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(3+1)维变系数 Kudryashov-Sinelshchikov(K-S)方程的同宿呼吸波解和高阶怪波解*

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摘要： 基于 Hirota 双线性方法,利用拓展的同宿呼吸检验法得到了(3+1)维变系数 Kudryashov-Sinelshchikov(K-S)方程的同宿呼吸波解,对该解的参数选取合适的数值,可得到不同结构的同宿呼吸波.通过对同宿呼吸波解的周期取极限,推导出方程的怪波解.最后,构造出一个特殊的高阶多项式作为测试函数,求得该方程的一阶怪波解和二阶怪波解.

关　键　词： (3+1)维变系数 Kudryashov-Sinelshchikov(K-S)方程；　Hirota 双线性方法；　呼吸解；　怪波解

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Homoclinic Breathing Wave Solutions and High-Order Rogue Wave Solutions of (3+1)-Dimensional Variable Coefficient Kudryashov-Sinelshchikov Equations

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Abstract: Based on the Hirota bilinear method, the homoclinic breathing wave solutions to the (3+1)-dimensional variable coefficient Kudryashov-Sinelshchikov (K-S) equations were obtained by means of the extended homoclinic breathing test method. Homoclinic breathing waves with different structures were given through selection of appropriate values for the parameters of the solution, and the rogue wave solutions to the equation were derived under the limit of the periodicity of the homoclinic breathing wave solutions. Finally, a special high-order polynomial was constructed as a test function to obtain the 1st-order and the 2nd-order rogue wave solutions.

Key words: (3+1)-dimensional variable coefficient Kudryashov-Sinelshchikov equation; Hirota bilinear method; breathing solution; rogue wave solution

引　言

随着科学技术的发展,非线性在自然科学领域和社会科学领域的作用越来越重要,因此科学家们越来越

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关注非线性问题的研究.在非线性问题中,非线性发展方程的求解占有很重要的地位.近年来,已经发现许多求解非线性发展方程的方法,比如 Hirota 双线性方法[1-2]、Darboux 变换法[3]、辅助方程法[4]和双曲函数法[5]等.其中,基于 Hirota 双线性方法我们可以获得方程的 lump 解、呼吸解、怪波解[6-8]等,由于 lump 解、呼吸解、怪波解在流体力学、等离子体物理和非线性光学中广泛应用而越来越受到研究者的关注,所以已经成为目前研究的热点之一.

Kudryashov 和 Sinelshchikov 考虑了含有气泡的液体中的非线性波,建立了描述非线性压力波的(3+1)维 K-S 方程[9]:

$$(u_t + uu_x + u_{xxx})_x + \frac{1}{2}(u_{yy} + u_{zz}) = 0. \tag{1}$$

目前,已有学者用 sine-cosine 法和修正的截断展开法[10]、Hirota 双线性方法[11]、修正的 tanh-coth 方法结合 Riccati 方程[12]等方法进行研究,得到了该方程的孤子解和精确解.非线性发展方程的系数是随着时间和空间变化的,因此变系数发展方程所描述的演化规律更接近实际情况,所以本文将方程(1)拓展为(3+1)维变系数 K-S 方程:

$$u_{xt} + \alpha(t)(uu_x)_x + \beta(t)u_{yy} + \gamma(t)u_{zz} - \rho(t)u_{xxxx} = 0, \tag{2}$$

其中 $u(x,y,z,t)$ 是关于 x,y,z,t 的函数, $\alpha(t),\beta(t),\gamma(t),\rho(t)$ 是关于 t 的任意函数型参数.文献[13]通过建立非线性波符号计算方法,给出了(3+1)维变系数 K-S 方程的自 Bäcklund 变换和孤子解.

