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具有反应扩散项的变时滞复数域神经网络的指数稳定性*

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摘要: 该文研究了一类具有反应扩散项的变时滞复数域神经网络的指数稳定性.首先在假设复数域激活函数可分解的情况下,将该系统分解为相应的实部系统和虚部系统.利用矢量 Lyapunov 函数法和 M 矩阵理论,得到了确保该系统平衡状态指数稳定性的充分条件.该条件不含有任何自由变量,相对现有结论具有较低的保守性.最后通过一个数值仿真算例验证了所得结论的正确性.

关 键 词: 复数域神经网络; 变时滞; 反应扩散项; 矢量 Lyapunov 函数法; 指数稳定性
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Exponential Stability of Complex-Valued Neural Networks With Time-Varying Delays and Reaction-Diffusion Terms

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Abstract: The exponential stability of complex-valued neural networks with time-varying delays and reaction-diffusion terms was studied. Firstly, the addressed systems were separated into their real parts with the complex-valued activation functions assumed to be divided into the real parts and imaginary parts. Secondly, some

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sufficient conditions for ensuring the exponential stability of the equilibrium states of the systems were established based on the vector Lyapunov function method and the M-matrix theory. The obtained criteria have no free variables and reduced conservatism compared with the existing results. A numerical example proves the correctness of the obtained results.

Key words: complex-valued neural network; time-varying delay; reaction-diffusion term; vector Lyapunov function method; exponential stability

引　　言

复数域神经网络因其在图像处理、联想记忆、模式识别等应用领域具有不可取代的优势[1],对复值神经网络的动态行为分析成为近年的一个研究热点,学者们取得了大量重要的研究成果[2-14].文献[2]研究了一类固定时滞复数域递归神经网络的平衡点的存在性、唯一性以及全局渐进稳定性和全局指数稳定性,并得到了相应的判定定理.文献[5,8,10]同样在复数域神经网络模型中考虑了固定时滞.固定时滞在神经元个数较少的网络模型中是一种合理的近似模型,但不适用于高阶神经网络系统.此外,鉴于不同神经元之间的通信时间因硬件条件和通道长短不同而不同,因此文献[3,6-7,9]在复数域神经网络模型中考虑了可变时滞,并采用 Lyapunov 函数法对各系统的平衡点的动态行为进行了研究.除时滞现象外,在实际神经网络系统中也会存在其他影响系统稳定性的不确定性因素,如脉冲干扰[11,13]、随机干扰[11-12]、区间参数不确定性[4,12,14]、Markov 跳变参数[14]、反应扩散现象[15-17]等.然而,关于具有反应扩散项的神经网络的动态行为研究主要集中在实数域神经网络模型[15-17],关于具有反应扩散项的复数域神经网络的研究极为少见,可参见文献[18].另一方面,学者们在研究神经网络的动态行为时,主要采用了加权 Lyapunov 函数法[5-8,15-16]和 LMI 技术[2,4,10].采用这两种方法所建立的稳定性条件是由自由变量和系统本身假设条件构成的.在实际应用时,自由变量和假设条件的组合很大程度上依赖于人们的经验和尝试.文献[3,9,13-14]采用矢量 Lyapunov 函数法研究了几类复数域神经网络的稳定性问题,建立的稳定性条件不含有任何自由变量,仅由系统的假设条件和参数构成,便于应用.

综上讨论,据笔者所知目前并没有学者对具有变时滞和反应扩散项的复数域神经网络的稳定性进行研究,因此本文将采用矢量 Lyapunov 函数法和 M 矩阵理论建立确保该系统指数稳定的相关判据.

1　模型描述和基本引理、假设

首先给出一些记号.令 $\mathbb{C}$ 表示复数集,令 $\tilde{u}=\tilde{u}^{\mathrm{R}}+\tilde{u}^{\mathrm{I}}\mathrm{i}$ 是复数,其中 $\mathrm{i}^2=-1$.记 $\Theta=\{1,2,\cdots,n\}$.对于任意 $\tilde{\boldsymbol{u}}\in\mathbb{C}^n$,令 $|\tilde{\boldsymbol{u}}|=(|\tilde{u}_1|,|\tilde{u}_2|,\cdots,|\tilde{u}_n|)^{\mathrm{T}}$ 表示 $\tilde{\boldsymbol{u}}$ 的模.假设 $\Omega\in\mathbb{R}^m$ 是一个具有光滑边界 $\partial\Omega$ 的有界紧集,其中 $\operatorname{mes}\Omega>0$.

考虑如下系统:

$$\begin{cases}\dfrac{\partial\tilde{u}_k(t,z)}{\partial t}=\pi_k\displaystyle\sum_{h=1}^{m}\frac{\partial}{\partial z_h}\left(\frac{\partial\tilde{u}_k(t,z)}{\partial z_h}\right)-d_k\tilde{u}_k(t,z)+\\ \qquad\displaystyle\sum_{j=1}^{n}\left[a_{kj}\tilde{g}_j(\tilde{u}_j(t,z))+b_{kj}\tilde{g}_j(\tilde{u}_j(t-\tau_{kj}(t)),z)\right]+J_k,\\ \tilde{u}_k(t,z)\,|_{z\in\partial\Omega}=0,\end{cases}\tag{1}$$

其中 $t\geqslant0,k\in\Theta,n\geqslant2$ 表示神经元的个数; $\tilde{u}_k(t,z)$ 表示第 k 个神经元, $z\in\Omega$; 符号 $L^2(\mathbb{C}\times\Omega)$ 表示 Lebesgue 可测函数定义在 $\mathbb{C}\times\Omega$ 上的复数集,显然有 $\|\tilde{u}_k(t,z)\|_{L^2}=\left(\int_\Omega|\tilde{u}_k(t,z)|^2\mathrm{d}z\right)^{1/2}<\infty,k\in\Theta;\pi_k>0$ 表示扩散系数, $k\in\Theta;z_h$ 代表空间变量,且 $|z_h|\leqslant\omega_h,h=1,2,\cdots,m;d_k>0$ 表示充电时间常数, $k\in\Theta;\boldsymbol{A}=(a_{kj})_{n\times n}$ 和 $\boldsymbol{B}=(b_{kj})_{n\times n}$ 表示定义在 $\mathbb{C}^{n\times n}$ 上的关联矩阵; $\tilde{\boldsymbol{g}}(\tilde{\boldsymbol{u}})=(\tilde{g}_1(\tilde{u}_1),\tilde{g}_2(\tilde{u}_2),\cdots,\tilde{g}_n(\tilde{u}_n))^{\mathrm{T}}$ 表示激活函数; J_k 表示第 k 个神经元的外部常输入, $k\in\Theta$; 时延 $\tau_{kj}(t)$ 为有界函数,且 $\tau=\max_{k,j\in\Theta}\sup_{t\geqslant0}\tau_{kj}(t)$.

令 $\boldsymbol{J}=(J_1,J_2,\cdots,J_n)^{\mathrm{T}}$, $\boldsymbol{\pi}=\operatorname{diag}(\pi_k)_{n\times n}$, $\boldsymbol{D}=\operatorname{diag}(d_k)_{n\times n}$.

假设系统(1)初始条件为 $\tilde{u}_k(s,z)=\tilde{\varphi}_k(s,z)$, 其中 $k\in\Theta, z\in\Omega$, 且 $\tilde{\boldsymbol{\varphi}}=[\tilde{\varphi}_1,\tilde{\varphi}_2,\cdots,\tilde{\varphi}_n]^{\mathrm{T}}\in C_{F_0}[[-\tau,0]\times\mathbb{R}^m,\mathbb{C}^n]$, 这里的 $F_0=F_s$ 定义在 $[-\tau,0]\times\mathbb{R}^m$, $C_{F_0}[[-\tau,0]\times\mathbb{R}^m,\mathbb{C}^n]$ 上,是一族连续复值函数,且满足 $\|\tilde{\varphi}_k(s,z)\|_{L^2}=\left(\int_\Omega|\tilde{\varphi}_k(s,z)|^2\mathrm{d}z\right)^{1/2}, k\in\Theta$.

