　http://www.applmathmech.cn

周期环境中捕食者具有尺度结构的三物种捕食-食饵系统的最优收获*

刘　荣[1]，　刘桂荣[2]

(1. 山西财经大学 应用数学学院, 太原 030006;
2. 山西大学 数学科学学院, 太原 030006)

摘要：　对种群动力学及相关控制问题的研究,不仅具有理论意义,而且与生物多样性保护、病虫害防治及可再生资源的开发利用密切相关.该文研究了一类周期环境中具有两相互竞争食饵和一捕食者的三物种捕食-食饵系统的最优收获,其中捕食者具有尺度结构且用一阶偏微分方程描述.运用不动点定理证明了系统非负有界解的存在唯一性,并讨论了解关于控制变量的连续依赖性.应用切-法锥技巧导出最优收获条件,并借助 Ekeland 变分原理讨论了最优策略的存在唯一性.这里目标泛函表示收获三物种产生的净经济效益.所得结果将有利于可再生资源的开发.

关　键　词：　周期环境;　尺度结构;　三物种;　捕食-食饵;　最优收获
中图分类号：　O175.22　　**文献标志码：**　A　　DOI: 10.21656/1000-0887.410285

Optimal Harvesting in a Periodic 3-Species Predator-Prey Model With Size Structure in Predators

LIU Rong[1],　LIU Guirong[2]
(1. *School of Applied Mathematics*, *Shanxi University of Finance and Economics*, *Taiyuan* 030006, *P.R.China*;
2. *School of Mathematical Sciences*, *Shanxi University*, *Taiyuan* 030006, *P.R.China*)

Abstract: The research on population dynamics and related control problems is not only of theoretical significance, but also closely related to biodiversity protection, pest control, and the development and utilization of renewable resources. The optimal harvesting problem was considered in a periodic 3-species predator-prey system with 1 predator and 2 competing preys, where the predator has size structure and was described with 1st-order partial differential equations. First, the existence of a unique non-negative solution of the controlled system was proven by means of the fixed-point reasoning, and the continuous dependence of the solution on the control variables was discussed. Then, the optimal harvesting conditions were given with the techniques of tangential-normal cones and the adjoint system. Finally, with Ekeland's variational principle, the existence of the

* **收稿日期：**　2020-09-21; **修订日期：**　2021-03-12
基金项目：　国家自然科学基金(12001341;11971279);山西省青年科技研究基金(201901D211410);山西省高等学校科技创新项目(2020L0258)
作者简介：　刘荣(1988—),男,博士(E-mail: rliu29@sxufe.edu.cn);
刘桂荣(1975—),男,教授,博士生导师(通讯作者. E-mail: lgr5791@sxu.edu.cn).
引用格式：　刘荣, 刘桂荣. 周期环境中捕食者具有尺度结构的三物种捕食-食饵系统的最优收获[J]. 应用数学和力学, 2021, **42**(5): 510-521.

optimal harvesting strategy was derived. Here the objective functional represents the net economic benefit in the harvesting of 3 species. The results obtained would be beneficial for exploration of renewable resources.

Key words: periodic environment; size structure; 3 species; predator-prey; optimal harvesting

引　　言

数学模型不仅可以刻画种群与环境、种群与种群之间的相互作用,而且可以描述、预测以至调节和控制种群的发展过程及发展趋势.目前,对不同种群模型的动力学研究成果有许多,如王双明等[1]利用动力系统理论分析了一类带有时滞的周期 logistic 反应扩散传染病模型的动力学;曹建智等[2]讨论了一类带有时滞的云杉蚜虫种群阶段结构模型,分析了模型正平衡点的存在唯一性,并给出平衡点的局部稳定性和出现 Hopf 分岔的充分条件.

然而,生态学研究表明三物种捕食-食饵系统是大型生态系统的重要组成部分.目前,关于三物种捕食-食饵系统的研究,多数是基于均匀混合的假设下建立的,即将生活在同一区域的同一物种视为一个整体,建立确定型或随机型常微分方程系统(可参见文献[3-6]及所附文献).

注意到种群个体间存在诸如年龄、生理尺度等结构差异.而结构化种群模型就是根据个体间的结构差异来确定种群个体的出生率、增长率、死亡率、个体之间以及个体与环境之间的相互作用等[7].近年来,关于依赖个体年龄的种群模型的动力学分析及相关调控问题的研究取得了大量成果(可参考文献[8-15]及相关文献).文献[12-13]分别讨论了食饵具有年龄分布的捕食-食饵模型的动力学行为和最优收获问题.然而,食饵通常处于食物链中较低的生态位,年龄依赖性对捕食者来说更为重要.基于此,文献[14]研究了一类捕食具有年龄结构的捕食-食饵系统的最优收获策略.文献[15]研究了如下周期环境中具有年龄结构的种群模型的最优收获问题:

$$\begin{cases}\dfrac{\partial p(a,t)}{\partial t}+\dfrac{\partial p(a,t)}{\partial a}=f(a,t)-\mu(a,t)p(a,t)-u(a,t)p(a,t), \\ \qquad\qquad\qquad\qquad\qquad (a,t)\in Q=[0,a_\dagger)\times(0,+\infty), \\ p(0,t)=\displaystyle\int_0^{a_\dagger}\beta(a,t)p(a,t)\,\mathrm{d}a, \qquad t\in\mathbf{R}_+, \\ p(a,t)=p(a,t+T), \qquad (a,t)\in Q,\end{cases}$$

其中生命参数 $\beta(a,t)$ 和 $\mu(a,t)$、迁入率 $f(a,t)$ 及收获努力度 $u(a,t)$ 均为关于时间 t 的 T- 周期函数.

进一步研究表明:个体尺度在很大程度上决定个体的出生率、死亡率、捕食能力和新陈代谢能力等,从而影响种群动力学行为[15].目前,对于具有尺度分布的种群模型的动力学分析及相关控制问题的研究多数是基于单种群模型[17-19].对多种群模型,刘炎和何泽荣在文献[20-21]中分别讨论了捕食者具有尺度结构的捕食-食饵系统和食饵种群带有尺度结构的种群系统的最优收获问题.曹雪靓[22]讨论了一类污染环境下具有尺度结构的捕食种群模型解的存在唯一性问题.

注意到三物种捕食-食饵系统是生态系统的重要组成部分.文献[3]将三物种捕食-食饵系统分为五类:两捕食者争夺一食饵、一捕食者捕食两食饵、食物链、杂食食物链和循环食物链.然而,目前仅文献[23]研究了一类周期环境中捕食者具有尺度结构的杂食食物链模型.基于此,本文研究了一类周期环境中具有两相互竞争食饵和一捕食者的三物种捕食-食饵系统的最优收获,其中捕食者具有尺度结构.为建立模型,记 $\mathbf{R}_+=(0,\infty)$, $Q=(0,l)\times\mathbf{R}_+$. $p(x,t)$ 表示 t 时刻尺度为 x 的捕食者个体数量; $q_1(t)$ 和 $q_2(t)$ 分别表示 t 时刻食饵 1 和食饵 2 的个体数量.于是,基于文献[15,23],本文建立并分析了如下周期环境中捕食者具有尺度结构的三物种捕食-食饵系统的最优收获问题:

$$
\begin{cases}
\dfrac{\partial p(x,t)}{\partial t} + \dfrac{\partial(g(x)p(x,t))}{\partial x} = f(x,t) - \mu(x,t)p(x,t) - \alpha_1(x,t)p(x,t), & (x,t) \in Q, \\
\dfrac{\mathrm{d}q_1(t)}{\mathrm{d}t} = r_1(q_1(t),t)q_1(t) - \Phi_1(P(t))q_1(t) - h_1(q_2(t))q_1(t) - \alpha_2(t)q_1(t), & t \in \mathbf{R}_+, \\
\dfrac{\mathrm{d}q_2(t)}{\mathrm{d}t} = r_2(q_2(t),t)q_2(t) - \Phi_2(P(t))q_2(t) - h_2(q_1(t))q_2(t) - \alpha_3(t)q_2(t), & t \in \mathbf{R}_+, \\
g(0)p(0,t) = [f_1(q_1(t)) + f_2(q_2(t))]\int_0^l \beta(x)p(x,t)\mathrm{d}x, & t \in \mathbf{R}_+, \\
p(x,t) = p(x,t+T), & (x,t) \in Q, \\
q_i(t) = q_i(t+T),\ q_i(0) = q_{i0} > 0, & i = 1,2,\ t \in \mathbf{R}_+,
\end{cases}
\tag{1}
$$