本文根据 Hirota 双线性方法,得到了(3+1)维变系数 K-S 方程的双线性形式,利用拓展的同宿呼吸检验法[14-15]求得该方程的同宿呼吸波解,通过给待定参数赋予不同的数值得到不同结构的同宿呼吸波.随后,借助双曲余弦函数的 Maclaurin 公式对待定参数赋值,并对同宿呼吸波解的周期取极限,从而得到了有理呼吸波解,发现此有理呼吸波解为方程(2)的怪波解.文献[16-17]引入一种可用于讨论非线性发展方程的 N 阶怪波的符号计算方法,基于上述方法,本文构造出了一个特殊的高阶多项式,利用此多项式同样可以得到 N 阶怪波,这里我们讨论了一阶怪波解和二阶怪波解,最后借助 MATHEMATICA 软件给出所求波解的图像.

1　同宿呼吸波解

设

$$u(x,y,z,t) = u(\xi,z), \tag{3}$$

其中 $\xi = kx + my + \omega t$,这里 k,m,ω 均为实数.将式(3)代入方程(2)求得

$$\omega k u_{\xi\xi} + \alpha(t)k^2(u_\xi^2 + uu_{\xi\xi}) + \beta(t)m^2 u_{\xi\xi} + \gamma(t)u_{zz} - \rho(t)k^4 u_{\xi\xi\xi\xi} = 0. \tag{4}$$

通过变换

$$u(\xi,z) = 2(\ln f)_{\xi\xi}, \tag{5}$$

可以将方程(4)转换为双线性形式:

$$[(k\omega + \beta(t)m^2)D_\xi^2 + \gamma(t)D_z^2 - \rho(t)k^4 D_\xi^4] f\cdot f = 0, \tag{6}$$

其中 $\alpha(t) = 6\rho(t)k^2$,这里的 D 为双线性微分算子,其定义为

$$D_t^m D_x^n a\cdot b \equiv \frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n} a(t+s,x+y)b(t-s,x-y)\mid_{s=0,y=0},\qquad m,n = 0,1,2,3,\cdots. \tag{7}$$

为了计算呼吸解,选择拓展的同宿测试函数[14-15]:

$$f(\xi,z) = 1 + A(t)(e^{-i\xi p(t)} + e^{i\xi p(t)})e^{\lambda(t)+\Omega(t)z} + B(t)e^{2(\lambda(t)+\Omega(t)z)}, \tag{8}$$

这里 $p(t),\lambda(t),\Omega(t),A(t),B(t)$ 是关于 t 的待定函数,将式(8)代入方程(6)中,再令 $e^{3\lambda(t)+3\Omega(t)z\pm i\xi p(t)}$, $e^{\lambda(t)+\Omega(t)z\pm i\xi p(t)}$ 和 $e^{2(\lambda(t)+\Omega(t))}$ 的系数为零,可得

$$\begin{cases} B(t) = \dfrac{A^2(t)(4k^4p^2(t)\rho(t) + k\omega + m^2\beta(t))}{k^4p^2(t)\rho(t) + k\omega + m^2\beta(t)}, \\ \Omega^2(t) = \dfrac{p^2(t)(k^4p^2(t)\rho(t) + k\omega + m^2\beta(t))}{\gamma(t)}. \end{cases} \tag{9}$$

