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基于 *n* 阶剪切变形理论的复合材料层合板屈曲分析*

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摘要： 采用 n 阶剪切变形理论分析了复合材料层合板的屈曲问题,根据虚功原理推导出了复合材料层合板在面内载荷作用下临界屈曲的控制微分方程.将此方法所得结果与其他文献中的结果进行了比较,结果表明,该理论具有较高的计算精度.

关　键　词： 复合材料层合板； 屈曲； 解析法； 面内载荷； n 阶剪切变形理论

中图分类号： O326　　**文献标志码：** A　　DOI：10.21656/1000-0887.410061

引　言

复合材料层合板具有较高的比强度,已经被广泛用于航空、航天、建筑、船舶等领域.随着复合材料在民用飞机中的广泛应用,对其结构的力学分析已成为普遍关注的重要问题之一,如复合材料结构屈曲与后屈曲问题是国内外复合材料界的研究热点[1].

很多学者研究了复合材料层合板的屈曲问题:袁坚锋等[2]对弯剪复合载荷作用下,复合材料层合板屈曲的强度校核方法进行了分析;彭林欣等[3]针对加肋板屈曲临界荷载的问题进行了求解;顾元宪等[4]在有限元分析基础上,研究了以屈曲稳定性作为约束条件或优化目标的复合材料层合板结构优化设计及其灵敏度分析方法;洪厚全等[5]采用模拟退火算法对复合材料层合板屈曲问题进行了优化设计研究;龚良贵等[6]采用广义 Fourier 级数法建立了具有弹性约束的复合材料矩形层板在面内载荷作用下的非线性稳定性控制方程组,并分析了层合板非线性屈曲问题;彭凡等[7]将瞬时弹性失稳荷载、持久临界荷载和后屈曲持久变形刚度作为复合材料层合板蠕变屈曲与变形的优化指标,对纤维铺设角进行了优化分析.

由于复合材料层合板结构的复杂性,严格按三维理论分析复合材料层合板结构将是非常困难的,故发展出了各种高阶剪切变形层合板理论.高阶剪切变形层合板理论是在一阶剪切变

* **收稿日期：** 2020-02-24；**修订日期：** 2020-04-16

基金项目： 航空科学基金（20173254006）；辽宁省自然科学基金指导计划（20180550824）；辽宁省教育厅重点攻关和服务地方项目（JYT19004）

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引用格式： 石峰，马洪英，孙义真，项松，王艳冰，栾婷婷. 基于 n 阶剪切变形理论的复合材料层合板屈曲分析[J]. 应用数学和力学，2020，**41**（12）：1346-1357.

形层合板理论的基础上发展起来的,是一种分析复合材料层合板结构有效的近似方法,不但能反映三维效应,而且计算精度较高,目前受到了广泛的重视.杨加明等[8]根据 Reddy 的高阶剪切变形理论,分析了复合材料层板的中等大挠度情况.

n 阶剪切变形理论是对 Reddy 的高阶剪切变形理论的概括,Reddy 的高阶剪切变形理论是 n 阶剪切变形理论在 $n = 3$ 时的特例.n 阶剪切变形理论已经由项松等[9-13]用于复合材料层合板、功能梯度板和复合材料夹层板的自由振动及静力分析.

本文采用 n 阶剪切变形理论分析了复合材料层合板的屈曲问题,采用虚功原理推导出了复合材料层合板在面内载荷作用下临界屈曲的控制微分方程.最后将本文方法所得结果与其他文献中的结果进行了比较.

1　n 阶剪切变形理论

n 阶剪切变形理论满足剪切应力条件,则其公式如下:

$$\begin{cases} U = u(x,y) + z\phi_x(x,y) - \dfrac{1}{n}\left(\dfrac{2}{h}\right)^{n-1} z^n\left(\phi_x(x,y) + \dfrac{\partial w(x,y)}{\partial x}\right), \\ V = v(x,y) + z\phi_y(x,y) - \dfrac{1}{n}\left(\dfrac{2}{h}\right)^{n-1} z^n\left(\phi_y(x,y) + \dfrac{\partial w(x,y)}{\partial y}\right), \\ W = w(x,y), \end{cases} \tag{1}$$

式中,u,v,w,ϕ_x 是位移函数,h 是板的厚度,$n = 3,5,7,9,\cdots$.3 阶理论是 n 阶剪切变形理论的特例.