其中 $P(t) = \int_0^l p(x,t)\mathrm{d}x$ 表示 t 时刻捕食者数量;生命参数 $\mu(x,t)$,$[f_1(q_1(t)) + f_2(q_2(t))]\beta(x)$ 和 $g(x)$ 分别表示捕食者的死亡率、出生率和个体尺度增长率; $f(x,t)$ 表示捕食者的迁入率;$r_i(q_i(t),t)$ 表示食饵种群 $q_i(i=1,2)$ 的内禀增长率;$\Phi_i(i=1,2)$ 为功能反应函数;$h_i(i=1,2)$ 为种间竞争系数;这里 $T \in \mathbf{R}_+$ 表示种群演变周期;控制变量 $\alpha_1(x,t)$,$\alpha_2(t)$,$\alpha_3(t)$ 为收获三物种的收获努力度,且属于如下允许控制集:

$$
\mathcal{U} = \left\{ (\alpha_1,\alpha_2,\alpha_3) \in L_T^\infty(Q) \times L_T^\infty(\mathbf{R}_+) \times L_T^\infty(\mathbf{R}_+) \;\middle|\; \begin{array}{l} 0 \leqslant \alpha_1(x,t) \leqslant N_1,\ \text{a.e. } (x,t) \in Q, \\ 0 \leqslant \alpha_i(t) \leqslant N_i,\ \text{a.e. } t \in \mathbf{R}_+, i = 2,3 \end{array} \right\},
$$

其中, $L_T^\infty(Q) = \{h \in L^\infty(Q): v(x,t) = v(x,t+T),\ \text{a.e.}(x,t) \in Q\}$ 且 $L_T^\infty(\mathbf{R}_+) = \{h \in L^\infty(\mathbf{R}_+): v(t) = v(t+T), \text{a.e. } t \in \mathbf{R}_+\}$.如果$(p,q_1,q_2)$ 为系统(1)相应于 $(\alpha_1,\alpha_2,\alpha_3) \in \mathcal{U}$ 的解,那么由系统(1)的最后两个方程可知 p,q_1,q_2 均为关于时间 t 的 T- 周期函数.本文考虑如下最优化问题:

$$
\max_{(\alpha_1,\alpha_2,\alpha_3) \in \mathcal{U}} J(\alpha_1,\alpha_2,\alpha_3),
\tag{2}
$$

其中

$$
J(\alpha_1,\alpha_2,\alpha_3) = \int_0^T\int_0^l \omega_1(x,t)\alpha_1(x,t)p(x,t)\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^T\int_0^l c_1\alpha_1^2(x,t)\mathrm{d}x\mathrm{d}t + \sum_{i=2}^3\int_0^T \omega_i(t)\alpha_i(t)q_{i-1}(t)\mathrm{d}t - \frac{1}{2}\sum_{i=2}^3\int_0^T c_i\alpha_i^2(t)\mathrm{d}t,
$$

权函数 $\omega_1(\cdot,t)$,$\omega_2(t)$ 和 $\omega_3(t)$ 表示三物种的经济价值且均为关于时间 t 的 T- 周期函数;正常数 c_1,c_2 和 c_3 为收获三物种的收获成本.于是,目标泛函 $J(\alpha_1,\alpha_2,\alpha_3)$ 表示在$[0,T]$ 时间段内收获所有物种所获得的净经济利益.为研究需要,本文采用如下假设:

(H1) $g \in C^1([0,l),\mathbf{R}_+)$ 为有界函数.$g(0) = 1$,$\lim_{x\uparrow l} g(x) = 0$ 且对任意 $x \in [0,l)$,有 $\dot{g}(x) \leqslant 0$.进一步,存在正常数 L_{V} 使得对任意的 $x_1,x_2 \in [0,l)$,有 $|g(x_1) - g(x_2)| \leqslant L_{\mathrm{V}}|x_1 - x_2|$.

(H2) 存在正常数 $\bar{\beta}$ 使得对任意的 $x \in (0,l)$,有 $0 \leqslant \beta(x) \leqslant \bar{\beta}$.

(H3) $\mu:[0,l) \times \mathbf{R}_+ \to \mathbf{R}_+$ 为可测函数,且对任意$(x,t) \in Q$,有 $\mu(x,t) = \mu(x,t+T) \geqslant 0$ 和 $\mu(x,t) + \dot{g}(x) \geqslant 0$ 成立.

(H4) 存在常数 $B_i > 0(i=1,2)$ 使得对所有的 $S,t \geqslant 0$,有 $0 \leqslant r_i(S,t) = r_i(S,t+T) \leqslant B_i$,且 r_i 关于第一个变量满足 Lipschitz 条件.即存在常数 $B_{r_i} > 0(i=1,2)$ 使得对所有的 $t,S_1,S_2 \geqslant 0$,有 $|r_i(S_1,t) - r_i(S_2,t)| \leqslant B_{r_i}|S_1 - S_2|$.

(H5) 存在正常数 C_i 和 $C_{\Phi_i}(i=1,2)$ 使得对任意 $S \geqslant 0$,有 $0 \leqslant \Phi_i(S) \leqslant C_i$,且对所有的 $S_1,S_2 \geqslant 0$,有

$|\Phi_i(S_1)-\Phi_i(S_2)|\leqslant C_{\Phi_i}|S_1-S_2|$.

(H6) 存在正常数 D_i 和 $D_{f_i}(i=1,2)$ 使得对任意 $S\geqslant 0$, 有 $0\leqslant f_i(S)\leqslant D_i$, 且对所有的 $S_1,S_2\geqslant 0$,有 $|f_i(S_1)-f_i(S_2)|\leqslant D_{f_i}|S_1-S_2|$.

(H7) 存在正常数 E_i 和 $E_{h_i}(i=1,2)$ 使得对任意 $S\geqslant 0$, 有 $0\leqslant h_i(S)\leqslant E_i$,且对所有的 $S_1,S_2\geqslant 0$,有 $|h_i(S_1)-h_i(S_2)|\leqslant E_{h_i}|S_1-S_2|$.

(H8) $f\in L_T^\infty(Q)$,且对任意的 $(x,t)\in Q$,有 $f(x,t)=f(x,t+T)\geqslant 0$.

1　状态系统的适定性

本节将利用不动点理论讨论系统(1)非负有界解的存在唯一性.本文考虑 $t>z^{-1}(l)\triangleq\bar{t}$ 的情况.为研究方便,引入如下定义.

定义 1　称初值问题 $\dot{x}(t)=g(x(t)),x(t_0)=x_0$ 的唯一解 $x=\varphi(t;t_0,x_0)$ 为通过点 (x_0,t_0) 的特征曲线.记 $z(t)=\varphi(t;0,0)$ 为 x-t 平面上通过点(0,0)的特征曲线.