将式(8)代入式(5)得到

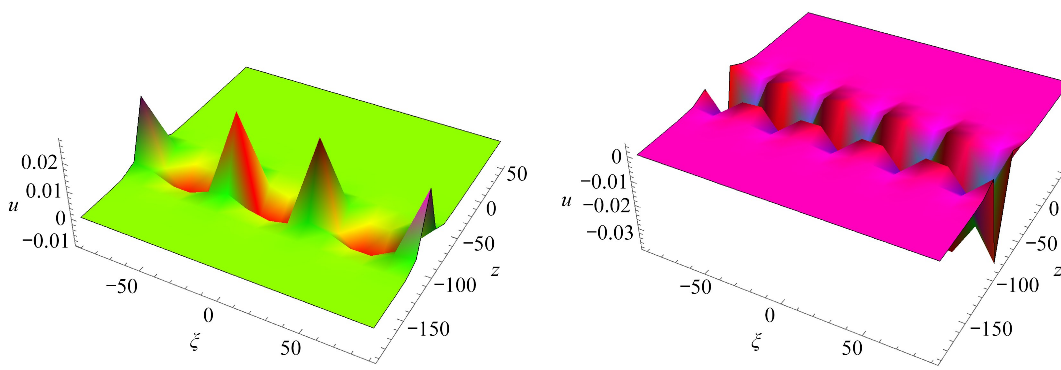
$$u(\xi,z)=-\frac{4A(t)p^2(t)\mathrm{e}^{\lambda(t)+\Omega(t)z}(2A(t)\mathrm{e}^{\lambda(t)+\Omega(t)z}+\cos(\xi p(t))(B(t)\mathrm{e}^{2(\lambda(t)+\Omega(t)z)}+1))}{(2A(t)\cos(\xi p(t))\mathrm{e}^{\lambda(t)+\Omega(t)z}+(B(t)\mathrm{e}^{2(\lambda(t)+\Omega(t)z)}+1))^2}. \tag{10}$$

上式也可以写成以下形式:

$$u(\xi,z)=-\frac{2A(t)p^2(t)(A(t)+\sqrt{B(t)}\cos(\xi p(t))\cosh(\ln(B(t))/2+\lambda(t)+\Omega(t)z))}{(A(t)\cos(\xi p(t))+\sqrt{B(t)}\cosh(\ln(B(t))/2+\lambda(t)+\Omega(t)z))^2}, \tag{11}$$

这里 $B(t)$ 和 $\Omega(t)$ 可由式(9)求得.式(11)是同宿波,也是呼吸波,由于同宿波和呼吸波在同一方向上相互作用,所以形成了同宿呼吸波.当 $z\to\pm\infty$ 时,$u(\xi,z)$ 趋于一个固定值 0,且同宿呼吸波的周期为 $2\pi/p(t)$.

情形 1 取 $p(t)=1/t, A(t)=t^2, \lambda(t)=t, \beta(t)=9t, \gamma(t)=t^2, \rho(t)=19t$,这里 $t>1$.图 1 为同宿呼吸波的图像,可以看出图 1(a)的呼吸波在波面之上,图 1(b)的呼吸波在波面之下.



(a) $m=0.7$, $t=9.14$, $\omega=8.3$, $k=2.4$ (b) $m=0.6$, $t=6.12$, $\omega=-1$, $k=0.7$

图 1 方程(2)同宿呼吸波的三维图

Fig. 1 The 3D map of the homoclinic respiratory wave of eq. (2)

