 Ginzburg-Landau 方程描述了光脉冲在非线性光纤中的传播，在光孤子通信等领域具有重大研究价值。时空分数阶复 Ginzburg-Landau 方程是非线性光学和光孤子通信研究的一个主要课题，且在 Benard 对流问题、Taylor-couette 流动、量子场论等领域中有着广泛应用，该方程的解也有助于研究与设计光纤扩大器和光脉冲压缩器。文献[13]通过同解变型法并结合高阶辅助方程法取得了该方程的亮孤子解、暗孤子解、奇异孤子解、周期解和扭结波解。文献[14]通过 exp-function 法取得了该方程的广义孤立波解、广义扭结型孤立波解、周期解。文献[15]通过 $(G'/G, 1/G)$ 展开法取得了该方程的双曲函数解、三角函数解、有理函数解。文献[16]通过扩展辅助方程取得了该方程的 Jacobi 椭圆函数解、双曲函数解、三角函数解。文献[17]通过新映射法取得了该方程的双曲函数解、三角函数解等。文献[18]通过修正的 Kudryashov 方法取得了该方程的精确解。文献[19]通过 Riccati-Bernoulli 辅助方程取得了该方程的三角函数解、暗孤子解、有理函数解、指数函数解。

由 Liu 提出的求非线性发展方程行波解的新方法：多项式完全判别系统法[20-22]是一种简单而又基本的方法，利用该方法能得到方程的所有单行波解的完整分类。从前面有限的综述可知，尚未见到用该方法来研究时空分数阶复 Ginzburg-Landau 方程的文献，本文则利用该方法来研究方程(1)，以图求得一些新的解结构。

1　预备知识

本节将介绍整合分数阶导数的相关性质。

设 $f: [0, \infty) \to \mathbf{R}$，$f$ 的 α 阶导数定义如下：

定义 1

$$D_t^{\alpha}f(t) = \lim_{\varepsilon \to 0}\frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \qquad t > 0,\ 0 < \alpha \leqslant 1.$$

设 $\alpha \in (0,1]$，$f(t), g(t)$ 在 $t > 0$ 时可微，则有如下性质：

1) $D_t^{\alpha}(\lambda) = 0$，$\lambda$ 为常数；

2) $D_t^{\alpha}(t^{\mu}) = \mu t^{\mu-\alpha}$，$\mu \in \mathbf{R}$；

3) $D_t^{\alpha}(af(t) + bg(t)) = aD_t^{\alpha}f(t) + bD_t^{\alpha}g(t)$，$a, b \in \mathbf{R}$；

4) $D_t^{\alpha}(f(t)g(t)) = f(t)D_t^{\alpha}g(t) + g(t)D_t^{\alpha}f(t)$；

5) $D_t^{\alpha}\left(\dfrac{f(t)}{g(t)}\right) = \dfrac{g(t)D_t^{\alpha}f(t) - f(t)D_t^{\alpha}g(t)}{g(t)^2}$，若 $f(t)$ 可微，则有 $D_t^{\alpha}(f(t)) = t^{1-\alpha}\dfrac{\mathrm{d}f(t)}{\mathrm{d}t}$.

2　时空分数阶复 Ginzburg-Landau 方程的精确解

设方程(1)的解为如下形式：

$$u(x, z, \tau) = v^{1/2}(x, \tau)\mathrm{e}^{\mathrm{i}[kz^{\delta}/\delta + \varphi(x, \tau)]}, \tag{2}$$

其中 $\varphi(x, \tau) = \varphi(\eta)$，$\eta = q_0x^{\beta}/\beta - q_1\tau^{\alpha}/\alpha$，$v(x, \tau) = v(\xi)$，$\xi = p_0x^{\beta}/\beta - p_1\tau^{\alpha}/\alpha$，$k$ 是实数。

将式(2)代入方程(1)得

$$(v')^2 = a_4v^4 + a_3v^3 + a_2v^2 + a_1v, \tag{3}$$

其中 $\qquad a_4 = \dfrac{8r_2}{3p_1^2},\ a_3 = \dfrac{-4r_1}{p_1^2},\ a_2 = \dfrac{3r_1^2}{2r_2p_1^2},\ a_1 = \dfrac{k_0}{r_2p_1^2}.$