定义 2　称函数 $u(x,t)$ 沿特征曲线 φ 在点 (x,t) 处的方向导数为

$$\mathrm{D}_\varphi u(x,t)=\lim_{h\to 0}\frac{u(\varphi(t+h;t,x),t+h)-u(x,t)}{h}.$$

定义 3　若三元函数组 $(p(x,t),q_1(t),q_2(t))\in L_T^\infty(Q)\times L_T^\infty(\mathbf{R}_+)\times L_T^\infty(\mathbf{R}_+)$ 且满足

$$p(x,t)=p(0,\varphi^{-1}(0;t,x))\Pi(t;t,x)+\int_\tau^t f(\varphi(s;t,x),s)\frac{\Pi(t;t,x)}{\Pi(s;t,x)}\mathrm{d}s,\tag{3}$$

$$q_1(t)=q_{10}\exp\left\{\int_0^t[r_1(q_1(s),s)-\Phi_1(P(s))-h_1(q_2(s))-\alpha_2(s)]\mathrm{d}s\right\},\tag{4}$$

$$q_2(t)=q_{20}\exp\left\{\int_0^t[r_2(q_2(s),s)-\Phi_2(P(s))-h_2(q_1(s))-\alpha_3(s)]\mathrm{d}s\right\},\tag{5}$$

则称其为系统(1)的解.这里 $\Pi(r;t,x)=\exp\left\{-\int_\tau^r[(\mu+\alpha_1)(\varphi(w;t,x),w)+\dot{g}(\varphi(w;t,x))]\mathrm{d}w\right\}$,且 $P(t)=\int_0^l p(x,t)\mathrm{d}x$.

记 $M\triangleq\max\{q_{10}\mathrm{e}^{B_1(\bar{t}+T)},q_{20}\mathrm{e}^{B_2(\bar{t}+T)},M_1(\bar{t}+T)+\|f(\cdot,\cdot)\|_{L^1(Q)}\}$, 其中 $M_1=(D_1+D_2)\bar{\beta}\|f(\cdot,\cdot)\|_{L^1(Q)}\mathrm{e}^{(D_1+D_2)\bar{\beta}(\bar{t}+T)}$.设 $\boldsymbol{X}=L_T^\infty(\mathbf{R}_+,L^1(0,l))\times L_T^\infty(\mathbf{R}_+)\times L_T^\infty(\mathbf{R}_+)$,对常数 $\lambda>0$,定义 X 上的等价范数

$$\|(p,q_1,q_2)\|_*=\operatorname*{ess\,sup}_{t\in[\bar{t},\bar{t}+T]}\left\{\mathrm{e}^{-\lambda t}\left[|q_1(t)|+|q_2(t)|+\int_0^l|p(x,t)|\mathrm{d}x\right]\right\}.$$

进一步地,定义解空间为

$$\mathcal{X}=\left\{(p,q_1,q_2)\in\boldsymbol{X}\,\middle|\,p(x,t)\geqslant 0,\ \text{a.e. }(x,t)\in Q,\right.$$
$$\left.\int_0^l p(x,t)\mathrm{d}x\leqslant M,\ 0\leqslant q_i(t)\leqslant M,\ \text{a.e. }t\in\mathbf{R}_+,i=1,2\right\},$$

算子 $\mathcal{A}:\mathcal{X}\to X$ 为

$$(\mathcal{A}(p,q_1,q_2))=(\mathcal{A}_1(p,q_1,q_2),\mathcal{A}_2(p,q_1,q_2),\mathcal{A}_3(p,q_1,q_2)),$$

其中 $\mathcal{A}_1(p,q_1,q_2),\mathcal{A}_2(p,q_1,q_2)$ 和 $\mathcal{A}_3(p,q_1,q_2)$ 分别由式(3)~(5)的右端所给出.显然,若 (p,q_1,q_2) 是映射 $\mathcal{A}$ 的不动点,则其必为系统(1)的解,反之亦然.

定理 1　对于任意给定的 $(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{U}$,系统(1) 有唯一解 $(p,q_1,q_2)\in\mathcal{X}$.

证明　首先证明 $\mathcal{A}$ 是从 $\mathcal{X}$ 到 $\mathcal{X}$ 的映射.由假设(H1)有 $g(0)=1$.记 $b(t)=p(0,t)$, 则

$$b(t)=g(0)p(0,t)=[f_1(q_1(t))+f_2(q_2(t))]\int_0^l\beta(x)p(x,t)\mathrm{d}x\leqslant$$
$$(D_1+D_2)\bar{\beta}\left[\int_0^l b(\varphi^{-1}(0;t,x))\mathrm{d}x+\int_0^l\int_\tau^t f(\varphi(s;t,x),s)\mathrm{d}s\mathrm{d}x\right]\triangleq I_1+I_2.\tag{6}$$

对于 I_1,令 $s=\varphi^{-1}(0;t,x)$.于是由定义 1 知,当 $x=0$ 时,有 $s=\varphi^{-1}(0;t,0)=t$;当 $x=l$ 时,有 $s=\varphi^{-1}(0;t,l)$

$=0$.进一步,由 $s=\varphi^{-1}(0;t,x)$ 知,$x=\varphi(t;s,0)$.由于 $\mathrm{d}x/\mathrm{d}s$ 是初值问题 $\mathrm{d}z/\mathrm{d}t=\dot g(\varphi(t;s,0))z, z(s)=-g(0)=-1$ 的解且其解为 $z(t)=-\exp\left\{\int_s^t \dot g(\varphi(r;s,0))\mathrm{d}r\right\}$,则

$$\mathrm{d}x=-\exp\left\{\int_s^t \dot g(\varphi(r;s,0))\mathrm{d}r\right\}\mathrm{d}s. \tag{7}$$

从而,由假设(H1)可得

$$I_1=\int_0^t b(s)\exp\left\{\int_s^t \dot g(\varphi(r;s,0))\mathrm{d}r\right\}\mathrm{d}s\leqslant\int_0^t b(s)\mathrm{d}s. \tag{8}$$

对于 I_2,令 $w=\varphi(s;t,x)$.于是由定义 1 知,当 $x=0$ 时,有 $w=\varphi(s;t,0)=0$;当 $x=l$ 时,有 $w=\varphi(s;t,l)=l$.注意到 $\mathrm{d}w/\mathrm{d}x$ 是初值问题 $\mathrm{d}z/\mathrm{d}s=\dot g(\varphi(s;t,x))z, z(t)=1$ 的解,且其解为 $z(s)=\exp\left\{\int_t^s \dot g(\varphi(r;t,x))\mathrm{d}r\right\}$,则

$$\mathrm{d}x=\exp\left\{\int_s^t \dot g(\varphi(r;t,x))\mathrm{d}r\right\}\mathrm{d}w. \tag{9}$$

从而,由假设(H1)可知

$$I_2=\int_0^l\int_\tau^t f(w,s)\exp\left\{\int_s^t \dot g(\varphi(r;t,x))\mathrm{d}r\right\}\mathrm{d}s\mathrm{d}w\leqslant\|f(\cdot,\cdot)\|_{L^1(Q)}. \tag{10}$$

进一步地,由式(6)、(8)和(10)可得

$$b(t)\leqslant(D_1+D_2)\bar\beta\|f(\cdot,\cdot)\|_{L^1(Q)}+(D_1+D_2)\bar\beta\int_0^t b(s)\mathrm{d}s.$$

由于变量 $b(t)$ 具有周期性,本文仅考虑 $t\in[\bar t,\bar t+T]$.于是,由 Gronwall 不等式可得

$$b(t)\leqslant(D_1+D_2)\bar\beta\|f(\cdot,\cdot)\|_{L^1(Q)}\mathrm{e}^{(D_1+D_2)\bar\beta(\bar t+T)}=M_1.$$

类似地,由式(3)、(8)和(10)得

$$\int_0^l|\mathscr{A}_1(p,q_1,q_2)|(x,t)\mathrm{d}x\leqslant\int_0^t b(s)\mathrm{d}s+\|f(\cdot,\cdot)\|_{L^1(Q)}\leqslant M_1(\bar t+T)+\|f(\cdot,\cdot)\|_{L^1(Q)}.$$

进一步地,由式(4)和(5)可得

$$|\mathscr{A}_2(p,q_1,q_2)|(t)\leqslant q_{10}\mathrm{e}^{B_1(\bar t+T)},\ |\mathscr{A}_3(p,q_1,q_2)|(t)\leqslant q_{20}\mathrm{e}^{B_2(\bar t+T)}.$$

从而可知,$\mathscr{A}$ 是从 $\mathscr{X}$ 到其自身的一个映射.