应变如下式所示:

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} + z\dfrac{\partial \phi_x}{\partial x} - \dfrac{1}{n}\left(\dfrac{2}{h}\right)^{n-1} z^n\left(\dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial^2 w}{\partial x^2}\right), \\ \varepsilon_y = \dfrac{\partial v}{\partial y} + z\dfrac{\partial \phi_y}{\partial y} - \dfrac{1}{n}\left(\dfrac{2}{h}\right)^{n-1} z^n\left(\dfrac{\partial \phi_y}{\partial y} + \dfrac{\partial^2 w}{\partial y^2}\right), \\ \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + z\left(\dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x}\right) - \dfrac{1}{n}\left(\dfrac{2}{h}\right)^{n-1} z^n\left(\dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} + 2\dfrac{\partial^2 w}{\partial x \partial y}\right), \\ \gamma_{yz} = \left(1 - \left(\dfrac{2z}{h}\right)^{n-1}\right)\left(\phi_y + \dfrac{\partial w}{\partial y}\right),\ \gamma_{xz} = \left(1 - \left(\dfrac{2z}{h}\right)^{n-1}\right)\left(\phi_x + \dfrac{\partial w}{\partial x}\right). \end{cases} \tag{2}$$

基于虚功原理可以获得如下的方程:

$$\begin{cases} \dfrac{\partial N_x}{\partial x} + \dfrac{\partial N_{xy}}{\partial y} = 0,\ \dfrac{\partial N_{xy}}{\partial x} + \dfrac{\partial N_y}{\partial y} = 0, \\ \dfrac{\partial Q_x}{\partial x} - C_2\dfrac{\partial R_x}{\partial x} + \dfrac{\partial Q_y}{\partial y} - C_2\dfrac{\partial R_y}{\partial y} + C_1\left(\dfrac{\partial^2 P_x}{\partial x^2} + 2\dfrac{\partial^2 P_{xy}}{\partial x \partial y} + \dfrac{\partial^2 P_y}{\partial y^2}\right) + \\ \qquad \bar{N}_{xx}\dfrac{\partial^2 w}{\partial x^2} + 2\bar{N}_{xy}\dfrac{\partial^2 w}{\partial x \partial y} + \bar{N}_{yy}\dfrac{\partial^2 w}{\partial y^2} = 0, \\ \dfrac{\partial M_x}{\partial x} - C_1\dfrac{\partial P_x}{\partial x} + \dfrac{\partial M_{xy}}{\partial y} - C_1\dfrac{\partial P_{xy}}{\partial y} - Q_x + C_2 R_x = 0, \\ \dfrac{\partial M_{xy}}{\partial x} - C_1\dfrac{\partial P_{xy}}{\partial x} + \dfrac{\partial M_y}{\partial y} - C_1\dfrac{\partial P_y}{\partial y} - Q_y + C_2 R_y = 0, \end{cases} \tag{3}$$

式中 $C_1 = \dfrac{1}{n}\left(\dfrac{2}{h}\right)^{n-1}$，$C_2 = \left(\dfrac{2}{h}\right)^{n-1}$；$\bar{N}_{xx},\bar{N}_{xy},\bar{N}_{yy}$ 是临界屈曲载荷.

$$\begin{cases} N_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} \mathrm{d}z, \\ M_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} z \mathrm{d}z, \\ P_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} z^n \mathrm{d}z, \\ Q_{\alpha} = \int_{-h/2}^{h/2} \sigma_{\alpha z} \mathrm{d}z, \\ R_{\alpha} = \int_{-h/2}^{h/2} \sigma_{\alpha z} z^{n-1} \mathrm{d}z. \end{cases} \tag{4}$$

应力-应变关系如下：

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{26} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}, \tag{5}$$

式中

$$\begin{cases} \bar{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta, \\ \bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta), \\ \bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta^3\cos\theta, \\ \bar{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta, \\ \bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta^3\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta, \\ \bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta), \\ \bar{Q}_{44} = Q_{44}\cos^2\theta + Q_{55}\sin^2\theta, \\ \bar{Q}_{45} = (Q_{55} - Q_{44})\cos\theta\sin\theta, \\ \bar{Q}_{55} = Q_{55}\cos^2\theta + Q_{44}\sin^2\theta, \end{cases} \tag{6}$$