令 $\tilde{u}_k(t,z)=\tilde{u}_k^{\mathrm{R}}(t,z)+\mathrm{i}\tilde{u}_k^{\mathrm{I}}(t,z)$, 且激活函数 $\tilde{g}_j(\tilde{u}_j(t,z))$ 可分解为如下形式:

$$\tilde{g}_j(\tilde{u}_j(t,z))=\tilde{g}_j^{\mathrm{R}}(\tilde{u}_j^{\mathrm{R}}(t,z),\tilde{u}_j^{\mathrm{I}}(t,z))+\mathrm{i}\tilde{g}_j^{\mathrm{I}}(\tilde{u}_j^{\mathrm{R}}(t,z),\tilde{u}_j^{\mathrm{I}}(t,z)),\tag{2}$$

其中

$$\tilde{g}_j^{\mathrm{R}}(\tilde{u}_j^{\mathrm{R}}(t,z),\tilde{u}_j^{\mathrm{I}}(t,z)):\mathbb{R}^2\to\mathbb{R},\ \tilde{g}_j^{\mathrm{I}}(\tilde{u}_j^{\mathrm{R}}(t,z),\tilde{u}_j^{\mathrm{I}}(t,z)):\mathbb{R}^2\to\mathbb{R},\qquad k,j\in\Theta.$$

进一步,系统(1)分解为

$$\begin{cases}\dfrac{\partial\tilde{u}_k^{\mathrm{R}}(t,z)}{\partial t}=\boldsymbol{\pi}_k\sum_{h=1}^m\dfrac{\partial}{\partial z_h}\left(\dfrac{\partial\tilde{u}_k^{\mathrm{R}}(t,z)}{\partial z_h}\right)-d_k\tilde{u}_k^{\mathrm{R}}(t,z)+\\ \quad\sum_{j=1}^n[a_{kj}^{\mathrm{R}}\tilde{g}_j^{\mathrm{R}}(\tilde{u}_j^{\mathrm{R}}(t,z),\tilde{u}_j^{\mathrm{I}}(t,z))-a_{kj}^{\mathrm{I}}\tilde{g}_j^{\mathrm{I}}(\tilde{u}_j^{\mathrm{R}}(t,z),\tilde{u}_j^{\mathrm{I}}(t,z))]+\\ \quad\sum_{j=1}^n[b_{kj}^{\mathrm{R}}\tilde{g}_j^{\mathrm{R}}(\tilde{u}_j^{\mathrm{R}}(t-\tau_{kj}(t),z),\tilde{u}_j^{\mathrm{I}}(t-\tau_{kj}(t),z))-\\ \quad b_{kj}^{\mathrm{I}}\tilde{g}_j^{\mathrm{I}}(\tilde{u}_j^{\mathrm{R}}(t-\tau_{kj}(t),z),\tilde{u}_j^{\mathrm{I}}(t-\tau_{kj}(t),z))]+J_k^{\mathrm{R}},\\ \tilde{u}_k^{\mathrm{R}}(t,z)|_{z\in\partial\Omega}=0,\end{cases}\tag{3}$$

$$\begin{cases}\dfrac{\partial\tilde{u}_k^{\mathrm{I}}(t,z)}{\partial t}=\boldsymbol{\pi}_k\sum_{h=1}^m\dfrac{\partial}{\partial z_h}\left(\dfrac{\partial\tilde{u}_k^{\mathrm{I}}(t,z)}{\partial z_h}\right)-d_k\tilde{u}_k^{\mathrm{I}}(t,z)+\\ \quad\sum_{j=1}^n[a_{kj}^{\mathrm{R}}\tilde{g}_j^{\mathrm{I}}(\tilde{u}_j^{\mathrm{R}}(t,z),\tilde{u}_j^{\mathrm{I}}(t,z))+a_{kj}^{\mathrm{I}}\tilde{g}_j^{\mathrm{R}}(\tilde{u}_j^{\mathrm{R}}(t,z),\tilde{u}_j^{\mathrm{I}}(t,z))]+\\ \quad\sum_{j=1}^n[b_{kj}^{\mathrm{R}}\tilde{g}_j^{\mathrm{I}}(\tilde{u}_j^{\mathrm{R}}(t-\tau_{kj}(t),z),\tilde{u}_j^{\mathrm{I}}(t-\tau_{kj}(t),z))+\\ \quad b_{kj}^{\mathrm{I}}\tilde{g}_j^{\mathrm{R}}(\tilde{u}_j^{\mathrm{R}}(t-\tau_{kj}(t),z),\tilde{u}_j^{\mathrm{I}}(t-\tau_{kj}(t),z))]+J_k^{\mathrm{I}},\\ \tilde{u}_k^{\mathrm{I}}(t,z)|_{z\in\partial\Omega}=0,\end{cases}\tag{4}$$

其中 $k\in\Theta$, $\boldsymbol{A}^{\mathrm{R}}=(a_{kj}^{\mathrm{R}})_{n\times n}$, $\boldsymbol{B}^{\mathrm{R}}=(b_{kj}^{\mathrm{R}})_{n\times n}$ 和 $\boldsymbol{J}^{\mathrm{R}}=(J_1^{\mathrm{R}},J_2^{\mathrm{R}},\cdots,J_n^{\mathrm{R}})^{\mathrm{T}}$ 分别表示 $\boldsymbol{A},\boldsymbol{B}$ 和 $\boldsymbol{J}$ 的实部; $\boldsymbol{A}^{\mathrm{I}}=(a_{kj}^{\mathrm{I}})_{n\times n}$, $\boldsymbol{B}^{\mathrm{I}}=(b_{kj}^{\mathrm{I}})_{n\times n}$ 和 $\boldsymbol{J}^{\mathrm{I}}=(J_1^{\mathrm{I}},J_2^{\mathrm{I}},\cdots,J_n^{\mathrm{I}})^{\mathrm{T}}$ 分别表示 $\boldsymbol{A},\boldsymbol{B}$ 和 $\boldsymbol{J}$ 的虚部.

令 $\tilde{\boldsymbol{u}}^{\#}=(\tilde{u}_1^{\#},\tilde{u}_2^{\#},\cdots,\tilde{u}_n^{\#})^{\mathrm{T}}$ 表示系统(1)的平衡状态, $\tilde{u}_k^{\#}=\tilde{u}_k^{\#\mathrm{R}}+\mathrm{i}\tilde{u}_k^{\#\mathrm{I}}, k\in\Theta$.

假设 1 假设激活函数 $\tilde{g}_j(\cdot)$ 满足如下条件:

(a) $\tilde{g}_j(\cdot)$ 存在连续的偏导数 $\partial\tilde{g}_j^{\mathrm{R}}/\partial\tilde{u}_j^{\mathrm{R}}$, $\partial\tilde{g}_j^{\mathrm{R}}/\partial\tilde{u}_j^{\mathrm{I}}$, $\partial\tilde{g}_j^{\mathrm{I}}/\partial\tilde{u}_j^{\mathrm{R}}$ 和 $\partial\tilde{g}_j^{\mathrm{I}}/\partial\tilde{u}_j^{\mathrm{I}}$;