将

$$f=(a_4)^{1/4}\left(v+\frac{a_3}{4a_4}\right),\ \xi_1=(a_4)^{1/4}\xi \tag{4}$$

代入式(3)得

$$f_{\xi_1}^2=f^4+b_2f^2+b_1f+b_0, \tag{5}$$

其中 $b_2=\dfrac{a_2-3a_3^2/(8a_4)}{\sqrt{a_4}},\ b_1=\left(\dfrac{a_3^3}{8a_4^2}-\dfrac{a_2a_3}{2a_4}+a_1\right)(a_4)^{-1/4},\ b_0=\dfrac{-3a_3^4}{256a_4^3}+\dfrac{a_2a_3^2}{16a_4^2}-\dfrac{a_1a_3}{4a_4}$.

所以我们有

$$\pm(\xi_1-\xi_0)=\int\frac{\mathrm{d}f}{\sqrt{f^4+b_2f^2+b_1f+b_0}}. \tag{6}$$

根据四次多项式的完全判别系统有

$$D_1=4,\ D_2=-b_2,\ D_3=-2b_2^3+8b_2b_0-9b_1^2,$$

$$D_4=-b_2^3b_1^2+4b_2^4b_0+36b_2b_1^2-32b_2^2b_0^2-\frac{27}{4}b_1^4+64b_0^3,\ E_2=9b_2^2-32b_2b_0.$$

分别讨论以下 9 种情况.

情况 1 当 $D_2=0,D_3=0,D_4=0$ 时,有 $(f')^2=f^4$, 可得

$$f_1=-\frac{1}{\xi_1-\xi_0}. \tag{7}$$

则方程(1)的精确解为

$$u_1=\left(\left(-\frac{1}{\xi_1-\xi_0}\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)}. \tag{8}$$

情况 2 当 $D_2<0,D_3=0,D_4=0$ 时,有 $(f')^2=(f^2+s^2)^2$, 其中 s 是正实数,可得

$$f_2=s\tan(s(\xi_1-\xi_0)). \tag{9}$$

则方程(1)的精确解为

$$u_2=\left(s\tan(s(\xi_1-\xi_0))a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)}. \tag{10}$$

情况 3 当 $D_2>0,D_3=0,D_4=0,E_2>0$ 时,有 $(f')^2=(f-s)^2(f-h)^2,s+h=0$, 其中 $s>0$, 则

$$(\xi_1-\xi_0)=\int\frac{\mathrm{d}f}{(f-s)(f-h)}=\frac{1}{f-s}\ln\left|\frac{f-h}{f-s}\right|. \tag{11}$$

可得

$$f_3=\frac{h-s}{2}\left[\coth\frac{(s-h)(\xi_1-\xi_0)}{2}-1\right]+s,\qquad f>s\quad \text{or}\quad f<h, \tag{12}$$

$$f_4=\frac{h-s}{2}\left[\tanh\frac{(h-s)(\xi_1-\xi_0)}{2}-1\right]+s,\qquad h<f<s. \tag{13}$$

则方程(1)的精确解为

$$u_3=\left(\left(\frac{h-s}{2}\left[\coth\frac{(s-h)(\xi_1-\xi_0)}{2}-1\right]+s\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)}, \tag{14}$$

$$u_4=\left(\left(\frac{h-s}{2}\left[\tanh\frac{(h-s)(\xi_1-\xi_0)}{2}-1\right]+s\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)}. \tag{15}$$

情况 4 当 $D_2>0,D_3=0,D_4=0,E_2=0$ 时,有 $(f')^2=(\xi_1-s)^3(\xi_1-h),h+3s=0$, 可得

$$f_5=\frac{4(s-h)}{(s-h)^2(\xi_1-\xi_0)^2-4}+s. \tag{16}$$

则方程(1)的精确解为

$$u_5=\left(\left(\frac{4(s-h)}{(s-h)^2(\xi_1-\xi_0)^2-4}+s\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)}. \tag{17}$$