接着,讨论 $\mathscr{A}$ 的压缩性.记 $W(t)\triangleq\int_0^t\left[\sum_{i=1}^2|q_i(s)-q_i'(s)|+\int_0^l|p(x,s)-p'(x,s)|\mathrm{d}x\right]\mathrm{d}s$.由式(3)和(7)知

$$\begin{aligned}
&\int_0^l|\mathscr{A}_1(p,q_1,q_2)-\mathscr{A}_1(p',q_1',q_2')|(x,t)\mathrm{d}x\leqslant\\
&\quad\int_0^l|b(\varphi^{-1}(0;t,x))-b'(\varphi^{-1}(0;t,x))|\mathrm{d}x\leqslant\int_0^t|b(s)-b'(s)|\mathrm{d}s=\\
&\quad\int_0^t g|[f_1(q_1(s))+f_2(q_2(s))]\int_0^l\beta(x)p(x,s)\mathrm{d}x-\\
&\quad[f_1(q_1'(s))+f_2(q_2'(s))]\int_0^l\beta(x)p'(x,s)\mathrm{d}xg|\mathrm{d}s\leqslant\\
&\quad\bar\beta(D_1+D_2)\int_0^t\int_0^l|p(x,s)-p'(x,s)|\mathrm{d}x\mathrm{d}s+\\
&\quad\bar\beta(D_{f_1}+D_{f_2})\int_0^t[|q_1(s)-q_1'(s)|+|q_2(s)-q_2'(s)|]\int_0^l p'(x,s)\mathrm{d}x\mathrm{d}s\leqslant M_2W(t),
\end{aligned}$$

其中 $M_2=\max\{\bar\beta(D_1+D_2),\bar\beta(D_{f_1}+D_{f_2})M\}$.由式(4)得

$$\begin{aligned}
&|\mathscr{A}_2(p,q_1,q_2)-\mathscr{A}_2(p',q_1',q_2')|(t)\leqslant\\
&\quad q_{10}\mathrm{e}^{2B_1(\bar t+T)}\int_0^t[|r_1(q_1(s),s)-r_1(q_1'(s),s)|+|\Phi_1(P(s))-\Phi_1(P'(s))|+\\
&\quad|h_1(q_2(s))-h_1(q_2'(s))|]\mathrm{d}s\leqslant M_3W(t),
\end{aligned}$$

其中 $M_3 = q_{10}\mathrm{e}^{2B_1(\bar{t}+T)}\max\{B_{r_1}, C_{\Phi_1}, E_{h_1}\}$.记 $M_4 = q_{20}\mathrm{e}^{2B_2(\bar{t}+T)}\max\{B_{r_2}, C_{\Phi_2}, E_{h_2}\}$.由式(5)得

$$|\mathscr{A}_3(p,q_1,q_2) - \mathscr{A}_3(p',q'_1,q'_2)|(t) \leqslant M_4 W(t).$$

于是

$$\begin{aligned}&\|\mathscr{A}(p,q_1,q_2) - \mathscr{A}(p',q'_1,q'_2)\|_* \leqslant \\ &M_5 \operatorname*{ess\,sup}_{t\in[\bar{t},\bar{t}+T]}\left\{\mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda s}\left[\mathrm{e}^{-\lambda s}\left(\sum_{i=1}^{2}|q_i(s) - q'_i(s)| + \int_0^l |p(x,s) - p'(x,s)|\mathrm{d}x\right)\right]\mathrm{d}s\right\} \leqslant \\ &\frac{M_5}{\lambda}\|(p-p', q_1-q'_1, q_2-q'_2)\|_*,\end{aligned}$$

其中 $M_5 = M_2 + M_3 + M_4$.选择 λ 充分大使得 $\lambda > M_5$,于是 $\mathscr{A}$ 为空间$(\mathscr{X}, \|\cdot\|_*)$上的压缩映射.从而,由不动点理论可知,映射 $\mathscr{A}$ 有唯一不动点.定理得证.

下面定理给出系统解关于控制变量的连续依赖性.

定理 2　对任意的$(\alpha_1,\alpha_2,\alpha_3)$,$(\alpha'_1,\alpha'_2,\alpha'_3) \in \mathscr{U}$,记$(p,q_1,q_2)$和$(p',q'_1,q'_2)$分别是系统(1)相应于$(\alpha_1,\alpha_2,\alpha_3)$和$(\alpha'_1,\alpha'_2,\alpha'_3)$的解.则存在正常数 K_1 和 K_2 使得

$$\begin{aligned}&\|p-p'\|_{L_T^\infty(\mathbf{R}_+;L^1(0,l))} + \sum_{i=1}^{2}\|q_i - q'_i\|_{L_T^\infty(\mathbf{R}_+)} \leqslant \\ &K_1 T\left[\|\alpha_1 - \alpha'_1\|_{L_T^\infty(\mathbf{R}_+;L^1(0,l))} + \sum_{i=2}^{3}\|\alpha_i - \alpha'_i\|_{L_T^\infty(\mathbf{R}_+)}\right],\end{aligned}$$

$$\|p-p'\|_{L_T^1(Q)} + \sum_{i=1}^{2}\|q_i - q'_i\|_{L_T^1(\mathbf{R}_+)} \leqslant K_2 T\left[\|\alpha_1 - \alpha'_1\|_{L_T^1(Q)} + \sum_{i=2}^{3}\|\alpha_i - \alpha'_i\|_{L_T^1(\mathbf{R}_+)}\right].$$

证明　由于(p,q_1,q_2)和(p',q'_1,q'_2)分别是系统(1)相应于$(\alpha_1,\alpha_2,\alpha_3)$和$(\alpha'_1,\alpha'_2,\alpha'_3)$的解.于是,由式(4)可知

$$\begin{aligned}&|q_1(t) - q'_1(t)| \leqslant q_{10}\mathrm{e}^{2B_1(\bar{t}+T)}\int_0^t\Big[|r_1(q_1(s),s) - r_1(q'_1(s),s)| + \\ &\left|\Phi_1\left(\int_0^l p(x,s)\mathrm{d}x\right) - \Phi_1\left(\int_0^l p'(x,s)\mathrm{d}x\right)\right| + \\ &|h_1(q_2(s)) - h_1(q'_2(s))| + |\alpha_2(s) - \alpha'_2(s)|\Big]\mathrm{d}s \leqslant \\ &q_{10}\mathrm{e}^{2B_1(\bar{t}+T)}\int_0^t\Big[B_{r_1}|q_1(s) - q'_1(s)| + C_{\Phi_1}\int_0^l |p(x,s) - p'(x,s)|\mathrm{d}x + \\ &E_{h_1}|q_2(s) - q'_2(s)| + |\alpha_2(s) - \alpha'_2(s)|\Big]\mathrm{d}s \leqslant \\ &M_3 W(t) + q_{10}\mathrm{e}^{2B_1(\bar{t}+T)}\int_0^t |\alpha_2(s) - \alpha'_2(s)|\mathrm{d}s.\end{aligned}$$

类似地,由式(5)有

$$|q_2(t) - q'_2(t)| \leqslant M_4 W(t) + q_{20}\mathrm{e}^{2B_2(\bar{t}+T)}\int_0^t |\alpha_3(s) - \alpha'_3(s)|\mathrm{d}s.$$

此外,由式(3)知

$$\begin{aligned}&\int_0^l |p(x,t) - p'(x,t)|\mathrm{d}x \leqslant \\ &\int_0^t |b(s) - b'(s)|\mathrm{d}s + M_1\int_0^l\int_0^t |\alpha_1(\varphi(w;t,x),w) - \alpha'_1(\varphi(w;t,x),w)|\mathrm{d}w\mathrm{d}x + \\ &\int_0^l\int_0^t f(\varphi(s;t,x),s)\int_s^t |\alpha_1(\varphi(w;t,x),w) - \alpha'_1(\varphi(w;t,x),w)|\mathrm{d}w\mathrm{d}s\mathrm{d}x \leqslant \\ &M_2\int_0^t\Big[|q_1(s) - q'_1(s)| + |q_2(s) - q'_2(s)| + \int_0^l |p(x,s) - p'(x,s)|\mathrm{d}x\Big]\mathrm{d}s + \\ &M_1\int_0^l\int_0^t |\alpha_1(\varphi(w;t,x),w) - \alpha'_1(\varphi(w;t,x),w)|\mathrm{d}w\mathrm{d}x + \\ &\int_0^l\int_0^t f(\varphi(s;t,x),s)\int_s^t |\alpha_1(\varphi(w;t,x),w) - \alpha'_1(\varphi(w;t,x),w)|\mathrm{d}w\mathrm{d}s\mathrm{d}x \leqslant\end{aligned}$$

$$M_2 W(t) + M_6 \int_0^l \int_0^t \mid \alpha_1(\varphi(w;t,x),w) - \alpha'_1(\varphi(w;t,x),w) \mid \mathrm{d}w\mathrm{d}x,$$

其中 $\Pi'(r;t,x) = \exp\left\{ -\int_\tau^r \left[(\mu + \alpha'_1)(\varphi(w;t,x),w) + \dot{g}(\varphi(w;t,x)) \right] \mathrm{d}w \right\}$, $M_6 = M_1 + \| f(\cdot,\cdot) \|_{L^1(Q)}$. 于是,由上述分析可得所需结果.定理得证.