(b) 存在正常数 $l_j^{\mathrm{RR}}, l_j^{\mathrm{RI}}, l_j^{\mathrm{IR}}$ 和 l_j^{II} 使得 $|\partial\tilde{g}_j^{\mathrm{R}}/\partial\tilde{u}_j^{\mathrm{R}}|\le l_j^{\mathrm{RR}}$, $|\partial\tilde{g}_j^{\mathrm{R}}/\partial\tilde{u}_j^{\mathrm{I}}|\le l_j^{\mathrm{RI}}$, $|\partial\tilde{g}_j^{\mathrm{I}}/\partial\tilde{u}_j^{\mathrm{R}}|\le l_j^{\mathrm{IR}}$, $|\partial\tilde{g}_j^{\mathrm{I}}/\partial\tilde{u}_j^{\mathrm{I}}|\le l_j^{\mathrm{II}}$.根据多变量函数均值定理,对任意 $\tilde{u}_j^{\mathrm{R}},\hat{u}_j^{\mathrm{R}},\tilde{u}_j^{\mathrm{I}},\hat{u}_j^{\mathrm{I}}\in\mathbb{R}$ 和 $j\in\Theta$, 有

$$|\tilde{g}_j^{\mathrm{R}}(\tilde{u}_j^{\mathrm{R}},\tilde{u}_j^{\mathrm{I}})-\tilde{g}_j^{\mathrm{R}}(\hat{u}_j^{\mathrm{R}},\hat{u}_j^{\mathrm{I}})|\le l_j^{\mathrm{RR}}|\tilde{u}_j^{\mathrm{R}}-\hat{u}_j^{\mathrm{R}}|+l_j^{\mathrm{RI}}|\tilde{u}_j^{\mathrm{I}}-\hat{u}_j^{\mathrm{I}}|,\tag{5}$$

$$|\tilde{g}_j^{\mathrm{I}}(\tilde{u}_j^{\mathrm{R}},\tilde{u}_j^{\mathrm{I}})-\tilde{g}_j^{\mathrm{I}}(\hat{u}_j^{\mathrm{R}},\hat{u}_j^{\mathrm{I}})|\le l_j^{\mathrm{IR}}|\tilde{u}_j^{\mathrm{R}}-\hat{u}_j^{\mathrm{R}}|+l_j^{\mathrm{II}}|\tilde{u}_j^{\mathrm{I}}-\hat{u}_j^{\mathrm{I}}|.\tag{6}$$

令 $\boldsymbol{L}^{\mathrm{RR}}=\mathrm{diag}(l_1^{\mathrm{RR}},l_2^{\mathrm{RR}},\cdots,l_n^{\mathrm{RR}})$, $\boldsymbol{L}^{\mathrm{RI}}=\mathrm{diag}(l_1^{\mathrm{RI}},l_2^{\mathrm{RI}},\cdots,l_n^{\mathrm{RI}})$, $\boldsymbol{L}^{\mathrm{IR}}=\mathrm{diag}(l_1^{\mathrm{IR}},l_2^{\mathrm{IR}},\cdots,l_n^{\mathrm{IR}})$, $\boldsymbol{L}^{\mathrm{II}}=\mathrm{diag}(l_1^{\mathrm{II}},l_2^{\mathrm{II}},\cdots,l_n^{\mathrm{II}})$.

定义 1 若存在常数 $\Gamma>0$ 和 $\lambda>0$, 对于所有的 $z\in\Omega$, $\boldsymbol{J}\in\mathbb{C}^n$ 和 $t\ge0$ 有 $\|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}^{\#}\|_{L^2}\le\sup_{s\in[-\tau,0]}\|\tilde{\boldsymbol{\varphi}}(s,z)-\tilde{\boldsymbol{u}}^{\#}\|_{L^2}\Gamma\mathrm{e}^{-\lambda t}$ 成立,则称系统(1)的平衡状态 $\tilde{\boldsymbol{u}}^{\#}$ 是全局指数稳定的.

引理 1[19] 令 $\Omega\in\mathbb{R}^m$ 是具有光滑边界 $\partial\Omega$ 的紧集.对于所有 $z\in\Omega$,令 $|z_h|\le\omega_h\le\hat{\omega}(h=1,2,\cdots,m)$.令 $u(z)\in\mathbb{R}$ 是属于 $C^1(\Omega)$ 的实值连续函数, 并且满足 Dirichlet 边界条件, 则有 $\int_\Omega u^2(z)\mathrm{d}z\le\dfrac{\hat{\omega}^2}{m}\int_\Omega\nabla u(z)\cdot\nabla u^{\mathrm{T}}(z)\mathrm{d}z$ 成立.

2　稳定性判据

定理 1　假设 1 是成立的.若对于所有 $\boldsymbol{J}\in\mathbb{C}^n, z\in\Omega$, 矩阵 $\tilde{\boldsymbol{T}}-\tilde{\boldsymbol{P}}\tilde{\boldsymbol{L}}$ 是 M 矩阵,则系统(1)的平衡状态是全局指数稳定的,其中

$$\tilde{\boldsymbol{P}}=\begin{bmatrix}|\boldsymbol{A}^{\mathrm{R}}|+|\boldsymbol{B}^{\mathrm{R}}| & |\boldsymbol{A}^{\mathrm{I}}|+|\boldsymbol{B}^{\mathrm{I}}|\\ |\boldsymbol{A}^{\mathrm{I}}|+|\boldsymbol{B}^{\mathrm{I}}| & |\boldsymbol{A}^{\mathrm{R}}|+|\boldsymbol{B}^{\mathrm{R}}|\end{bmatrix},\ \tilde{\boldsymbol{L}}=\begin{bmatrix}\boldsymbol{L}^{\mathrm{RR}} & \boldsymbol{L}^{\mathrm{RI}}\\ \boldsymbol{L}^{\mathrm{IR}} & \boldsymbol{L}^{\mathrm{II}}\end{bmatrix},$$

$$\tilde{\boldsymbol{T}}=\mathrm{diag}(\tilde{T}_k)_{2n\times 2n}=\begin{bmatrix}\boldsymbol{T} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{T}\end{bmatrix},\ \boldsymbol{T}=\mathrm{diag}(T_k)_{n\times n},$$

这里

$$T_k=2\frac{m\pi_k}{\hat{\omega}^2}+2d_k-\sum_{j=1}^{n}\left[(l_j^{\mathrm{RR}}+l_j^{\mathrm{RI}})(|a_{kj}^{\mathrm{R}}|+|b_{kj}^{\mathrm{R}}|)+(l_j^{\mathrm{IR}}+l_j^{\mathrm{II}})(|a_{kj}^{\mathrm{I}}|+|b_{kj}^{\mathrm{I}}|)\right],\qquad k\in\Theta.$$

证明　为了分析方便,首先进行坐标平移.令 $u_k=\tilde{u}_k-\tilde{u}_k^{\#}=u_k^{\mathrm{R}}+\mathrm{i}u_k^{\mathrm{I}}, k\in\Theta$, 则系统(3)和(4)变为

$$\begin{cases}\dfrac{\partial u_k^{\mathrm{R}}(t,z)}{\partial t}=\pi_k\sum\limits_{h=1}^{m}\dfrac{\partial}{\partial z_h}\left[\dfrac{\partial u_k^{\mathrm{R}}(t,z)}{\partial z_h}\right]-d_ku_k^{\mathrm{R}}(t,z)+\\ \quad\sum\limits_{j=1}^{n}\left[a_{kj}^{\mathrm{R}}g_j^{\mathrm{R}}(u_j^{\mathrm{R}}(t,z),u_j^{\mathrm{I}}(t,z))-a_{kj}^{\mathrm{I}}g_j^{\mathrm{I}}(u_j^{\mathrm{R}}(t,z),u_j^{\mathrm{I}}(t,z))\right]+\\ \quad\sum\limits_{j=1}^{n}\left[b_{kj}^{\mathrm{R}}g_j^{\mathrm{R}}(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z),u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))-\right.\\ \quad\left.b_{kj}^{\mathrm{I}}g_j^{\mathrm{I}}(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z),u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))\right],\\ u_k^{\mathrm{R}}(t,z)|_{z\in\partial\Omega}=0,\end{cases}\tag{7}$$

$$\begin{cases}\dfrac{\partial u_k^{\mathrm{I}}(t,z)}{\partial t}=\pi_k\sum\limits_{h=1}^{m}\dfrac{\partial}{\partial z_h}\left[\dfrac{\partial u_k^{\mathrm{I}}(t,z)}{\partial z_h}\right]-d_ku_k^{\mathrm{I}}(t,z)+\\ \quad\sum\limits_{j=1}^{n}\left[a_{kj}^{\mathrm{R}}g_j^{\mathrm{I}}(u_j^{\mathrm{R}}(t,z),u_j^{\mathrm{I}}(t,z))+a_{kj}^{\mathrm{I}}g_j^{\mathrm{R}}(u_j^{\mathrm{R}}(t,z),u_j^{\mathrm{I}}(t,z))\right]+\\ \quad\sum\limits_{j=1}^{n}\left[b_{kj}^{\mathrm{R}}g_j^{\mathrm{I}}(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z),u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))+\right.\\ \quad\left.b_{kj}^{\mathrm{I}}g_j^{\mathrm{R}}(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z),u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))\right],\\ u_k^{\mathrm{I}}(t,z)|_{z\in\partial\Omega}=0,\end{cases}\tag{8}$$