情况 5　当 $D_2D_3<0, D_4=0$ 时，有 $(f')^2=(f-s)^2[(f^2+s)^2+h^2]$，其中 s 和 h 是正数，则有

$$\xi_1-\xi_0=\int\frac{\mathrm{d}f}{(f-s)\sqrt{(f+s)^2+h^2}}=\frac{1}{\sqrt{4s^2+h^2}}\ln\left|\frac{\pi f+\sigma-\sqrt{(f+s)^2+h^2}}{f-s}\right|, \tag{18}$$

其中　$\pi=\dfrac{3s}{\sqrt{4s^2+h}},\ \sigma=\sqrt{4s^2+h^2}-\dfrac{3s^2}{\sqrt{4s^2+h^2}}$.

可得

$$f_6=\frac{\mathrm{e}^{\sqrt{(4s^2+h^2)}(\xi_1-\xi_0)}-\pi+\sqrt{(4s^2+h^2)}(2-\pi)}{(\mathrm{e}^{\sqrt{(4s^2+h^2)}(\xi_1-\xi_0)}-\pi)^2-1}. \tag{19}$$

则方程(1)的精确解为

$$u_6=\left(\left(\frac{\mathrm{e}^{\sqrt{(4s^2+h^2)}(\xi_1-\xi_0)}-\pi+\sqrt{(4s^2+h^2)}(2-\pi)}{(\mathrm{e}^{\sqrt{(4s^2+h^2)}(\xi_1-\xi_0)}-\pi)^2-1}\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)}. \tag{20}$$

情况 6　当 $D_2>0, D_3>0, D_4>0$ 时，有

$$(f')^2=(f-\alpha_1)(f-\alpha_2)(f-\alpha_3)(f-\alpha_4),\qquad \alpha_1+\alpha_2+\alpha_3+\alpha_4=0,\ \alpha_1>\alpha_2>\alpha_3>\alpha_4.$$

当 $\alpha_4>0, f>\alpha_1$ 或 $\alpha_4>0, f<\alpha_4$ 时，可得

$$f_7=\frac{\alpha_2(\alpha_1-\alpha_4)\mathrm{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_1\right)-\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_4)\mathrm{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_1\right)-(\alpha_2-\alpha_4)}. \tag{21}$$

当 $\alpha_4>0, \alpha_3<f<\alpha_2$ 时，可得

$$f_8=\frac{\alpha_4(\alpha_2-\alpha_3)\mathrm{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_1\right)-\alpha_3(\alpha_2-\alpha_4)}{(\alpha_2-\alpha_3)\mathrm{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_1\right)-(\alpha_2-\alpha_4)}. \tag{22}$$

当 $\alpha_4<0, \alpha_2<f<\alpha_1$ 时，可得

$$f_9=\frac{\alpha_3(\alpha_1-\alpha_2)\mathrm{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_2\right)-\alpha_2(\alpha_1-\alpha_3)}{(\alpha_1-\alpha_2)\mathrm{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_2\right)-(\alpha_1-\alpha_3)}. \tag{23}$$

当 $\alpha_4<0, f>\alpha_1$ 或 $\alpha_4<0, f<\alpha_4$ 时，可得

$$f_{10}=\frac{\alpha_1(\alpha_3-\alpha_4)\mathrm{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_2\right)-\alpha_4(\alpha_3-\alpha_1)}{(\alpha_3-\alpha_4)\mathrm{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_2\right)-(\alpha_3-\alpha_1)}. \tag{24}$$

则方程(1)的精确解为

$$u_7=\left(\left(\frac{\alpha_2(\alpha_1-\alpha_4)\mathrm{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_1\right)-\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_4)\mathrm{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_1\right)-(\alpha_2-\alpha_4)}\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\times$$

$$\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)}, \tag{25}$$

$$u_8=\left(\left(\frac{\alpha_4(\alpha_2-\alpha_3)\operatorname{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_1\right)-\alpha_3(\alpha_2-\alpha_4)}{(\alpha_2-\alpha_3)\operatorname{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_1\right)-(\alpha_2-\alpha_4)}\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\times$$