2 最优性条件

本节将利用非线性泛函分析中的切线-法锥技术来推导最优收获控制的必要条件.对任意的 $(\alpha_1,\alpha_2,\alpha_3) \in \mathcal{U}$,记 $\mathcal{T}_{\mathcal{U}}(\alpha_1,\alpha_2,\alpha_3)$ 和 $\mathcal{N}_{\mathcal{U}}(\alpha_1,\alpha_2,\alpha_3)$ 分别为集合 $\mathcal{U}$ 在点 $(\alpha_1,\alpha_2,\alpha_3)$ 处的切锥和法锥.考虑系统(1)的共轭系统

$$\begin{cases} \mathrm{D}_\varphi \xi = [\mu(x,t) + \alpha_1(x,t)]\xi(x,t) - [f_1(q_1(t)) + f_2(q_2(t))]\beta(x)\xi(0,t) + \\ \qquad \Phi'_1(P(t))q_1(t)\eta_1(t) + \Phi'_2(P(t))q_2(t)\eta_2(t) + \omega_1(x,t)\alpha_1(x,t), \\ \dfrac{\mathrm{d}\eta_1}{\mathrm{d}t} = -\left[r_1(q_1(t),t) + q_1(t)\left(\dfrac{\partial r_1(q_1(t),t)}{\partial q_1} - h'_1(q_2(t)) \right) - \Phi_1(P) - h_1(q_2(t)) - \alpha_2 \right]\eta_1 + \\ \qquad \omega_2(t)\alpha_2(t) - f'_1(q_1(t))\xi_1(0,t)\int_0^l \beta(x)p(x,t)\mathrm{d}x, \\ \dfrac{\mathrm{d}\eta_2}{\mathrm{d}t} = -\left[r_2(q_2(t),t) + q_2(t)\left(\dfrac{\partial r_2(q_2(t),t)}{\partial q_2} - h'_2(q_1(t)) \right) - \Phi_2(P) - h_2(q_1(t)) - \alpha_3 \right]\eta_2 + \\ \qquad \omega_3(t)\alpha_3(t) - f'_2(q_2(t))\xi_1(0,t)\int_0^l \beta(x)p(x,t)\mathrm{d}x, \end{cases} \tag{11}$$

其中

$$\begin{cases} \xi(x,t) = \xi(x,t+T),\ \eta_i(t) = \eta_i(t+T), \qquad i = 1,2, \\ \xi(l,t) = 0,\ P(t) = \int_0^l p(x,t)\mathrm{d}x. \end{cases}$$

对系统(11),类似定理 1 和 2 的证明可得如下结论.

定理 3 对任意的 $(\alpha_1,\alpha_2,\alpha_3) \in \mathcal{U}$, 系统(11)有唯一的有界解 (ξ,η_1,η_2).进一步地, 存在正常数 K_3 使得

$$\| \xi - \xi' \|_{L^\infty_T(Q)} + \sum_{i=1}^2 \| \eta_i - \eta'_i \|_{L^\infty_T(\mathbf{R}_+)} \leqslant$$
$$K_3 T \left[\| \alpha_1 - \alpha'_1 \|_{L^\infty_T(Q)} + \sum_{i=2}^3 \| \alpha_i - \alpha'_i \|_{L^\infty_T(\mathbf{R}_+)} \right], \tag{12}$$

其中 (ξ,η_1,η_2) 和 (ξ',η'_1,η'_2) 分别为系统(11)相应于 $(\alpha_1,\alpha_2,\alpha_3)$ 和 $(\alpha'_1,\alpha'_2,\alpha'_3) \in \mathcal{U}$ 的解.

定理 4 设 $(\alpha_1^*,\alpha_2^*,\alpha_3^*)$ 是最优控制,(p^*,q_1^*,q_2^*) 为相应的最优状态,则

$$\alpha_1^*(x,t) = \mathcal{F}_1\left[\frac{[\omega_1(x,t) + \xi(x,t)]p^*(x,t)}{c_1} \right], \tag{13}$$

$$\alpha_2^*(t) = \mathcal{F}_2\left[\frac{[\omega_2(t) + \eta_1(t)]q_1^*(t)}{c_2} \right], \tag{14}$$

$$\alpha_3^*(t) = \mathcal{F}_3\left[\frac{[\omega_3(t) + \eta_2(t)]q_2^*(t)}{c_3} \right], \tag{15}$$

其中

$$(\mathcal{F}_i)(S) = \begin{cases} 0, & S < 0, \\ S, & 0 \leqslant S \leqslant N_i,\ i = 1,2,3, \\ N_i, & S > N_i, \end{cases} \tag{16}$$

且 $(\xi(x,t),\eta_1(t),\eta_2(t))$ 满足系统

$$\begin{cases} D_\varphi \xi = [\mu(x,t)+\alpha_1^*(x,t)]\xi(x,t) - [f_1(q_1^*(t))+f_2(q_2^*(t))]\beta(x)\xi(0,t) + \\ \quad \Phi_1'(P^*(t))q_1^*(t)\eta_1(t) + \Phi_2'(P^*(t))q_2^*(t)\eta_2(t) + \omega_1(x,t)\alpha_1^*(x,t), \\ \dfrac{d\eta_1}{dt} = -\left[r_1(q_1^*(t),t) + q_1^*(t)\left(\dfrac{\partial r_1(q_1^*(t),t)}{\partial q_1} - h_1'(q_2^*(t))\right) - \Phi_1(P^*) - h_1(q_2^*) - \alpha_2^*\right]\eta_1 + \\ \quad \omega_2(t)\alpha_2^*(t) - f_1'(q_1^*(t))\xi_1(0,t)\displaystyle\int_0^l \beta(x)p^*(x,t)dx, \\ \dfrac{d\eta_2}{dt} = -\left[r_2(q_2^*(t),t) + q_2^*(t)\left(\dfrac{\partial r_2(q_2^*(t),t)}{\partial q_2} - h_2'(q_1^*(t))\right) - \Phi_2(P^*) - h_2(q_1^*) - \alpha_3^*\right]\eta_2 + \\ \quad \omega_3(t)\alpha_3^*(t) - f_2'(q_2^*(t))\xi_1(0,t)\displaystyle\int_0^l \beta(x)p^*(x,t)dx, \end{cases} \tag{17}$$

其中

$$\begin{cases} \xi(x,t) = \xi(x,t+T),\ \eta_i(t) = \eta_i(t+T), \quad i = 1,2, \\ \xi(l,t) = 0,\ P^*(t) = \displaystyle\int_0^l p^*(x,t)dx. \end{cases}$$