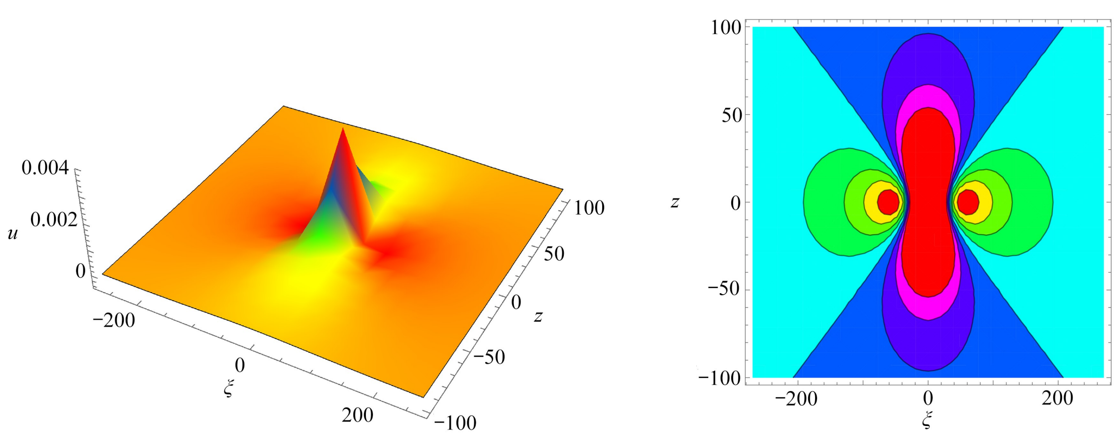


图 2 方程(2)在 $m=3.7$, $t=30$, $\omega=-13.5$, $k=-6.9$ 时怪波的三维图和等高线图

Fig. 2 The 3D map and the contour map of the rogue wave of eq. (2) for $m=3.7$, $t=30$, $\omega=-13.5$, $k=-6.9$

情形 2 根据双曲余弦函数的 Maclaurin 公式:

$$\cosh x=1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+\frac{x^8}{40\,320}+\cdots+\frac{x^{2n}}{(2n)!}+o(x^{2n+1}), \tag{12}$$

取

$$A(t)=-\cosh(kp(t))=-1-\frac{k^2p^2(t)}{2}-\frac{k^4p^4(t)}{24}-\frac{k^6p^6(t)}{720}-\frac{k^8p^8(t)}{40\,320}, \tag{13}$$

另外,取 $\lambda(t)=0, \beta(t)=9t, \gamma(t)=t^2, \rho(t)=19t$,当同宿周期波的周期 $2\pi/p(t)$ 趋于无穷,即 $p(t)\to 0$ 时,可以得到有理呼吸波解为

$$u(\xi,z)=\frac{4t^2(kl+9m^2t)((kl+9m^2t)^2z^2-(kl+9m^2t)t^2\xi^2+57k^4t^3)}{((kl+9m^2t)^2z^2+(kl+9m^2t)t^2\xi^2+57k^4t^3)^2}. \tag{14}$$

如图 2 所示,根据图像可以看出其波高远远大于其周围波的波高,由于波高大于有效波高 2 倍(或 2.2 倍)的单波可以称为怪波,所以可以说图 2 是方程(2)的怪波,因此式(14)为方程(2)的怪波解.

2　高阶怪波解

为了构造方程(2)的高阶怪波解,根据文献[16-17]的符号计算方法,构造一个新的高阶多项式:

$$f = F_n(\xi,z) = \sum_{j=0}^{n(n+1)/2}\sum_{i=0}^{j}\left(a_{n(n+1)-2j,2i}(t)\xi^{2i}z^{n(n+1)-2j} + a_{n(n+1)/2-j,3i}(t)\xi^{i}z^{n(n+1)/2-j}\right), \tag{15}$$

这里 $a_{n(n+1)-2j,2i}(t)$ 和 $a_{n(n+1)/2-j,3i}(t)$ 是关于 t 的待定函数.

2.1　一阶怪波解

当 $n = 1$ 时,可得

$$f = F_1(\xi,z) = a_{0,2}(t)\xi^2 + a_{0,3}(t)\xi + a_{2,0}(t)z^2 + a_{1,0}(t)z + 2a_{0,0}(t). \tag{16}$$

将式(16)代入方程(6)中并取 ξ 和 z 的所有系数为零,得到

$$\begin{cases} a_{1,0}(t) = -\dfrac{\sqrt{(8a_{0,0}(t)a_{0,2}(t) - a_{0,3}^2(t))(k\omega + m^2\beta(t)) - 12k^4a_{0,2}^2(t)\rho(t)}}{\sqrt{\gamma(t)}}, \\ a_{2,0}(t) = \dfrac{a_{0,2}(t)(k\omega + m^2\beta(t))}{\gamma(t)}. \end{cases} \tag{17}$$