其中

$$g_j^{\mathrm{R}}(u_j^{\mathrm{R}},u_j^{\mathrm{I}})=\tilde{g}_j^{\mathrm{R}}(\tilde{u}_j^{\mathrm{R}},\tilde{u}_j^{\mathrm{I}})-\tilde{g}_j^{\mathrm{R}}(\tilde{u}_j^{\#\mathrm{R}},\tilde{u}_j^{\#\mathrm{I}}),\ g_j^{\mathrm{I}}(u_j^{\mathrm{R}},u_j^{\mathrm{I}})=\tilde{g}_j^{\mathrm{I}}(\tilde{u}_j^{\mathrm{R}},\tilde{u}_j^{\mathrm{I}})-\tilde{g}_j^{\mathrm{I}}(\tilde{u}_j^{\#\mathrm{R}},\tilde{u}_j^{\#\mathrm{I}}),\qquad j\in\Theta.$$

系统(7)和(8)的初始条件变为 $\varphi_k(s,z)=\tilde{\varphi}_k(s,z)-\tilde{u}_k^{\#}, k\in\Theta, s\in[-\tau,0]$.

令 $\widehat{\Theta}=\{1,2,\cdots,n,n+1,n+2,\cdots,2n\}$, $\boldsymbol{\alpha}=((\boldsymbol{u}^{\mathrm{R}})^{\mathrm{T}},(\boldsymbol{u}^{\mathrm{I}})^{\mathrm{T}})^{\mathrm{T}}$.

由于 $\tilde{\boldsymbol{T}}-\tilde{\boldsymbol{P}}\tilde{\boldsymbol{L}}$ 是 M 矩阵,根据引理 1[19]可知存在一个正的向量 $\boldsymbol{\Lambda}=(\zeta_1,\zeta_2,\cdots,\zeta_{2n})$ 使得不等式(9)和(10)成立,这里:当 $k'\in\Theta$ 时, $\zeta'_k=\beta_k$ 且 $k=k'$;当 $k'\in\widehat{\Theta}-\Theta$ 时, $\zeta'_k=\gamma_k$ 且 $k=k'-n$,

$$\begin{aligned}&\left\{\left(-2\frac{m\pi_k}{\hat{\omega}^2}-2d_k\right)+\sum_{j=1}^{n}\left[(l_j^{\mathrm{RR}}+l_j^{\mathrm{RI}})(|a_{kj}^{\mathrm{R}}|+|b_{kj}^{\mathrm{R}}|)+(l_j^{\mathrm{IR}}+l_j^{\mathrm{II}})(|a_{kj}^{\mathrm{I}}|+|b_{kj}^{\mathrm{I}}|)\right]\right\}\beta_k+\\&\quad\sum_{j=1}^{n}\beta_j\left[(|a_{kj}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{IR}})+(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{IR}})\right]+\\&\quad\sum_{j=1}^{n}\gamma_j\left[(|a_{kj}^{\mathrm{R}}|l_k^{\mathrm{RI}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{II}})+(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{RI}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{II}})\right]<0,\end{aligned}\tag{9}$$

$$\left\{-2\frac{m\pi_k}{\hat{\omega}^2}-2d_k+\sum_{j=1}^{n}\left[(l_j^{\mathrm{IR}}+l_j^{\mathrm{II}})(|a_{kj}^{\mathrm{R}}|+|b_{kj}^{\mathrm{R}}|)+(l_j^{\mathrm{RR}}+l_j^{\mathrm{RI}})(|a_{kj}^{\mathrm{I}}|+|b_{kj}^{\mathrm{I}}|)\right]\right\}\gamma_k+$$

$$\sum_{j=1}^{n}\beta_j[(|a_{kj}^{\mathrm{R}}|l_j^{\mathrm{IR}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{RR}})+(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{IR}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{RR}})]+$$

$$\sum_{j=1}^{n}\gamma_j[(|a_{kj}^{\mathrm{R}}|l_k^{\mathrm{II}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{RI}})+(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{II}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{RI}})]<0. \tag{10}$$

根据式(9)和(10)构造如下函数:

$$F_k^{\mathrm{R}}(\varepsilon)=\left\{\varepsilon-2\frac{m\pi_k}{\hat{\omega}^2}-2d_k+\sum_{j=1}^{n}[(l_j^{\mathrm{RR}}+l_j^{\mathrm{RI}})(|a_{kj}^{\mathrm{R}}|+|b_{kj}^{\mathrm{R}}|)+(l_j^{\mathrm{IR}}+l_j^{\mathrm{II}})(|a_{kj}^{\mathrm{I}}|+|b_{kj}^{\mathrm{I}}|)]\right\}\beta_k+$$

$$\sum_{j=1}^{n}[(|a_{kj}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{IR}})+(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{IR}})\mathrm{e}^{\varepsilon\tau}]\beta_j+$$

$$\sum_{j=1}^{n}[(|a_{kj}^{\mathrm{R}}|l_k^{\mathrm{RI}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{II}})+(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{RI}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{II}})\mathrm{e}^{\varepsilon\tau}]\gamma_j, \tag{11}$$

$$F_k^{\mathrm{I}}(\varepsilon)=\left\{\varepsilon-2\frac{m\pi_k}{\hat{\omega}^2}-2d_k+\sum_{j=1}^{n}[(l_j^{\mathrm{IR}}+l_j^{\mathrm{II}})(|a_{kj}^{\mathrm{R}}|+|b_{kj}^{\mathrm{R}}|)+(l_j^{\mathrm{RR}}+l_j^{\mathrm{RI}})(|a_{kj}^{\mathrm{I}}|+|b_{kj}^{\mathrm{I}}|)]\right\}\gamma_k+$$

$$\sum_{j=1}^{n}[(|a_{kj}^{\mathrm{R}}|l_j^{\mathrm{IR}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{RR}})+(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{IR}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{RR}})\mathrm{e}^{\varepsilon\tau}]\beta_j+$$

$$\sum_{j=1}^{n}[(|a_{kj}^{\mathrm{R}}|l_k^{\mathrm{II}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{RI}})+(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{II}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{RI}})\mathrm{e}^{\varepsilon\tau}]\gamma_j. \tag{12}$$

由于 $F_k^{\mathrm{R}}(\varepsilon)$ 和 $F_k^{\mathrm{I}}(\varepsilon)$ 是关于 ε 的连续函数,根据式(9)和(10)有 $F_k^{\mathrm{R}}(0)<0$ 和 $F_k^{\mathrm{I}}(0)<0$.显然,存在常数 $\lambda>0$ 使得 $F_k^{\mathrm{R}}(\lambda)<0$ 和 $F_k^{\mathrm{I}}(\lambda)<0$ 成立.

选择如下向量 Lyapunov 函数:

$$V'_k(t,\alpha_{k'}(t,z))=\begin{cases}\dfrac{1}{2}\mathrm{e}^{\lambda t}\|\alpha_k(t,z)\|_{L^2}^2=\dfrac{1}{2}\mathrm{e}^{\lambda t}\displaystyle\int_\Omega(u_k^{\mathrm{R}}(t,z))^2\mathrm{d}z, & k'\in\Theta,\ k=k',\\ \dfrac{1}{2}\mathrm{e}^{\lambda t}\|\alpha_{k'}(t,z)\|_{L^2}^2=\dfrac{1}{2}\mathrm{e}^{\lambda t}\displaystyle\int_\Omega(u_k^{\mathrm{I}}(t,z))^2\mathrm{d}z, & k'\in\widehat{\Theta}-\Theta,\ k=k'-n.\end{cases} \tag{13}$$

在不引起混淆的情况下,令 $V_{k'}(t,z)$ 表示 $V_{k'}(t,\alpha_{k'}(t,z)),k'\in\widehat{\Theta}$.