θ 是纤维方向角,

$$\begin{cases} Q_{11} = \dfrac{E_1}{1 - \nu_{12}\nu_{21}},\ Q_{22} = \dfrac{E_2}{1 - \nu_{12}\nu_{21}},\ Q_{12} = \nu_{21}Q_{11}, \\ Q_{66} = G_{12},\ Q_{44} = G_{23},\ Q_{55} = G_{13},\ \nu_{21}E_1 = \nu_{12}E_2. \end{cases} \tag{7}$$

将式(5)代入式(4),得到如下方程：

$$N_x = A_{11}\frac{\partial u}{\partial x} + B_{11}\frac{\partial \phi_x}{\partial x} - C_1E_{11}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + A_{12}\frac{\partial v}{\partial y} + B_{12}\frac{\partial \phi_y}{\partial y} -$$

$$C_1E_{12}\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+A_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+B_{16}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)-$$
$$C_1E_{16}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right), \quad (8)$$

$$N_y=A_{12}\frac{\partial u}{\partial x}+B_{12}\frac{\partial \phi_x}{\partial x}-C_1E_{12}\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+A_{22}\frac{\partial v}{\partial y}+B_{22}\frac{\partial \phi_y}{\partial y}-$$
$$C_1E_{22}\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+A_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+B_{26}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)-$$
$$C_1E_{26}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right), \quad (9)$$

$$N_{xy}=A_{16}\frac{\partial u}{\partial x}+B_{16}\frac{\partial \phi_x}{\partial x}-C_1E_{16}\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+A_{26}\frac{\partial v}{\partial y}+B_{26}\frac{\partial \phi_y}{\partial y}-$$
$$C_1E_{26}\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+B_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)-$$
$$C_1E_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right), \quad (10)$$

$$M_x=B_{11}\frac{\partial u}{\partial x}+D_{11}\frac{\partial \phi_x}{\partial x}-C_1E_{E_{11}}\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+B_{12}\frac{\partial v}{\partial y}+D_{12}\frac{\partial \phi_y}{\partial y}-$$
$$C_1E_{E_{12}}\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+B_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+D_{16}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)-$$
$$C_1E_{E_{16}}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right), \quad (11)$$

$$M_y=B_{12}\frac{\partial u}{\partial x}+D_{12}\frac{\partial \phi_x}{\partial x}-C_1E_{E_{12}}\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+B_{22}\frac{\partial v}{\partial y}+D_{22}\frac{\partial \phi_y}{\partial y}-$$
$$C_1E_{E_{22}}\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+B_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+D_{26}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)-$$
$$C_1E_{E_{26}}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right), \quad (12)$$

$$M_{xy}=B_{16}\frac{\partial u}{\partial x}+D_{16}\frac{\partial \phi_x}{\partial x}-C_1E_{E_{16}}\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+B_{26}\frac{\partial v}{\partial y}+D_{26}\frac{\partial \phi_y}{\partial y}-$$
$$C_1E_{E_{26}}\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+B_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+D_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)-$$
$$C_1E_{E_{66}}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right), \quad (13)$$

$$P_x=E_{11}\frac{\partial u}{\partial x}+E_{E_{11}}\frac{\partial \phi_x}{\partial x}-C_1H_{11}\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+E_{12}\frac{\partial v}{\partial y}+E_{E_{12}}\frac{\partial \phi_y}{\partial y}-$$
$$C_1H_{12}\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+E_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+E_{E_{16}}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)-$$

$$C_1H_{16}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right), \tag{14}$$

$$\begin{aligned}P_y = & E_{12}\frac{\partial u}{\partial x}+E_{E_{12}}\frac{\partial\phi_x}{\partial x}-C_1H_{12}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+E_{22}\frac{\partial v}{\partial y}+E_{E_{22}}\frac{\partial\phi_y}{\partial y}-\\ & C_1H_{22}\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+E_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+E_{E_{26}}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)-\\ & C_1H_{26}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right),\end{aligned} \tag{15}$$