$$\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)},\tag{26}$$

$$u_9=\left(\left(\frac{\alpha_3(\alpha_1-\alpha_2)\operatorname{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_2\right)-\alpha_2(\alpha_1-\alpha_3)}{(\alpha_1-\alpha_2)\operatorname{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_2\right)-(\alpha_1-\alpha_3)}\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\times$$

$$\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)},\tag{27}$$

$$u_{10}=\left(\left(\frac{\alpha_1(\alpha_3-\alpha_4)\operatorname{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_2\right)-\alpha_4(\alpha_3-\alpha_1)}{(\alpha_3-\alpha_4)\operatorname{sn}^2\left(\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi_1-\xi_0),m_2\right)-(\alpha_3-\alpha_1)}\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\times$$

$$\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)},\tag{28}$$

其中 $m_1^2=\dfrac{(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)},\ m_2^2=\dfrac{(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}$.

情况 7 当 $D_2D_3\geqslant 0,D_4<0$ 时,有 $(f')^2=(f-a_1)(f-a_2)[(f-l)^2+s^2]$,$a_1>a_2,\ s>0$,且有如下定义:

$$a=\frac{1}{2}(a_1+a_2)c-\frac{1}{2}(a_1-a_2)d,\ b=\frac{1}{2}(a_1+a_2)d-\frac{1}{2}(a_1-a_2)c,$$

$$c=a_1-l-\frac{s}{m_1},\ d=a_1-l-sm_1,\ E=\frac{s^2+(a_1-l)(a_2-l)}{s(a_1-a_2)},\ m_1=E+\sqrt{E^2+1}.$$

可得

$$f_{11}=\frac{ac\mathrm{n}\left(\frac{\sqrt{2sm_1(a_1-a_2)}}{2mm_1}(\xi_1-\xi_0),m\right)+b}{cc\mathrm{n}\left(\frac{\sqrt{2sm_1(a_1-a_2)}}{2mm_1}(\xi_1-\xi_0),m\right)+d}.\tag{29}$$

则方程(1)的精确解为

$$u_{11}=\left(\left(\frac{ac\mathrm{n}\left(\frac{\sqrt{2sm_1(a_1-a_2)}}{2mm_1}(\xi_1-\xi_0),m\right)+b}{cc\mathrm{n}\left(\frac{\sqrt{2sm_1(a_1-a_2)}}{2mm_1}(\xi_1-\xi_0),m\right)+d}\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)},\tag{30}$$

其中 $m^2=\dfrac{1}{1+m_1^2}$.

情况 8 当 $D_2D_3\leqslant 0,D_4>0$ 时,有$(f')^2=[(f-a_1)^2+s^2][(f-a_2)^2+l^2]$,$s\geqslant l>0$,且有如下定义:

$$a=a_1c+sd,\ b=a_1d-sc,\ c=-s-l/m_1,$$

$$d=a_1-a_2,\ E=\frac{(a_1-a_2)^2+s^2+l^2}{2sl},\ m_1=E+\sqrt{E^2-1}.$$

可得

$$f_{12}=\frac{a\mathrm{sn}(\xi(\xi_1-\xi_0),m)+b\mathrm{cn}(\xi(\xi_1-\xi_0),m)}{c\mathrm{sn}(\xi(\xi_1-\xi_0),m)+d\mathrm{cn}(\xi(\xi_1-\xi_0),m)}. \tag{31}$$

则方程(1)的精确解为

$$u_{12}=\left(\left(\frac{a\mathrm{sn}(\zeta(\xi_1-\xi_0),m)+b\mathrm{cn}(\zeta(\xi_1-\xi_0),m)}{c\mathrm{sn}(\zeta(\xi_1-\xi_0),m)+d\mathrm{cn}(\zeta(\xi_1-\xi_0),m)}\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)}, \tag{32}$$

其中　$m^2=\dfrac{m_1^2-1}{m_1^2},\ \zeta=\dfrac{\sqrt{(c^2+d^2)(m_1^2c^2+d^2)}}{c^2+d^2}$.