① 当 $k'\in\Theta$, 考虑到式(7),计算 $\mathrm{D}^+V_{k'}(t,\alpha_{k'}(t))$ 如下:

$$\begin{aligned}&\mathrm{D}^+V_{k'}(t,z)=\\&\quad 0.5\lambda\mathrm{e}^{\lambda t}\int_\Omega(u_k^{\mathrm{R}}(t,z))^2\mathrm{d}z+\mathrm{e}^{\lambda t}\pi_k\sum_{h=1}^{m}\int_\Omega u_k^{\mathrm{R}}(t,z)\frac{\partial}{\partial z_h}\left[\frac{\partial u_k^{\mathrm{R}}(t,z)}{\partial z_h}\right]\mathrm{d}z+\\&\quad \mathrm{e}^{\lambda t}\sum_{h=1}^{m}\int_\Omega u_k^{\mathrm{R}}(t,z)\Big\{-d_ku_k^{\mathrm{R}}(t,z)+\sum_{j=1}^{n}[a_{kj}^{\mathrm{R}}g_j^{\mathrm{R}}(u_j^{\mathrm{R}}(t,z),u_j^{\mathrm{I}}(t,z))-a_{kj}^{\mathrm{I}}g_j^{\mathrm{I}}(u_j^{\mathrm{R}}(t,z),u_j^{\mathrm{I}}(t,z))]+\\&\quad \sum_{j=1}^{n}[b_{kj}^{\mathrm{R}}g_j^{\mathrm{R}}(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z),u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))-\\&\quad b_{kj}^{\mathrm{I}}g_j^{\mathrm{I}}(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z),u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))]\Big\}\mathrm{d}z.\end{aligned}$$

利用分部积分,并考虑到引理 1 和假设 1,有

$$\begin{aligned}&\mathrm{D}^+V_{k'}(t,z)=\\&\quad 0.5\lambda\mathrm{e}^{\lambda t}\int_\Omega(u_k^{\mathrm{R}}(t,z))^2\mathrm{d}z-\mathrm{e}^{\lambda t}\pi_k\sum_{h=1}^{m}\int_\Omega\frac{\partial u_k^{\mathrm{R}}(t,z)}{\partial z_h}\mathrm{d}z+\\&\quad \mathrm{e}^{\lambda t}\int_\Omega u_k^{\mathrm{R}}(t,z)\Big\{-d_ku_k^{\mathrm{R}}(t,z)+\sum_{j=1}^{n}[a_{kj}^{\mathrm{R}}g_j^{\mathrm{R}}(u_j^{\mathrm{R}}(t,z),u_j^{\mathrm{I}}(t,z))-a_{kj}^{\mathrm{I}}g_j^{\mathrm{I}}(u_j^{\mathrm{R}}(t,z),u_j^{\mathrm{I}}(t,z))]+\\&\quad \sum_{j=1}^{n}[b_{kj}^{\mathrm{R}}g_j^{\mathrm{R}}(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z),u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))-\end{aligned}$$

$$
\begin{aligned}
&b_{kj}^{\mathrm{I}} g_j^{\mathrm{I}}(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z),u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))]\Big\}\mathrm{d}z \leqslant \\
&0.5\lambda \mathrm{e}^{\lambda t}\int_\Omega (u_k^{\mathrm{R}}(t,z))^2\mathrm{d}z - \mathrm{e}^{\lambda t}\int_\Omega\left(\frac{m\pi_k}{\hat{\omega}^2}+d_k\right)(u_k^{\mathrm{R}}(t,z))^2\mathrm{d}z + \\
&\mathrm{e}^{\lambda t}\int_\Omega\Big\{|u_k^{\mathrm{R}}(t,z)|\sum_{j=1}^n[|a_{kj}^{\mathrm{R}}|[l_j^{\mathrm{RR}}|u_j^{\mathrm{R}}(t,z)|+l_j^{\mathrm{RI}}|u_j^{\mathrm{I}}(t,z)|]+ \\
&|a_{kj}^{\mathrm{I}}|[l_j^{\mathrm{IR}}|u_j^{\mathrm{R}}(t,z)|+l_j^{\mathrm{II}}|u_j^{\mathrm{I}}(t,z)|]]+ \\
&\sum_{j=1}^n[|b_{kj}^{\mathrm{R}}|[l_j^{\mathrm{RR}}|u_j^{\mathrm{R}}(t-\tau_{kj}(t),z)|+l_j^{\mathrm{RI}}|u_j^{\mathrm{I}}(t-\tau_{kj}(t),z)|]+ \\
&|b_{kj}^{\mathrm{I}}|[l_j^{\mathrm{IR}}|u_j^{\mathrm{R}}(t-\tau_{kj}(t),z)|+l_j^{\mathrm{II}}|u_j^{\mathrm{I}}(t-\tau_{kj}(t),z)|]]\Big\}\mathrm{d}z.
\end{aligned}
$$

利用 Hölder 不等式,进一步推导有

$$
\begin{aligned}
&\mathrm{D}^+V_{k'}(t,\alpha_{k'}(t,z)) \leqslant \\
&0.5\lambda \mathrm{e}^{\lambda t}\int_\Omega (u_k^{\mathrm{R}}(t,z))^2\mathrm{d}z - \mathrm{e}^{\lambda t}\int_\Omega\left(\frac{m\pi_k}{\hat{\omega}^2}+d_k\right)(u_k^{\mathrm{R}}(t,z))^2\mathrm{d}z + \\
&\mathrm{e}^{\lambda t}\int_\Omega\Big\{0.5\sum_{j=1}^n\{|a_{kj}^{\mathrm{R}}|\{l_j^{\mathrm{RR}}[(u_k^{\mathrm{R}}(t,z))^2+(u_j^{\mathrm{R}}(t,z))^2]+l_j^{\mathrm{RI}}[(u_k^{\mathrm{R}}(t,z))^2+(u_j^{\mathrm{I}}(t,z))^2]\}+ \\
&|a_{kj}^{\mathrm{I}}|\{l_j^{\mathrm{IR}}[(u_k^{\mathrm{R}}(t,z))^2+(u_j^{\mathrm{R}}(t,z))^2]+l_j^{\mathrm{II}}[(u_k^{\mathrm{R}}(t,z))^2+(u_j^{\mathrm{I}}(t,z))^2]\}\}+ \\
&0.5\sum_{j=1}^n\{|b_{kj}^{\mathrm{R}}|\{l_j^{\mathrm{RR}}[(u_k^{\mathrm{R}}(t,z))^2+(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z))^2]+ \\
&l_j^{\mathrm{RI}}[(u_k^{\mathrm{R}}(t,z))^2+(u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))^2]\}+ \\
&|b_{kj}^{\mathrm{I}}|\{l_j^{\mathrm{IR}}[(u_k^{\mathrm{R}}(t,z))^2+(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z))^2]+ \\
&l_j^{\mathrm{II}}[(u_k^{\mathrm{R}}(t,z))^2+(u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))^2]\}\}\Big\}\mathrm{d}z \leqslant \\
&0.5\mathrm{e}^{\lambda t}\Big\{\Big\{\lambda-2\frac{m\pi_k}{\hat{\omega}^2}-2d_k+\sum_{j=1}^n(l_j^{\mathrm{RR}}+l_j^{\mathrm{RI}})(|a_{kj}^{\mathrm{R}}|+|b_{kj}^{\mathrm{R}}|)+(l_j^{\mathrm{IR}}+l_j^{\mathrm{II}})(|a_{kj}^{\mathrm{I}}|+|b_{kj}^{\mathrm{I}}|)\Big\}\times \\
&\int_\Omega (u_k^{\mathrm{R}}(t,z))^2\mathrm{d}z+\sum_{j=1}^n\Big[(|a_{kj}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{IR}})\int_\Omega(u_j^{\mathrm{R}}(t,z))^2\mathrm{d}z+ \\
&(|a_{kj}^{\mathrm{R}}|l_j^{\mathrm{RI}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{II}})\int_\Omega(u_j^{\mathrm{I}}(t,z))^2\mathrm{d}z\Big]+ \\
&\sum_{j=1}^n[(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{IR}})\int_\Omega(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z))^2\mathrm{d}z+ \\
&(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{RI}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{II}})\int_\Omega(u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))^2\mathrm{d}z]\Big\} \leqslant \\
&\Big\{\lambda-2\frac{m\pi_k}{\hat{\omega}^2}-2d_k+\sum_{j=1}^n[(l_j^{\mathrm{RR}}+l_j^{\mathrm{RI}})(|a_{kj}^{\mathrm{R}}|+|b_{kj}^{\mathrm{R}}|)+(l_j^{\mathrm{IR}}+l_j^{\mathrm{II}})(|a_{kj}^{\mathrm{I}}|+|b_{kj}^{\mathrm{I}}|)]\Big\}V_k(t,z)+ \\
&\sum_{j=1}^n[(|a_{kj}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{IR}})V_j(t,z)+(|a_{kj}^{\mathrm{R}}|l_j^{\mathrm{RI}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{II}})V_{j+n}(t,z)]+ \\
&\mathrm{e}^{\lambda\tau}\sum_{j=1}^n[(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{IR}})V_j(t-\tau_{kj}(t),z)+ \\
&(|b_{kj}^{\mathrm{R}}|l_j^{\mathrm{RI}}+|b_{kj}^{\mathrm{I}}|l_j^{\mathrm{II}})V_{j+n}(t-\tau_{kj}(t),z)].
\end{aligned}
\tag{14}
$$