证明　对任意 $(v_1,v_2,v_3) \in \mathcal{T}_{\mathcal{U}}(\alpha_1^*,\alpha_2^*,\alpha_3^*)$ 及充分小 $\varepsilon > 0$,有 $(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon) \triangleq (\alpha_1^* + \varepsilon v_1, \alpha_2^* + \varepsilon v_2, \alpha_3^* + \varepsilon v_3) \in \mathcal{U}$.设 $(p^\varepsilon,q_1^\varepsilon,q_2^\varepsilon)$ 是系统(1)相应于 $(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)$ 的解.由 $(\alpha_1^*,\alpha_2^*,\alpha_3^*)$ 的最优性可得, $J(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon) \leqslant J(\alpha_1^*,\alpha_2^*,\alpha_3^*)$.从而有

$$\begin{aligned} 0 \geqslant & \int_0^T\int_0^l (\omega_1\alpha_1^* z_1)(x,t)dxdt + \int_0^T\int_0^l [(\omega_1 p^* - c_1\alpha_1^*)v_1](x,t)dxdt + \\ & \sum_{i=2}^3 \int_0^T (\omega_i\alpha_i^* z_i)(t)dt + \sum_{i=2}^3 \int_0^T [(\omega_i q_{i-1}{}^* - c_i\alpha_i^*)v_i](t)dt, \end{aligned} \tag{18}$$

其中

$$z_1(x,t) = \lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}[p^\varepsilon - p^*](x,t),\ z_2(t) = \lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}[q_1^\varepsilon - q_1^*](t),\ z_3(t) = \lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}[q_2^\varepsilon - q_2^*](t).$$

由文献[24]中定理 2(P18)知, $z_1(x,t)$, $z_2(t)$ 和 $z_3(t)$ 有意义.且由系统(1)知 (z_1,z_2,z_3) 满足

$$\begin{cases} \dfrac{\partial z_1}{\partial t} + \dfrac{\partial(g(x)z_1)}{\partial x} = -[\mu(x,t)+\alpha_1^*(x,t)]z_1(x,t) - v_1(x,t)p^*(x,t), \\ \dfrac{dz_2}{dt} = \left[r_1(q_1^*(t),t) + q_1^*(t)g\left(\dfrac{\partial r_1(q_1^*(t),t)}{\partial q_1} - h_1'(q_2^*(t))g\right) - \Phi_1(P^*) - \right. \\ \quad \left. h_1(q_2^*(t)) - \alpha_2^*\right]z_2 - \left[\Phi_1'(P^*(t))\displaystyle\int_0^l z_1(x,t)dx + v_2(t)\right]q_1^*(t), \\ \dfrac{dz_3}{dt} = \left[r_2(q_2^*(t),t) + q_2^*(t)g\left(\dfrac{\partial r_2(q_2^*(t),t)}{\partial q_2} - h_2'(q_1^*(t))g\right) - \Phi_2(P^*) - \right. \\ \quad \left. h_2(q_1^*(t)) - \alpha_3^*\right]z_3 - \left[\Phi_2'(P^*(t))\displaystyle\int_0^l z_1(x,t)dx + v_3(t)\right]q_2^*(t), \\ g(0)z_1(0,t) = [f_1(q_1^*(t)) + f_2(q_2^*(t))]\displaystyle\int_0^l \beta(x)z_1(x,t)dx + \\ \quad [f_1'(q_1^*(t))z_2(t) + f_2'(q_2^*(t))z_3(t)]\displaystyle\int_0^l \beta(x)p^*(x,t)dx, \\ P^*(t) = \displaystyle\int_0^l p^*(x,t)dx,\ z_1(x,t) = z_1(x,t+T),\ z_i(t) = z_i(t+T), \quad i = 2,3. \end{cases} \tag{19}$$

系统(19)的前三个方程分别乘以 $\xi(x,t)$, $\eta_1(t)$ 和 $\eta_2(t)$ 后, 并分别在 $Q_T \triangleq [0,l)\times[0,T]$, $[0,T]$ 和 $[0,T]$ 上进行积分, 然后利用式(11)可得

$$\int_0^T\int_0^l(\omega_1\alpha_1^* z_1)(x,t)\,\mathrm{d}x\mathrm{d}t + \sum_{i=2}^3\int_0^T(\omega_i\alpha_i^* z_i)(t)\,\mathrm{d}t =$$
$$\int_0^T\int_0^l(\xi p^* v_1)(x,t)\,\mathrm{d}x\mathrm{d}t + \sum_{i=2}^3\int_0^T(\eta_{i-1}q_{i-1}^* v_i)(t)\,\mathrm{d}t. \tag{20}$$

将式(20)代入式(18)可知,对任意的 $(v_1,v_2,v_3)\in\mathcal{T}_{\mathcal{U}}(\alpha_1^*,\alpha_2^*,\alpha_3^*)$,有

$$0\geqslant\int_0^T\int_0^l\{[\omega_1(x,t)+\xi(x,t)]p^*(x,t)-c_1\alpha_1^*(x,t)\}v_1(x,t)\,\mathrm{d}x\mathrm{d}t +$$
$$\sum_{i=2}^3\int_0^T\{[\omega_i(t)+\eta_{i-1}(t)]q_{i-1}^*(t)-c_i\alpha_i^*(t)\}v_i(t)\,\mathrm{d}t.$$

于是,$([(\omega_1+\xi)p^*-c_1\alpha_1^*](x,t),[(\omega_2+\eta_1)q_1^*-c_2\alpha_2^*](t),[(\omega_3+\eta_2)q_2^*-c_3\alpha_3^*](t))\in\mathcal{N}_{\mathcal{U}}(\alpha_1^*,\alpha_2^*,\alpha_3^*)$.因此,定理得证.

3　最优收获的存在唯一性

为应用 Ekeland 变分原理证明最优收获的存在唯一性,本节给出如下映射:

$$\tilde J(\alpha_1,\alpha_2,\alpha_3)=\begin{cases}J(\alpha_1,\alpha_2,\alpha_3), & (\alpha_1,\alpha_2,\alpha_3)\in\mathcal{U},\\ -\infty, & (\alpha_1,\alpha_2,\alpha_3)\notin\mathcal{U}.\end{cases}$$

引理 1　映射 $\tilde J(\alpha_1,\alpha_2,\alpha_3)$ 是上半连续的.

证明　假设当 $n\to+\infty$ 时,有$(\alpha_1^n,\alpha_2^n,\alpha_3^n)\to(\alpha_1,\alpha_2,\alpha_3)$.记$(p_n,q_{1n},q_{2n})$和$(p,q_1,q_2)$分别是系统(1)对应于$(\alpha_1^n,\alpha_2^n,\alpha_3^n)$和$(\alpha_1,\alpha_2,\alpha_3)$的解.由 Riesz 定理知,存在$\{(\alpha_1^n,\alpha_2^n,\alpha_3^n)\}$的子序列,仍记作$\{(\alpha_1^n,\alpha_2^n,\alpha_3^n)\}$,使得当 $n\to\infty$ 时,有

$$(\alpha_1^n(x,t))^2\to(\alpha_1(x,t))^2,\ \text{a.s. }(x,t)\in Q;\ (\alpha_i^n(t))^2\to(\alpha_i(t))^2,\ \text{a.s. }t\in(0,T),\qquad i=2,3.$$

由 Lebesgue 控制收敛定理可得

$$\lim_{n\to+\infty}\int_0^T\int_0^l(\alpha_1^n(x,t))^2\mathrm{d}x\mathrm{d}t=\int_0^T\int_0^l(\alpha_1(x,t))^2\mathrm{d}x\mathrm{d}t,$$
$$\lim_{n\to+\infty}\int_0^T(\alpha_i^n(t))^2\mathrm{d}t=\int_0^T(\alpha_i(t))^2\mathrm{d}t,\qquad i=2,3.$$