情况 9　当 $D_2,D_3>0,D_4=0$ 时,有 $(f')^2=(f-\alpha_1)(f-\alpha_2)(f-\alpha_3)^2$,$\alpha_1>\alpha_2$,且 α_i 是三个不同的实数.令

$$s=\frac{\alpha_1-\alpha_2}{2}\left(\frac{\alpha_1+\alpha_2}{2}-\alpha_3\right),$$

可得

$$f_{13}=\frac{2s_1}{1-\mathrm{e}^{-\sqrt{s^2-1}(\xi_1-\xi_0)}}+s,\qquad s^2-1>0, \tag{33}$$

$$f_{14}=\frac{1-s}{1+s}\tan\frac{\xi_1-\xi_0}{\sqrt{1-s^2}},\qquad s^2-1<0. \tag{34}$$

则方程(1)的精确解为

$$u_{13}=\left(\left(\frac{2s_1}{1-\mathrm{e}^{-\sqrt{s^2-1}(\xi_1-\xi_0)}}+s\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)}, \tag{35}$$

$$u_{14}=\left(\left(\frac{1-s}{1+s}\tan\frac{\xi_1-\xi_0}{\sqrt{1-s^2}}\right)a_4^{-1/4}-\frac{a_3}{4a_4}\right)^{1/2}\mathrm{e}^{\mathrm{i}\left(\frac{kz^\delta}{\delta}+\frac{q_0x^\beta}{\beta}-\frac{q_1\tau^\alpha}{\alpha}\right)}. \tag{36}$$

3　结　　论

本文首先通过整合分数阶导数与分数阶复变换,将时空分数阶复 Ginzburg-Landau 方程转化为常微分方程,再利用多项式的完全判别系统方法得到了时空分数阶复 Ginzburg-Landau 方程的 14 组精确解,具体与前述文献[11-17]的结果比较所得,解 $u_2,u_5,u_6,u_7,u_8,u_9,u_{10},u_{11},u_{12},u_{13},u_{14}$ 在上述文献中并未出现.所得结论验证了多项式完全判别系统方法的可行性,此方法对于探究更多分数阶非线性偏微分方程的精确解具有普适性.

参考文献(References):

[1] EKICI M, MIRZAZADEH M, SONMEZOGLU A, et al. Dark and singular optical solitons with Kundu-Eckhaus equation by extended trial equation method and extended G'/G-expansion scheme[J]. *Optik*, 2016, **127**(22): 10490-10497.

[2] VONG S, LYU P. On numerical contour integral method for fractional diffusion equations with variable coefficients[J]. *Applied Mathematics Letters*, 2017, **64**: 137-142.

[3] SAHADEVAN R, PRAKASH P. Exact solutions and maximal dimension of invariant subspaces of time fractional coupled nonlinear partial differential equations[J]. *Communications in Nonlinear Science and Numerical Simulation*, 2017, **42**: 158-177.

[4] ESLAMI M, REZAZADEH H, REZAZADEH M, et al. Exact solutions to the space-time fractional Schrödinger-Hirota equation and the space-time modified KDV-Zakharov-Kuznetsov equation[J]. *Optical & Quantum Electronics*, 2017, **49**(8): 279-293.

[5] GUNER O, BEKIR A. Theexp-function method for solving nonlinear space-time fractional differential equations in mathematical physics[J]. *Journal of the Association of Arab Universities for Basic and Applied Sciences*, 2017, **24**: 277-282.

[6] SHAMSELDEEN S. Approximate solution of space and time fractional higher order phase field equation[J]. *Physica*, 2018, **494**: 308-316.

[7] HASSEINE A, BART H J. Adomian decomposition method solution of population balance equations for aggregation, nucleation, growth and breakup processes[J]. *Applied Mathematical Modelling*, 2015, **39**(7): 1975-1984.