定义曲线 $\Upsilon=\{\boldsymbol{\eta}(\chi):\eta_{k'}=\zeta_{k'}\chi,\chi>0,k'\in\widehat{\Theta}\}$ 和集合 $\Pi(\boldsymbol{\eta})=\{\boldsymbol{w}:\boldsymbol{0}\leqslant\boldsymbol{w}\leqslant\boldsymbol{\eta},\boldsymbol{\eta}\in\boldsymbol{\zeta}\}$.显然当 $\chi>\chi'$ 时,有 $\Upsilon(\boldsymbol{\eta}(\chi))\supset\Upsilon(\boldsymbol{\eta}(\chi'))$.

下面将证明存在 $\Gamma>0$ 和 $\lambda>0$ 使得 $\|\boldsymbol{u}^{\mathrm{R}}(t,z)\|_{L^2}\leqslant\Gamma\|\boldsymbol{\varphi}(z)\|_{L^2}\mathrm{e}^{-0.5\lambda t}$ 成立,其中 $\|\boldsymbol{\varphi}(z)\|_{L^2}=\sup_{s\in[-\tau,0]}\|\boldsymbol{\varphi}(s,z)\|_{L^2}$,$t\geqslant 0$.

令 $\bar{\upsilon}=\max_{k\in\Theta}\{\beta_k,\gamma_k\}$，$\underline{\upsilon}=\min_{k\in\Theta}\{\beta_k,\gamma_k\}$，$\chi_0=\theta\|\boldsymbol{\varphi}(z)\|_{L_2}^2/\underline{\upsilon}$，这里 $\theta>1$.则

$$\{\boldsymbol{V}:\boldsymbol{V}(s,z)=0.5\mathrm{e}^{\lambda s}\|\boldsymbol{\alpha}(s,z)\|_{L^2}^2,\ -\tau\leqslant s\leqslant 0\}\subset Y(\boldsymbol{\eta}_0(\chi_0)),$$

也就是当 $-\tau\leqslant s\leqslant 0$ 时，有

$$V_j(s,z)=0.5\mathrm{e}^{\lambda s}\|u_j^{\mathrm{R}}(s,z)\|_{L_2}^2<\beta_j\chi_0,\ V_{j+n}(s,z)=0.5\mathrm{e}^{\lambda s}\|u_j^{\mathrm{I}}(s,z)\|_{L_2}^2<\gamma_j\chi_0,\qquad j\in\Theta.$$

进而有 $V_j(t,z)<\beta_j\chi_0$ 和 $V_{j+n}(t,z)<\gamma_j\chi_0$ 始终成立，其中 $j\in\Theta,t\geqslant 0$.若不成立，则存在某个 $\hat{k}\in\Theta$ 和时刻 $t_1>0$ 使得 $V_{\hat{k}}(t_1,z)=\beta_{\hat{k}}\chi_0$，$\mathrm{D}^+(V_{\hat{k}}(t_1,z))\geqslant 0$，$V_j(t_1,z)\leqslant\beta_j\chi_0$，$V_{j+n}(t_1,z)\leqslant\gamma_j\chi_0$，$j\neq\hat{k}$，$j\in\Theta$.将其代入到式(14)并考虑到 $F_k^{\mathrm{R}}(\lambda)<0$，有

$$\begin{aligned}
&\mathrm{D}^+V_{\hat{k}}(t_1,\alpha_{\hat{k}}(t_1,z))\leqslant\\
&\quad\Big\{\lambda-2\frac{m\pi_{\hat{k}}}{\hat{\omega}^2}-2d_{\hat{k}}+\sum_{j=1}^{n}[(l_j^{\mathrm{RR}}+l_j^{\mathrm{RI}})(|a_{\hat{k}j}^{\mathrm{R}}|+|b_{\hat{k}j}^{\mathrm{R}}|)+\\
&\quad(l_j^{\mathrm{IR}}+l_j^{\mathrm{II}})(|a_{\hat{k}j}^{\mathrm{I}}|+|b_{\hat{k}j}^{\mathrm{I}}|)]\Big\}V_{\hat{k}}(t_1,z)+\sum_{j=1}^{n}[(|a_{\hat{k}j}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|a_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{IR}})V_j(t_1,z)+\\
&\quad(|a_{\hat{k}j}^{\mathrm{R}}|l_j^{\mathrm{RI}}+|a_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{II}})V_{j+n}(t_1,z)]+\mathrm{e}^{\lambda\tau}\sum_{j=1}^{n}[(|b_{\hat{k}j}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|b_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{IR}})V_j(t_1-\tau_{\hat{k}j}(t),z)+\\
&\quad(|b_{\hat{k}j}^{\mathrm{R}}|l_j^{\mathrm{RI}}+|b_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{II}})V_{j+n}(t_1-\tau_{\hat{k}j}(t_1),z)]\leqslant\\
&\quad\Big\{\lambda-2\frac{m\pi_{\hat{k}}}{\hat{\omega}^2}-2d_{\hat{k}}+\sum_{j=1}^{n}[(l_j^{\mathrm{RR}}+l_j^{\mathrm{RI}})(|a_{\hat{k}j}^{\mathrm{R}}|+|b_{\hat{k}j}^{\mathrm{R}}|)+(l_j^{\mathrm{IR}}+l_j^{\mathrm{II}})(|a_{\hat{k}j}^{\mathrm{I}}|+|b_{\hat{k}j}^{\mathrm{I}}|)]\Big\}\beta_{\hat{k}}\chi_0+\\
&\quad\sum_{j=1}^{n}[(|a_{\hat{k}j}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|a_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{IR}})\beta_j\chi_0+(|a_{\hat{k}j}^{\mathrm{R}}|l_k^{\mathrm{RI}}+|a_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{II}})\gamma_j\chi_0]+\\
&\quad\mathrm{e}^{\lambda\tau}\sum_{j=1}^{n}[(|b_{\hat{k}j}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|b_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{IR}})\beta_j\chi_0+(|b_{\hat{k}j}^{\mathrm{R}}|l_j^{\mathrm{RI}}+|b_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{II}})\gamma_j\chi_0]=\\
&\quad\Big\{\lambda-2\frac{m\pi_{\hat{k}}}{\hat{\omega}^2}-2d_{\hat{k}}+\sum_{j=1}^{n}[(l_j^{\mathrm{RR}}+l_j^{\mathrm{RI}})(|a_{\hat{k}j}^{\mathrm{R}}|+|b_{\hat{k}j}^{\mathrm{R}}|)+(l_j^{\mathrm{IR}}+l_j^{\mathrm{II}})(|a_{\hat{k}j}^{\mathrm{I}}|+|b_{\hat{k}j}^{\mathrm{I}}|)]\Big\}\beta_{\hat{k}}\chi_0+\\
&\quad\sum_{j=1}^{n}[(|a_{\hat{k}j}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|a_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{IR}})+(|b_{\hat{k}j}^{\mathrm{R}}|l_j^{\mathrm{RR}}+|b_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{IR}})\mathrm{e}^{\lambda\tau}]\beta_j\chi_0+\\
&\quad\sum_{j=1}^{n}[(|a_{\hat{k}j}^{\mathrm{R}}|l_k^{\mathrm{RI}}+|a_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{II}})+(|b_{\hat{k}j}^{\mathrm{R}}|l_j^{\mathrm{RI}}+|b_{\hat{k}j}^{\mathrm{I}}|l_j^{\mathrm{II}})\mathrm{e}^{\lambda\tau}]\gamma_j\chi_0<0.
\end{aligned}\tag{15}$$