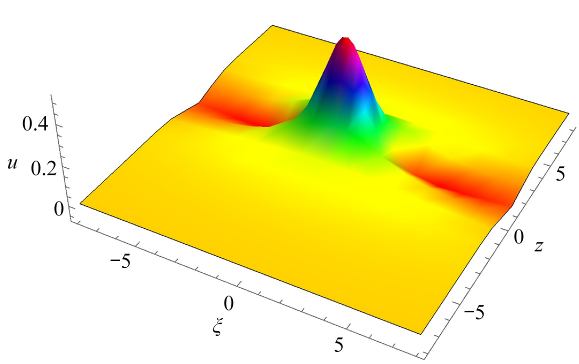
则式(16)可以写成

$$f = F_1(\xi,z) = -\frac{z\sqrt{(8a_{0,0}(t)a_{0,2}(t) - a_{0,3}^2(t))(k\omega + m^2\beta(t)) - 12k^4a_{0,2}^2(t)\rho(t)}}{\sqrt{\gamma(t)}} + \frac{z^2a_{0,2}(t)(k\omega + m^2\beta(t))}{\gamma(t)} + a_{0,2}(t)\xi^2 + a_{0,3}(t)\xi + 2a_{0,0}(t). \tag{18}$$

将式(18)代入式(5),可得到方程(2)的一阶怪波解为

$$u(\xi,z) = 2\ln(F_1(\xi,z))_{\xi\xi}. \tag{19}$$

图 3 展示了方程(2)在 $a_{0,2}(t) = 6t, a_{0,3}(t) = t, a_{0,0}(t) = t^3, \beta(t) = 9t, \gamma(t) = t^2, \rho(t) = 19t$ 时一阶怪波解的三维图和等高线图.



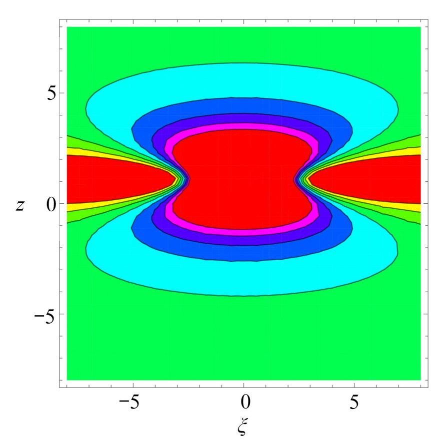


图 3　方程(2)在 $m = 3.65$, $t = 8.56$, $\omega = 4.25$, $k = 2$ 时一阶怪波的三维图和等高线图

Fig. 3　The 3D map and the contour map of the 1st-order rogue wave of eq. (2) for $m = 3.65$, $t = 8.56$, $\omega = 4.25$, $k = 2$

2.2　二阶怪波解

当 $n = 2$ 时,可得

$$f = F_2(\xi,z) = a_{0,6}(t)\xi^6 + a_{0,4}(t)\xi^4 + a_{0,9}(t)\xi^3 + a_{0,2}(t)\xi^2 + a_{0,6}(t)\xi^2 + a_{0,3}(t)\xi + a_{6,0}(t)z^6 + a_{4,2}(t)\xi^2z^4 + a_{4,0}(t)z^4 + a_{3,0}(t)z^3 + a_{2,4}(t)\xi^4z^2 + a_{2,2}(t)\xi^2z^2 + a_{2,3}(t)\xi z^2 + 2a_{2,0}(t)z^2 +$$