进一步地,由定理 2 可知

$$\left|\int_0^T\omega_2(t)\alpha_2^n(t)q_{1n}(t)\,\mathrm{d}t-\int_0^T\omega_2(t)\alpha_2(t)q_1(t)\,\mathrm{d}t\right|\leqslant$$
$$\int_0^T\omega_2(t)q_{1n}(t)\,|\,\alpha_2^n(t)-\alpha_2(t)\,|\,\mathrm{d}t+\int_0^T\omega_2(t)\alpha_2(t)\,|\,q_{1n}(t)-q_1(t)\,|\,\mathrm{d}t\leqslant$$
$$M\|\omega_2\|_{L^\infty(0,T)}\|\alpha_2^n-\alpha_2\|_{L^1(0,T)}+$$
$$\|\omega_2\|_{L^\infty(0,T)}N_2K_2T[\|\alpha_1^n-\alpha_1\|_{L^1(Q)}+\|\alpha_2^n-\alpha_2\|_{L^1(0,T)}+\|\alpha_3^n-\alpha_3\|_{L^1(0,T)}].$$

于是

$$\lim_{n\to+\infty}\int_0^T\omega_2(t)\alpha_2^n(t)q_{1n}(t)\,\mathrm{d}t=\int_0^T\omega_2(t)\alpha_2(t)q_1(t)\,\mathrm{d}t.$$

类似地,

$$\lim_{n\to+\infty}\int_0^T\int_0^l\omega_1(x,t)\alpha_1^n(x,t)p_n(x,t)\,\mathrm{d}x\mathrm{d}t=\int_0^T\int_0^l\omega_1(x,t)\alpha_1(x,t)p(x,t)\,\mathrm{d}x\mathrm{d}t,$$
$$\lim_{n\to+\infty}\int_0^T\omega_3(t)\alpha_3^n(t)q_{2n}(t)\,\mathrm{d}t=\int_0^T\omega_3(t)\alpha_3(t)q_2(t)\,\mathrm{d}t.$$

从而,由 Fatou 引理可得 $\limsup_{n\to+\infty}\tilde J(\alpha_1^n,\alpha_2^n,\alpha_3^n)\leqslant\tilde J(\alpha_1,\alpha_2,\alpha_3)$.引理得证.

定理 5　如果 $T(1/c_1+1/c_2+1/c_3)$ 充分小,则控制问题(2)有唯一最优解 $(\alpha_1^*,\alpha_2^*,\alpha_3^*)\in\mathcal{U}$.

证明　由引理 1 和 Ekeland 变分原理知,对任意的 $\varepsilon>0$,存在$(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)\in\mathcal{U}$ 使得

$$\tilde{J}(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon) \geqslant \sup_{(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{U}} \tilde{J}(\alpha_1,\alpha_2,\alpha_3) - \varepsilon, \tag{21}$$

$$\tilde{J}(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon) \geqslant \sup_{(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{U}} \Big\{ \tilde{J}(\alpha_1,\alpha_2,\alpha_3) - \sqrt{\varepsilon}\,\|\alpha_1^\varepsilon-\alpha_1\|_{L_T^1(Q)} - \sqrt{\varepsilon}\sum_{i=2}^{3}\|\alpha_i^\varepsilon-\alpha_i\|_{L_T^1(\mathbf{R}_+)} \Big\}. \tag{22}$$

因此,扰动泛函 $\tilde{J}_\varepsilon(\alpha_1,\alpha_2,\alpha_3)=\tilde{J}(\alpha_1,\alpha_2,\alpha_3)-\sqrt{\varepsilon}\,\|\alpha_1^\varepsilon-\alpha_1\|_{L_T^1(Q)}-\sqrt{\varepsilon}\sum_{i=2}^{3}\|\alpha_i^\varepsilon-\alpha_i\|_{L_T^1(\mathbf{R}_+)}$ 在点$(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)$ 处取得极大值.类似定理 4 的讨论,有

$$\alpha_1^\varepsilon(x,t)=\mathcal{F}_1\left[\frac{[\omega_1(x,t)+\xi^\varepsilon(x,t)]p^\varepsilon(x,t)}{c_1}+\frac{\sqrt{\varepsilon}\,\theta_1(x,t)}{c_1}\right],$$

$$\alpha_i^\varepsilon(t)=\mathcal{F}_i\left[\frac{[\omega_i(t)+\eta_{i-1}^\varepsilon(t)]q_{i-1}^\varepsilon(t)}{c_i}+\frac{\sqrt{\varepsilon}\,\theta_i(t)}{c_i}\right],\qquad i=2,3,$$

其中 $\theta_1\in L_T^\infty(Q)$, $\theta_2\in L_T^\infty(\mathbf{R}_+)$, $\theta_3\in L_T^\infty(\mathbf{R}_+)$, $|\theta_1(x,t)|\leqslant 1$, $|\theta_2(t)|\leqslant 1$, $|\theta_3(t)|\leqslant 1$.

定义映射 $\mathcal{C}:\mathcal{U}\to L_T^\infty(Q)\times L_T^\infty(\mathbf{R}_+)\times L_T^\infty(\mathbf{R}_+)$,

$$\mathcal{C}(\alpha_1,\alpha_2,\alpha_3)=\left(\mathcal{F}_1\left[\frac{(\omega_1+\xi_1)p}{c_1}\right],\mathcal{F}_2\left[\frac{(\omega_2+\eta_1)q_1}{c_2}\right],\mathcal{F}_3\left[\frac{(\omega_3+\eta_2)q_2}{c_3}\right]\right),$$

其中 (p,q_1,q_2) 和(ξ,η_1,η_2) 分别是状态系统和共轭系统相应于控制变量$(\alpha_1,\alpha_2,\alpha_3)$ 的解.对任意的$(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{U}$, 由式(16)知, $(0,0,0)\leqslant\mathcal{C}(\alpha_1,\alpha_2,\alpha_3)\leqslant(N_1,N_2,N_3)$.从而,$\mathcal{C}$ 是从 $\mathcal{U}$ 到 $\mathcal{U}$ 的映射.由定理 2 和 3 知,(p_1,p_2,q) 和(ξ_1,ξ_1,η) 均关于控制变量连续.于是

$$\begin{aligned}&\|\mathcal{C}(\alpha_1,\alpha_2,\alpha_3)-\mathcal{C}(\alpha_1',\alpha_2',\alpha_3')\|\leqslant\\&\quad c_1^{-1}\|(\omega_1+\xi)p-(\omega_1+\xi')p'\|_{L_T^\infty(Q)}+\sum_{i=2}^{3}c_i^{-1}\|(\omega_i+\eta_{i-1})q_{i-1}-(\omega_i+\eta_{i-1}')q_{i-1}'\|_{L_T^\infty(\mathbf{R}_+)}\leqslant\\&\quad K_4T(c_1^{-1}+c_2^{-1}+c_3^{-1})(\|\alpha_1-\alpha_1'\|_{L_T^\infty(Q)}+\|\alpha_2-\alpha_2'\|_{L_T^\infty(\mathbf{R}_+)}+\|\alpha_3-\alpha_3'\|_{L_T^\infty(\mathbf{R}_+)}),\end{aligned}$$

其中 K_4 为正常数.显然,若 $K_4T(c_1^{-1}+c_2^{-1}+c_3^{-1})<1$,则 $\mathcal{C}$ 有唯一不动点$(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)\in\mathcal{U}$.