显然不等式(15)与假设 $\mathrm{D}^+(V_{\hat{k}}(t_1,z))\geqslant 0$ 是矛盾的，即对所有 $t\geqslant 0,z\in\Omega$，有 $V_k(t,z)<\beta_k\chi_0$，$V_{k+n}(t,z)<\gamma_k\chi_0$，$k\in\Theta$.也就是说 $V_{k'}(t,z)<\zeta_{k'}\chi_0$，$k'\in\widehat{\Theta}$.

② 当 $k'\in\widehat{\Theta}-\Theta$，根据式(8)计算 $\mathrm{D}^+V_{k'}(t,z)$，并考虑到假设 1 和引理 1，有

$$\begin{aligned}
&\mathrm{D}^+V_{k'}(t,z)=\\
&\quad 0.5\lambda\mathrm{e}^{\lambda t}\int_\Omega(u_k^{\mathrm{I}}(t,z))^2\mathrm{d}z+\mathrm{e}^{\lambda t}\pi_k\sum_{h=1}^{m}\int_\Omega u_k^{\mathrm{I}}(t,z)\frac{\partial}{\partial z_h}\left[\frac{\partial u_k^{\mathrm{I}}(t,z)}{\partial z_h}\right]\mathrm{d}z+\\
&\quad\mathrm{e}^{\lambda t}\sum_{h=1}^{m}\int_\Omega u_k^{\mathrm{I}}(t,z)\Big\{-d_ku_k^{\mathrm{I}}(t,z)\sum_{j=1}^{n}[a_{kj}^{\mathrm{R}}g_j^{\mathrm{I}}(u_j^{\mathrm{R}}(t,z),u_j^{\mathrm{I}}(t,z))+a_{kj}^{\mathrm{I}}g_j^{\mathrm{R}}(u_j^{\mathrm{R}}(t,z),u_j^{\mathrm{I}}(t,z))]+\\
&\quad\sum_{j=1}^{n}[b_{kj}^{\mathrm{R}}g_j^{\mathrm{I}}(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z),u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))+\\
&\quad b_{kj}^{\mathrm{I}}g_j^{\mathrm{R}}(u_j^{\mathrm{R}}(t-\tau_{kj}(t),z),u_j^{\mathrm{I}}(t-\tau_{kj}(t),z))]\Big\}\mathrm{d}z\leqslant\\
&\quad\Big\{\lambda-2\frac{m\pi_k}{\hat{\omega}^2}-2d_k+\sum_{j=1}^{n}[(l_j^{\mathrm{IR}}+l_j^{\mathrm{II}})(|a_{kj}^{\mathrm{R}}|+|b_{kj}^{\mathrm{R}}|)+\\
&\quad(l_j^{\mathrm{RR}}+l_j^{\mathrm{RI}})(|a_{kj}^{\mathrm{I}}|+|b_{kj}^{\mathrm{I}}|)]\Big\}V_{k+n}(t,z)+\sum_{j=1}^{n}[(|a_{kj}^{\mathrm{R}}|l_j^{\mathrm{IR}}+|a_{kj}^{\mathrm{I}}|l_j^{\mathrm{RR}})V_j(t,z)+
\end{aligned}$$

$$(|a_{kj}^{R}|l_j^{II}+|a_{kj}^{I}|l_j^{RI})V_{j+n}(t,z)]+e^{\lambda\tau}\sum_{j=1}^{n}[(|b_{kj}^{R}|l_j^{IR}+|b_{kj}^{I}|l_j^{RR})V_j(t-\tau_{kj}(t),z)+$$
$$(|b_{kj}^{R}|l_j^{II}+|b_{kj}^{I}|l_j^{RI})V_{j+n}(t-\tau_{kj}(t),z)]. \tag{16}$$

通过类似于①的分析过程，可以得到对于所有 $t\geqslant 0,k\in\Theta,k'\in\widehat{\Theta}-\Theta$，不等式 $V_k(t,z)<\beta_k\chi_0$ 和 $V_{k'}(t,z)<\gamma_k\chi_0$ 是成立的.

综合①和②有：$V_{k'}(t)<\zeta_{k'}\chi_0,k'\in\widehat{\Theta}$.也就是 $0.5\|u_k^{R}(t,z)\|_{L_2}^2e^{\lambda t}<\beta_k\chi_0,0.5\|u_k^{I}(t,z)\|_{L_2}^2e^{\lambda t}<\gamma_k\chi_0,k\in\Theta$.

进一步有 $\|u_k^{R}(t,z)\|_{L_2}^2<2\theta\beta_k e^{-\lambda t}\|\varphi(s,z)\|_{L_2}^2/\underline{\upsilon}$，$\|u_k^{I}(t,z)\|_{L_2}^2<2\theta\gamma_k e^{-\lambda t}\|\varphi(s,z)\|_{L_2}^2/\underline{\upsilon},k\in\Theta$.令 $\Gamma=\sqrt{2\theta\bar{\upsilon}/\underline{\upsilon}}$，则有 $\|u_k^{R}(t,z)\|_{L_2}<\Gamma\|\varphi(z)\|_{L_2}e^{-0.5\lambda t}$，$\|u_k^{I}(t,z)\|_{L_2}<\Gamma\|\varphi(z)\|_{L_2}e^{-0.5\lambda t},k\in\Theta$.

根据定义 1 可知由系统(7)和(8)所构成的反应扩散复数域神经网络的零解是全局指数稳定的，也就是说由系统(3)和(4)所构成的应扩散复数域神经网络的平衡状态是全局指数稳定的.证毕.

注 1　文献[5,8]和文献[2,4,10]分别采用加权 Lyapunov 函数法和 LMI 方法研究了几类复数域神经网络的动态行为，并得到了判定相应系统稳定的充分条件.文献[2,4-5,8,10]所建立的充分条件中包含自由变量，在实际应用时对自由变量的选择很大程度上依赖于经验和不断地尝试.换言之，若所选择的自由变量不合适，将无法判断出系统的稳定性.本文基于矢量 Lyapunov 函数法所建立的稳定性条件是矩阵的紧凑形式，其中不含有任何自由变量，在应用时不依赖于个人经验，且计算简单.

注 2　文献[15-16,19]在神经网络模型中引入了反应扩散项，并得到了确保系统稳定的判据.判据中包含的扩散项显示其对系统稳定性的促进作用.然而，文献[15-16,19]中的结论仅适用于实数域神经网络，所得到的结论不适用于本文所考虑的复数域神经网络.此外，本文所建立的稳定性条件中也体现了扩散项对系统稳定性的影响.