$$\begin{aligned}P_{xy} = & E_{16}\frac{\partial u}{\partial x}+E_{E_{16}}\frac{\partial\phi_x}{\partial x}-C_1H_{16}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+E_{26}\frac{\partial v}{\partial y}+E_{E_{26}}\frac{\partial\phi_y}{\partial y}-\\ & C_1H_{26}\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+E_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+E_{E_{66}}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)-\\ & C_1H_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right),\end{aligned} \tag{16}$$

$$\begin{cases}Q_x=(A_{45}-C_2D_{D_{45}})\left(\phi_y+\dfrac{\partial w}{\partial y}\right)+(A_{55}-C_2D_{D_{55}})\left(\phi_x+\dfrac{\partial w}{\partial x}\right),\\ Q_y=(A_{44}-C_2D_{D_{44}})\left(\phi_y+\dfrac{\partial w}{\partial y}\right)+(A_{45}-C_2D_{D_{45}})\left(\phi_x+\dfrac{\partial w}{\partial x}\right),\\ R_x=(D_{D_{45}}-C_2F_{45})\left(\phi_y+\dfrac{\partial w}{\partial y}\right)+(D_{D_{55}}-C_2F_{55})\left(\phi_x+\dfrac{\partial w}{\partial x}\right),\\ R_y=(D_{D_{44}}-C_2F_{44})\left(\phi_y+\dfrac{\partial w}{\partial y}\right)+(D_{D_{45}}-C_2F_{45})\left(\phi_x+\dfrac{\partial w}{\partial x}\right),\end{cases} \tag{17}$$

$$\begin{cases}A_{ij}=\sum\limits_{k=1}^{N_L}\int_{z_k}^{z_{k+1}}\bar{Q}_{ij}^{(k)}\mathrm{d}z,\ B_{ij}=\sum\limits_{k=1}^{N_L}\bar{Q}_{ij}^{(k)}\int_{z_k}^{z_{k+1}}z\mathrm{d}z,\\ D_{ij}=\sum\limits_{k=1}^{N_L}\bar{Q}_{ij}^{(k)}\int_{z_k}^{z_{k+1}}z^2\mathrm{d}z,\ D_{D_{ij}}=\sum\limits_{k=1}^{N_L}\bar{Q}_{ij}^{(k)}\int_{z_k}^{z_{k+1}}z^{n-1}\mathrm{d}z,\\ E_{ij}=\sum\limits_{k=1}^{N_L}\bar{Q}_{ij}^{(k)}\int_{z_k}^{z_{k+1}}z^{n}\mathrm{d}z,\ E_{E_{ij}}=\sum\limits_{k=1}^{N_L}\bar{Q}_{ij}^{(k)}\int_{z_k}^{z_{k+1}}z^{n+1}\mathrm{d}z,\\ F_{ij}=\sum\limits_{k=1}^{N_L}\bar{Q}_{ij}^{(k)}\int_{z_k}^{z_{k+1}}z^{2n-2}\mathrm{d}z,\ H_{ij}=\sum\limits_{k=1}^{N_L}\bar{Q}_{ij}^{(k)}\int_{z_k}^{z_{k+1}}z^{2n}\mathrm{d}z.\end{cases} \tag{18}$$

将式(8)~(17)代入式(3),得到用位移表示的控制微分方程如下:

$$\begin{aligned}& A_{11}\frac{\partial^2 u}{\partial x^2}+A_{12}\frac{\partial^2 v}{\partial x\partial y}+A_{16}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+B_{11}\frac{\partial^2\phi_x}{\partial x^2}+B_{12}\frac{\partial^2\phi_y}{\partial x\partial y}+B_{16}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)-\\ & C_1E_{11}\left(\frac{\partial^2\phi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)-C_1E_{12}\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)-C_1E_{16}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)+\\ & A_{16}\frac{\partial^2 u}{\partial x\partial y}+A_{26}\frac{\partial^2 v}{\partial y^2}+A_{66}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+B_{16}\frac{\partial^2\phi_x}{\partial x\partial y}+B_{26}\frac{\partial^2\phi_y}{\partial y^2}+B_{66}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)-\\ & C_1E_{16}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial x^2\partial y}\right)-C_1E_{26}\left(\frac{\partial^2\phi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)-\end{aligned}$$