[8] 张雪, 孙峪怀. (3+1)维时间分数阶 KdV-Zakharov-Kuznetsov 方程的分支分析及其行波解[J]. 应用数学和力学, 2019, **40**(12): 1345-1355.(ZHANG Xue, SUN Yuhuai. Dynamical analysis and solutions for (3+1)-dimensional time fractional KdV-Zakharov-Kuznetsov equations[J]. *Applied Mathematics and Mechanics*, 2019, **40**(12): 1345-1355.(in Chinese))

[9] 江林, 孙峪怀, 张雪, 等. (2+1)维时空分数阶 Nizhnik-Novikov-Veslov 方程的精确行波解及其分支[J]. 应用数学和力学, 2018, **39**(11): 1313-1322.(JIANG Lin, SUN Yuhuai, ZHANG Xue, et al. Exact traveling wave solutions and bifurcations of (2+1)-dimensional space-time fractional-order Nizhnik-Novikov-Veslov equations[J]. *Applied Mathematics and Mechanics*, 2018, **39**(11): 1313-1322.(in Chinese))

[10] SAKAGUCHI H, MALOMED B A. Stable localized pulses and zigzag stripes in a two-dimensional diffractive-diffusive Ginzburg-Landau equation[J]. *Physica D*, 2001, **159**(1/2): 91-100.

[11] SHI Y Q, DAI Z D, HAN S. Exact solutions for 2D cubic-quintic Ginzburg-Landau equation[J]. *Journal of Physics Conference*, 2008, **96**: 012148.

[12] WAINBLAT G, MALOMED B A. Interactions between two-dimensional solitons in the diffractive-diffusive Ginzburg-Landau equation with the cubic-quintic nonlinearity[J]. *Physica D*, 2009, **238**(14): 1143-1151.

[13] 施业琼. 2+1 维 Ginzburg-Landau 方程的精确波解[J]. 数学的实践与认识, 2009, **39**(16): 247-251.(SHI Yeqiong. The exact wave solutions for 2+1 dimensional cubic-quintic Ginzburg-Landau equation[J]. *Mathematics in Practice and Theory*, 2009, **39**(16): 247-251.(in Chinese))

[14] SHI Y Q, DAI Z D, LI D L. Application of exp-function method for 2D cubic-quintic Ginzburg-Landau equation[J]. *Applied Mathematics and Computation*, 2009, **210**(1): 269-275.

[15] ZAYED E M E, ALURRFI K A E. On solving two higher-order nonlinear PDEs describing the propagation of optical pulses in optic fibers using the (G'/G, $1/G$)-expansion method[J]. *Ricerche di Matematica*, 2015, **64**(1): 167-194.

[16] ZAYED E M E, ALURRFI K A E. Extended auxiliary equation method and its applications for finding the exact solutions for a class of nonlinear Schrödinger-type equations[J]. *Applied Mathematics & Computation*, 2016, **289**: 111-131.

[17] ZAYED E M E, AL-NOWEHY A G. Solitons and other exact solutions for a class of nonlinear Schrödinger-type equations[J]. *Optik*, 2017, **130**, 1295-1311.

[18] SIRISUBTAWEE S, KOONPRASERT S, SUNGNUL S, et al. Exact traveling wave solutions of the space-time fractional complex Ginzburg-Landau equation and the space-time fractional Phi-4 equation using reliable methods[J]. *Advances in Difference Equations*, 2019, **2019**(1): 219. DOI:10.1186/s13662-019-2154-9.

[19] 石兰芳, 王明灿, 钱正雅. 应用 Riccati-Bernoulli 辅助方程求解广义非线性 Schrödinger 方程和(2+1)维非线性 Ginzburg-Landau 方程[J]. 应用数学和力学, 2020, **41**(7): 786-795.(SHI Lanfang, WANG Mingcan, QIAN Zhengya. Solution of generalized nonlinear Schrödinger equations and (2+1)-dimensional nonlinear Ginzburg-Landau equations with a Riccati-Bernoulli auxiliary equation method[J]. *Applied Mathematics and Mechanics*, 2020, **41**(7): 786-795.(in Chinese))

[20] LIU C S. Exact travelling wave solutions for (1+1)-dimensional dispersive long wave equation[J]. *Chinese Physics B*, 2005, **14**(9): 1710-1715.

[21] LIU C S. Classification of all single travelling wave solutions to Calogero-Degasperis-Focas equation[J]. *Communications in Theoretical Physics*, 2007, **48**(10): 601-604.

[22] LIU C S. The classification of travelling wave solutions and superposition of multi-solutions to Camassa-Holm equation with dispersion[J]. *Chinese Physics B*, 2007, **16**(7): 1832-1837.