下面证明 $(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)$ 为最优策略,即当 $\varepsilon\to0^+$ 时,有$(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)\to(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)$.由于

$$\begin{aligned}&\|\mathcal{C}(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)-(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)\|_\infty\leqslant\\&\quad c_1^{-1}\|(\omega_1+\xi^\varepsilon)p^\varepsilon-(\omega_1+\xi^\varepsilon)p^\varepsilon-\sqrt{\varepsilon}\,\theta_1\|_\infty+\\&\quad\sum_{i=2}^{3}c_i^{-1}\|(\omega_i+\eta_{i-1}^\varepsilon)q_{i-1}^\varepsilon-(\omega_i+\eta_{i-1}^\varepsilon)q_{i-1}^\varepsilon-\sqrt{\varepsilon}\,\theta_i\|_\infty\leqslant\\&\quad c_1^{-1}\sqrt{\varepsilon}\,\|\theta_1(x,t)\|_\infty+c_2^{-1}\sqrt{\varepsilon}\,\|\theta_2(t)\|_\infty+c_3^{-1}\sqrt{\varepsilon}\,\|\theta_3(t)\|_\infty\leqslant\sqrt{\varepsilon}(c_1^{-1}+c_2^{-1}+c_3^{-1}),\end{aligned}$$

于是

$$\|(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)-(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)\|_\infty\leqslant(c_1^{-1}+c_2^{-1}+c_3^{-1})\left(K_4T\sum_{i=1}^{3}\|\bar{\alpha}_i-\alpha_i^\varepsilon\|_\infty+\sqrt{\varepsilon}\right).$$

此外,由于 $\|(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)-(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)\|_\infty=\|\bar{\alpha}_1-\alpha_1^\varepsilon\|_\infty+\|\bar{\alpha}_2-\alpha_2^\varepsilon\|_\infty+\|\bar{\alpha}_3-\alpha_3^\varepsilon\|_\infty$, 于是可得

$$\begin{aligned}&\|(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)-(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)\|_\infty\leqslant\\&\quad K_4T(c_1^{-1}+c_2^{-1}+c_3^{-1})\|(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)-(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)\|_\infty+\sqrt{\varepsilon}(c_1^{-1}+c_2^{-1}+c_3^{-1}),\end{aligned}$$

即

$$\|(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)-(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)\|_\infty\leqslant\frac{\sqrt{\varepsilon}(c_1^{-1}+c_2^{-1}+c_3^{-1})}{1-K_4T(c_1^{-1}+c_2^{-1}+c_3^{-1})}.$$

从而,当 $\varepsilon\to0^+$ 时,有$(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)\to(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)$.由引理 1 知,$\tilde{J}(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)=\sup_{(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{U}}\tilde{J}(\alpha_1,\alpha_2,\alpha_3)$,即$(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)\in\mathcal{U}$ 是最优收获策略.定理得证.

4　结论与注记

本文研究了一类周期环境中具有两相互竞争食饵和一捕食者的三物种捕食-食饵系统的最优收获问题，其中捕食者依赖个体尺度.讨论了系统非负有界解的存在唯一性及解关于控制变量的连续依赖性,更重要的结果是给出了最优策略的存在唯一性,为模型的实际应用奠定理论基础.对于最优策略结构,定理 4 利用共轭变量,提出了一个反馈策略.由于模型的高度非线性,本文很难给出一个精确的最优控制器.然而,反馈策略将有助于优化策略的数值计算,笔者拟将另文探讨.

参考文献(References)：

[1] 王双明, 张明军, 樊馨蔓. 一类具时滞的周期 logistic 传染病模型空间动力学研究[J]. 应用数学和力学, 2018, **39**(2)：226-238.(WANG Shuangming, ZHANG Mingjun, FAN Xinman. Spatial dynamics of periodic reaction-diffusion epidemic models with delay and logistic growth[J]. *Applied Mathematics and Mechanics*, 2018, **39**(2)：226-238.(in Chinese))

[2] 曹建智, 谭军, 王培光. 一类具有时滞的云杉蚜虫种群模型的 Hopf 分岔分析[J]. 应用数学和力学, 2019, **40**(3)：332-342.(CAO Jianzhi, TAN Jun, WANG Peiguang. Hopf bifurcation analysis of a model for spruce budworm populations with delays[J]. *Applied Mathematics and Mechanics*, 2019, **40**(3)：332-342.(in Chinese))

[3] HSU S B, RUAN S G, YANG T H. Analysis of three species Lotka-Volterra food web models with omnivory[J]. *Journal of Mathematical Analysis and Applications*, 2015, **426**(2)：659-687.

[4] SEN D, GHORAI S, BANERJEE M. Complex dynamics of a three species prey-predator model with intraguild predation[J]. *Ecological Complexity*, 2018, **34**：9-22.

[5] LIU R, LIU G R. Dynamics of a stochastic three species prey-predator model with intraguild predation[J]. *Journal of Applied Analysis and Computation*, 2020, **10**(1)：81-103.

[6] LIU M, BAI C. Analysis of a stochastic tri-trophic food-chain model with harvesting[J]. *Journal of Mathematical Biology*, 2016, **73**：597-625.

[7] MAGAL P, RUAN S G. *Structured Population Models in Biology and Epidemiology*[M]. Berlin：Springer, 2008.

[8] ANI A S. *Analysis and Control of Age-Dependent Population Dynamics*[M]. Dordrecht：Kluwer Academic Publishers, 2000.

[9] LUO Z X, HE Z R. Optimal control for age-dependent population hybrid system in a polluted environment[J]. *Applied Mathematics and Computation*, 2014, **228**：68-76.

[10] LI L, FERREIRA C, AINSEBA B. Optimal control of an age-structured problem modelling mosquito plasticity[J]. *Nonlinear Analysis：Real World Applications*, 2019, **45**：157-169.

[11] ANI A L, ANI A S. Note on some periodic optimal harvesting problems for age-structured population dynamics[J]. *Applied Mathematics and Computation*, 2016, **276**：21-30.

[12] LU Y, LIU S. Threshold dynamics of a predator-prey model with age-structured prey[J]. *Advances in Difference Equations*, 2018, **2018**(1)：164. DOI:10.1186/s13662-018-1614-y.

[13] FISTER K, LENHART S. Optimal harvesting in an age-structured predator-prey model[J]. *Applied Mathematics and Optimization*, 2006, **54**：1-15.

[14] 何泽荣, 刘荣, 刘丽丽. 捕食者具有年龄结构的种群系统的最优收获策略[J]. 数学进展, 2013, **42**(5)：691-700.(HE Zerong, LIU Rong, LIU Lili. Optimal harvesting for population system with age-dependent predator[J]. *Advances in Mathematics*, 2013, **42**(5)：691-700.(in Chinese))

[15] ANI A S, IANNELLI M, KIM M Y, et al. Optimal harvesting for periodic age-dependent population dynamics[J]. *SIAM Journal on Applied Mathematics*, 1998, **58**：1648-1666.

[16] SAUER J R, SLADE N A. Size-based demography of vertebrates[J]. *Annual Review of Ecology and Systematics*, 1987, **18**：71-90.

[17] CHU J X, DUCROT A, MAGAL P, et al. Hopf bifurcation in a size structured population dynamic model with random growth[J]. *Journal of Differential Equations*, 2009, **247**(3): 956-1000.

[18] YAN D, CAO Y, FU X. Asymptotic analysis of a size-structured population model with infinite states-at-birth [J]. *Applicable Analysis*, 2019, **98**(5): 913-933.

[19] LI Y, ZHANG Z, LV Y, et al. Optimal harvesting for a size-stage-structured population model[J]. *Nonlinear Analysis: Real World Applications*, 2018, **44**: 616-630.

[20] 刘炎, 何泽荣. 具有 size 结构的捕食种群系统的最优收获策略[J]. 数学物理学报, 2012, **32A**(1): 90-102.(LIU Yan, HE Zerong. Optimal harvesting of a size-structured predator-prey model[J]. *Acta Mathematical Scientia*, 2012, **32A**(1): 90-102.(in Chinese))

[21] 刘炎, 何泽荣. 一类基于尺度结构的种群系统的最优收获[J]. 系统科学与数学, 2015, **35**(4): 459-471.(LIU Yan, HE Zerong. Optimal harvesting for a population system with body size[J]. *Journal of Systems Science and Complexity*, 2015, **35**(4): 459-471.(in Chinese))

[22] 曹雪靓. 污染环境下具有尺度结构的捕食种群模型解的存在唯一性[J]. 应用数学进展, 2018, **7**(7): 758-765. (CAO Xuejing. Existence and uniqueness of solution for predator-prey population model with size-structured in polluted environment[J]. *Advances in Applied Mathematics*, 2018, **7**(7): 758-765.(in Chinese))

[23] ZHANG F Q, LIU R, CHEN Y M. Optimal harvesting in a periodic food chain model with size structures in predators[J]. *Applied Mathematics and Optimization*, 2017, **75**: 229-251.

[24] BARBU V. *Mathematical Methods in Optimization of Differential Systems*[M]. Dordrecht: Kluwer Academic Publishers, 1994.