$$C_1E_{66}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}+2\,\frac{\partial^3 w}{\partial x\partial y^2}\right)=0, \tag{19}$$

$$A_{16}\frac{\partial^2 u}{\partial x^2}+A_{26}\frac{\partial^2 v}{\partial x\partial y}+A_{66}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+B_{16}\frac{\partial^2\phi_x}{\partial x^2}+B_{26}\frac{\partial^2\phi_y}{\partial x\partial y}+B_{66}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)-$$

$$C_1E_{16}\left(\frac{\partial^2\phi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)-C_1E_{26}\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)-$$

$$C_1E_{66}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}+2\,\frac{\partial^3 w}{\partial x^2\partial y}\right)+A_{12}\frac{\partial^2 u}{\partial x\partial y}+A_{22}\frac{\partial^2 v}{\partial y^2}+A_{26}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+$$

$$B_{12}\frac{\partial^2\phi_x}{\partial x\partial y}+B_{22}\frac{\partial^2\phi_y}{\partial y^2}+B_{26}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)-C_1E_{12}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial x^2\partial y}\right)-$$

$$C_1E_{22}\left(\frac{\partial^2\phi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)-C_1E_{26}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}+2\,\frac{\partial^3 w}{\partial x\partial y^2}\right)=0, \tag{20}$$

$$A_{45}\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial^2 w}{\partial x\partial y}\right)+A_{55}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)-C_2D_{D_{45}}\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial^2 w}{\partial x\partial y}\right)-C_2D_{D_{55}}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+$$

$$A_{44}\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+A_{45}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial^2 w}{\partial x\partial y}\right)-C_2D_{D_{44}}\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)-$$

$$C_2D_{D_{45}}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial^2 w}{\partial x\partial y}\right)-C_2\left[D_{D_{45}}\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial^2 w}{\partial x\partial y}\right)+D_{D_{55}}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)-\right.$$

$$\left.C_2F_{45}\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial^2 w}{\partial x\partial y}\right)\right]-C_2F_{55}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+D_{D_{44}}\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+$$

$$D_{D_{45}}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial^2 w}{\partial x\partial y}\right)-C_2F_{44}\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)-C_2F_{45}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial^2 w}{\partial x\partial y}\right)+$$

$$C_1\left[E_{11}\frac{\partial^3 u}{\partial x^3}+E_{12}\frac{\partial^3 v}{\partial x^2\partial y}+E_{16}\left(\frac{\partial^3 u}{\partial x^2\partial y}+\frac{\partial^3 v}{\partial x^3}\right)+E_{E_{11}}\frac{\partial^3\phi_x}{\partial x^3}+E_{E_{12}}\frac{\partial^3\phi_y}{\partial x^2\partial y}+\right.$$

$$E_{E_{16}}\left(\frac{\partial^3\phi_x}{\partial x^2\partial y}+\frac{\partial^3\phi_y}{\partial x^3}\right)-C_1H_{11}\left(\frac{\partial^3\phi_x}{\partial x^3}+\frac{\partial^4 w}{\partial x^4}\right)-C_1H_{12}\left(\frac{\partial^3\phi_y}{\partial x^2\partial y}+\frac{\partial^4 w}{\partial x^2\partial y^2}\right)-$$

$$C_1H_{16}\left(\frac{\partial^3\phi_x}{\partial x^2\partial y}+\frac{\partial^3\phi_y}{\partial x^3}+2\,\frac{\partial^4 w}{\partial x^3\partial y}\right)+E_{12}\frac{\partial^3 u}{\partial x\partial y^2}+$$

$$E_{22}\frac{\partial^3 v}{\partial y^3}+E_{26}\left(\frac{\partial^3 u}{\partial y^3}+\frac{\partial^3 v}{\partial x\partial y^2}\right)+E_{E_{12}}\frac{\partial^3\phi_x}{\partial x\partial y^2}+E_{E_{22}}\frac{\partial^3\phi_y}{\partial y^3}+E_{E_{26}}\left(\frac{\partial^3\phi_x}{\partial y^3}+\frac{\partial^3\phi_y}{\partial x\partial y^2}\right)-$$