注 3　若系统(1)中的神经元状态和矩阵都定义在实数域中，可直接得到确保该系统指数稳定的充分条件，即对所有的 $\boldsymbol{J}\in\mathbb{R}^n$ 和 $z\in\Omega$，要求矩阵 $\boldsymbol{T}-(|\boldsymbol{A}|+|\boldsymbol{B}|)\boldsymbol{L}$ 是一个 M 矩阵，其中 $\boldsymbol{L}=\boldsymbol{L}^{RR}=\mathrm{diag}(l_j)_{n\times n}$，$\boldsymbol{T}=\mathrm{diag}(T_k)_{n\times n}$，这里 $T_k=2(m\pi_k/\hat{\omega}^2)+2d_k-\sum_{j=1}^{n}l_j(|a_{kj}|+|b_{kj}|),k\in\Theta$.

3　数值算例

考虑如下系统：

$$\begin{cases}\dfrac{\partial u_k(t,z)}{\partial t}=\boldsymbol{\pi}_k\sum\limits_{h=1}^{2}\dfrac{\partial}{\partial z_h}\left(\dfrac{\partial u_k(t,z)}{\partial z_h}\right)-d_ku_k(t,z)+\\ \quad\sum\limits_{j=1}^{2}[a_{kj}g_j(u_j(t,z))+b_{kj}g_j(u_j(t-\tau_{kj}(t)),z)],\\ u_k(t,z)|_{z\in\partial\Omega}=0,\end{cases} \tag{17}$$

其中初始条件为 $u_1^{R}(t,0)=u_2^{R}(t,0)=u_1^{I}(t,0)=u_2^{I}(t,0)=u_1^{R}(t,5)=u_2^{R}(t,5)=u_1^{I}(t,5)=u_2^{I}(t,5)=0,t\geqslant 0$，且 $u_1^{R}(0,z)=2\sin z+3,u_1^{I}(0,z)=\tanh(2z),u_2^{R}(0,z)=\cos(2z),u_2^{I}(0,z)=\cos(1.5z)+0.5$，这里 $z_h\in(0,5)$，$h=1,2$.令系统(17)中的延时为 $\tau_{1j}=0.5+0.2\sin t,\tau_{2j}=0.8+0.4\cos t$，$j=1,2,t\geqslant 0$.

假设

$$\boldsymbol{D}=\begin{bmatrix}4&0\\0&5\end{bmatrix},\boldsymbol{A}=\begin{bmatrix}0.1-0.2i&0.1+0.4i\\-0.1-0.1i&0.2+0.3i\end{bmatrix},\boldsymbol{B}=\begin{bmatrix}0.3+0.4i&0.2-0.1i\\-0.15+0.2i&0.5-0.2i\end{bmatrix},\boldsymbol{\pi}=\begin{bmatrix}0.8&0\\0&1.2\end{bmatrix}.$$

假设激活函数为

$$g_1(u_1)=0.5\tanh(u_1^{R})+i\frac{1}{1+e^{-u_1^{I}}},\ g_2(u_2)=0.6\frac{1-e^{-u_2^{I}}}{1+e^{-u_2^{I}}}+0.2(|u_2^{R}+1|-|u_2^{R}-1|)i.$$

经过计算有

$$\boldsymbol{L}^{RR}=\begin{bmatrix}0.5&0\\0&0\end{bmatrix},\boldsymbol{L}^{RI}=\begin{bmatrix}0&0\\0&0.3\end{bmatrix},\boldsymbol{L}^{IR}=\begin{bmatrix}0&0\\0&0.4\end{bmatrix},\boldsymbol{L}^{II}=\begin{bmatrix}0.25&0\\0&0\end{bmatrix},$$

进而可以得到

$$\tilde{\boldsymbol{T}} - \tilde{\boldsymbol{P}}\tilde{\boldsymbol{L}} = \begin{bmatrix} 7.29 & -0.20 & -0.15 & -0.09 \\ -0.13 & 9.38 & -0.08 & -0.21 \\ -0.30 & -0.12 & 7.39 & -0.15 \\ -0.15 & -0.28 & -0.06 & 9.43 \end{bmatrix}.$$

进一步计算得到矩阵 $\tilde{\boldsymbol{T}} - \tilde{\boldsymbol{P}}\tilde{\boldsymbol{L}}$ 的特征值向量为(7.09,7.56,9.19,9.65),根据引理 1[19] 可知矩阵 $\tilde{\boldsymbol{T}} - \tilde{\boldsymbol{P}}\tilde{\boldsymbol{L}}$ 是一个 M 矩阵.依据定理 1 断定系统(17)的平衡状态是全局指数稳定的.

图 1~4 给出了系统(17)状态的仿真结果.图 1 和图 3 给出了系统(17)实部的状态曲面,图 2 和图 4 给出了系统(17)虚部的状态曲面.仿真结果显示该系统的平衡状态是全局收敛的,该结果进一步验证了本文结论的正确性.

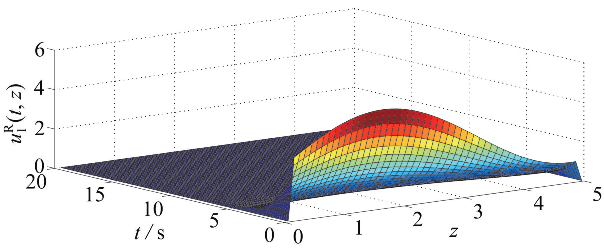

图 1 系统(17)的神经元 $u_1^{\mathrm{R}}(t,z)$ 的状态曲面

Fig. 1 The state curved surface of $u_1^{\mathrm{R}}(t,z)$ in system (17)

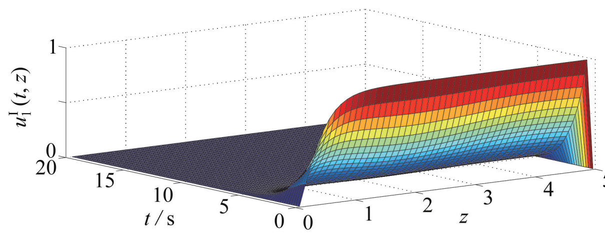

图 2 系统(17)的神经元 $u_1^{\mathrm{I}}(t,z)$ 的状态曲面

Fig. 2 The state curved surface of $u_1^{\mathrm{I}}(t,z)$ in system (17)

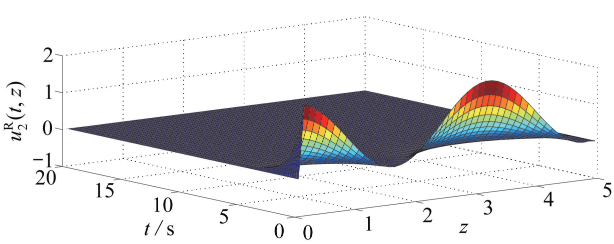

图 3 系统(17)的神经元 $u_2^{\mathrm{R}}(t,z)$ 的状态曲面

Fig. 3 The state curved surface of $u_2^{\mathrm{R}}(t,z)$ in system (17)

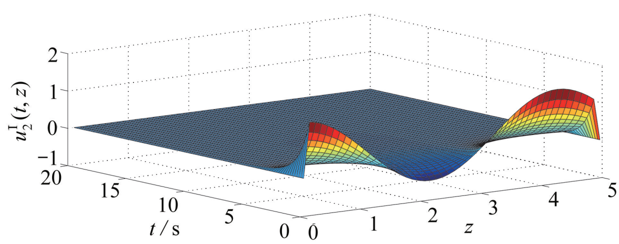

图 4 系统(17)的神经元 $u_2^{\mathrm{I}}(t,z)$ 的状态曲面

Fig. 4 The state curved surface of $u_2^{\mathrm{I}}(t,z)$ in system (17)

4 结论与展望

本文针对一类具有反应扩散项的变时滞复数域神经网络的指数稳定性进行了分析.利用矢量 Lyapunov 函数法和 M 矩阵理论,得到了确保该系统平衡状态指数稳定性的充分条件.该条件中体现了扩散项对系统稳定性的影响,并且不含有任何自由变量,降低了现有结论的保守性.最后通过数值仿真算例验证了所得结论的正确性.本文将该复数域神经网络分解为相应的实部系统和虚部系统.基于矢量 Lyapunov 函数法可以进一步研究另一类具有扩散项的复数域神经网络系统,即考虑系统在不可分解的情况下的平衡状态的模指数稳定性,该研究后续将进行开展.

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