$$a_{1,6}(t)\xi^2 z + a_{1,3}(t)\xi z + a_{1,0}(t)z + 2a_{0,0}(t). \tag{20}$$

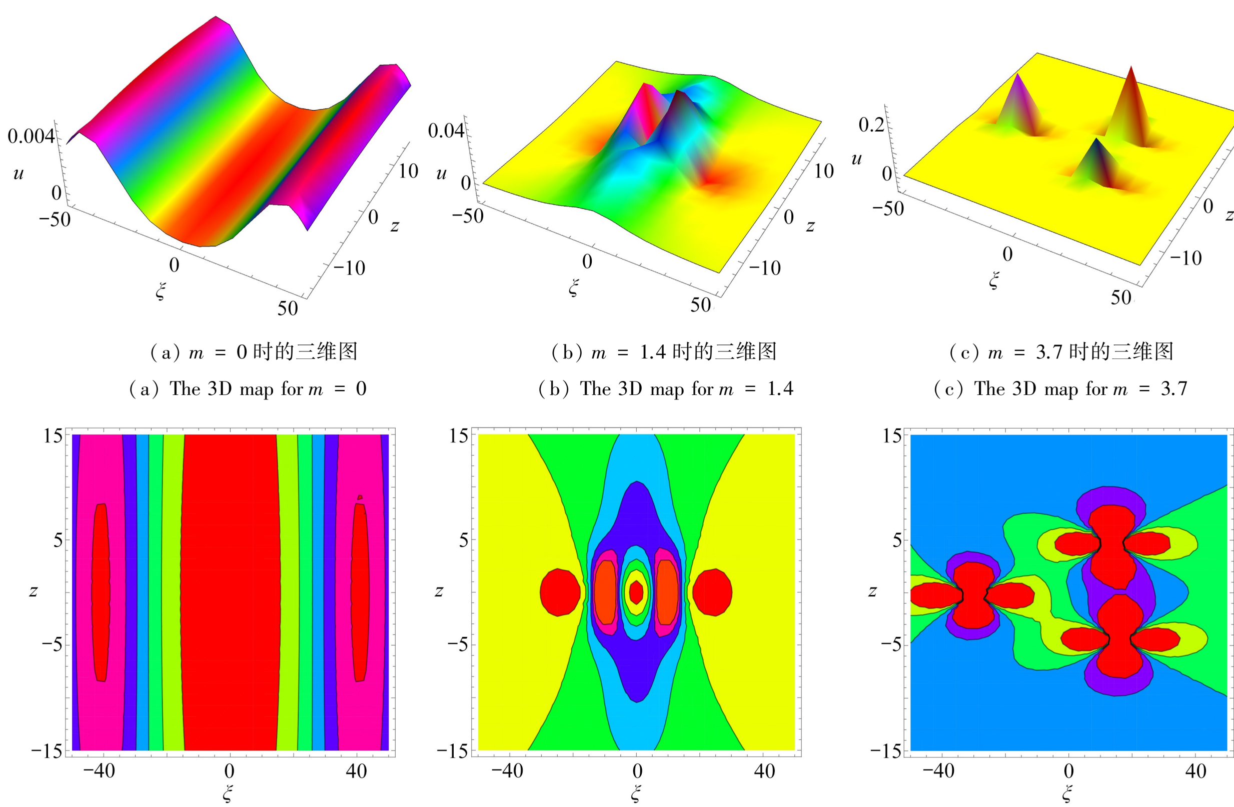
将式(20)代入方程(6)中并取 ξ 和 z 的所有系数为零,可以得到参数的一组约束关系:

$$\begin{cases} a_{0,0}(t) = \dfrac{67\,500k^{12}a_{6,0}^2(t)\gamma^6(t)\rho^3(t) + a_{1,6}^2(t)\gamma(t)(k\omega + m^2\beta(t))^8 + 9a_{0,9}^2(t)(k\omega + m^2\beta(t))^9}{72a_{6,0}(t)\gamma^3(t)(k\omega + m^2\beta(t))^6}, \\ a_{0,2}(t) = \dfrac{a_{6,0}(t)\gamma^3(t)(-125k^8\rho^2(t) - (k\omega + m^2\beta(t))^2)}{(k\omega + m^2\beta(t))^5}, \ a_{0,3}(t) = -\dfrac{k^4a_{0,9}(t)\rho(t)}{k\omega + m^2\beta(t)}, \\ a_{0,4}(t) = \dfrac{25k^4a_{6,0}(t)\gamma^3(t)\rho(t)}{(k\omega + m^2\beta(t))^4}, \ a_{0,6}(t) = \dfrac{a_{6,0}(t)\gamma^3(t)}{(k\omega + m^2\beta(t))^3}, \ a_{1,0}(t) = \dfrac{5k^4a_{1,6}(t)\rho(t)}{3(k\omega + m^2\beta(t))}, \\ a_{1,3}(t) = 0, \ a_{2,0}(t) = \dfrac{475k^8a_{6,0}(t)\gamma^2(t)\rho^2(t)}{2(k\omega + m^2\beta(t))^4}, \ a_{2,2}(t) = \dfrac{90k^4a_{6,0}(t)\gamma^2(t)\rho(t)}{(k\omega + m^2\beta(t))^3}, \\ a_{2,3}(t) = -\dfrac{3a_{0,9}(t)(k\omega + m^2\beta(t))}{\gamma(t)}, \ a_{2,4}(t) = \dfrac{3a_{6,0}(t)\gamma^2(t)}{(k\omega + m^2\beta(t))^2}, \\ a_{3,0}(t) = -\dfrac{a_{1,6}(t)(k\omega + m^2\beta(t))}{\gamma(t)}, \ a_{4,0}(t) = \dfrac{17k^4a_{6,0}(t)\gamma^2(t)\rho(t)}{(k\omega + m^2\beta(t))^2}, \ a_{2,4}(t) = \dfrac{3a_{6,0}(t)\gamma(t)}{k\omega + m^2\beta(t)}. \end{cases} \tag{21}$$

因此方程(2)的二阶怪波解可以利用变换(5),得到

$$u(\xi,z) = 2\ln(F_2(\xi,z))_{\xi\xi}, \tag{22}$$

这里的 $F_2(\xi,z)$ 可由式(20)求得.取 $a_{0,9}(t) = 6, a_{1,6}(t) = t^2, a_{6,0}(t) = t, \beta(t) = 9t, \gamma(t) = t^2, \rho(t) = 19t$.



(a) $m = 0$ 时的三维图
(a) The 3D map for $m = 0$

(b) $m = 1.4$ 时的三维图
(b) The 3D map for $m = 1.4$

(c) $m = 3.7$ 时的三维图
(c) The 3D map for $m = 3.7$

(d) $m = 0$ 时的等高线图
(d) The contour map for $m = 0$

(e) $m = 1.4$ 时的等高线图
(e) The contour map for $m = 1.4$

(f) $m = 3.7$ 时的等高线图
(f) The contour map for $m = 3.7$

图 4 方程(2)在 $t = 3.8, \omega = 2, k = 2.05$ 时二阶怪波的图像

Fig. 4 Profiles of the 2nd-order rogue wave of eq. (2) for $t = 3.8, \omega = 2, k = 2.05$

图 4 给出了二阶怪波解的图像,从图中可以看出怪波集中在(0,0)附近,且 m 的取值主要影响二阶怪波解的波峰数量,使其由两个波峰变成三个波峰.

3　结　　论

本文基于 Hirota 双线性方法,将拓展的同宿呼吸检验法推广到(3+1)维变系数 K-S 方程中,得到了它的同宿呼吸解,发现该解所含的参数影响着同宿呼吸波的结构,即波的周期、位置和形状随着参数的取值而变化.随后通过对同宿呼吸极波的周期取极限求得方程的怪波解.最后,构造一个特殊的高阶多项式作为测试函数,得到了该方程的一阶怪波解和二阶怪波解,借助 MATHEMATICA 软件作出的图像可以看出,参数 m 影响二阶怪波解的波峰,使其由两个怪波变成三个呈三角形的独立怪波.(3+1)维变系数 K-S 方程的研究在医学、自然科学和工程领域有着重要的意义.

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