$$C_1H_{12}\left(\frac{\partial^3\phi_x}{\partial x\partial y^2}+\frac{\partial^4 w}{\partial x^2\partial y^2}\right)-C_1H_{22}\left(\frac{\partial^3\phi_y}{\partial y^3}+\frac{\partial^4 w}{\partial y^4}\right)-C_1H_{26}\left(\frac{\partial^3\phi_x}{\partial y^3}+\frac{\partial^3\phi_y}{\partial x\partial y^2}+2\,\frac{\partial^4 w}{\partial x\partial y^3}\right)+$$

$$2E_{16}\frac{\partial^3 u}{\partial x^2\partial y}+2E_{26}\frac{\partial^3 v}{\partial x\partial y^2}+2E_{66}\left(\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial^3 v}{\partial x^2\partial y}\right)+2E_{E_{16}}\frac{\partial^3\phi_x}{\partial x^2\partial y}+$$

$$2E_{E_{26}}\frac{\partial^3\phi_y}{\partial x\partial y^2}+2E_{E_{66}}\left(\frac{\partial^3\phi_x}{\partial x\partial y^2}+\frac{\partial^3\phi_y}{\partial x^2\partial y}\right)-2C_1H_{16}\left(\frac{\partial^3\phi_x}{\partial x^2\partial y}+\frac{\partial^4 w}{\partial x^3\partial y}\right)-$$

$$\left.2C_1H_{26}\left(\frac{\partial^3\phi_y}{\partial x\partial y^2}+\frac{\partial^4 w}{\partial x\partial y^3}\right)-2C_1H_{66}\left(\frac{\partial^3\phi_x}{\partial x\partial y^2}+\frac{\partial^3\phi_y}{\partial x^2\partial y}+2\,\frac{\partial^4 w}{\partial x^2\partial y^2}\right)\right]+$$

$$\bar{N}_{xx}\frac{\partial^2 w}{\partial x^2}+2\bar{N}_{xy}\frac{\partial^2 w}{\partial x\partial y}+\bar{N}_{yy}\frac{\partial^2 w}{\partial y^2}=0, \tag{21}$$

$$\begin{aligned}
&B_{11}\frac{\partial^2 u}{\partial x^2}+B_{12}\frac{\partial^2 v}{\partial x\partial y}+B_{16}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+D_{11}\frac{\partial^2 \phi_x}{\partial x^2}+D_{12}\frac{\partial^2 \phi_y}{\partial x\partial y}+D_{16}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^2 \phi_y}{\partial x^2}\right)-\\
&C_1E_{E_{11}}\left(\frac{\partial^2 \phi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)-C_1E_{E_{12}}\left(\frac{\partial^2 \phi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)-\\
&C_1E_{E_{16}}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^2 \phi_y}{\partial x^2}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)+B_{16}\frac{\partial^2 u}{\partial x\partial y}+B_{26}\frac{\partial^2 v}{\partial y^2}+\\
&B_{66}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+D_{16}\frac{\partial^2 \phi_x}{\partial x\partial y}+D_{26}\frac{\partial^2 \phi_y}{\partial y^2}+D_{66}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial x\partial y}\right)-\\
&C_1E_{E_{16}}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial x^2\partial y}\right)-C_1E_{E_{26}}\left(\frac{\partial^2 \phi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)-\\
&C_1E_{E_{66}}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial x\partial y}+2\frac{\partial^3 w}{\partial x\partial y^2}\right)-A_{45}\left(\phi_y+\frac{\partial w}{\partial y}\right)-A_{55}\left(\phi_x+\frac{\partial w}{\partial x}\right)+\\
&2C_2D_{D_{45}}\left(\phi_y+\frac{\partial w}{\partial y}\right)+2C_2D_{D_{55}}\left(\phi_x+\frac{\partial w}{\partial x}\right)-C_2^2F_{45}\left(\phi_y+\frac{\partial w}{\partial y}\right)-\\
&C_2^2F_{55}\left(\phi_x+\frac{\partial w}{\partial x}\right)-C_1\left[E_{11}\frac{\partial^2 u}{\partial x^2}+E_{12}\frac{\partial^2 v}{\partial x\partial y}+E_{16}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+\right.\\
&E_{E_{11}}\frac{\partial^2 \phi_x}{\partial x^2}+E_{E_{12}}\frac{\partial^2 \phi_y}{\partial x\partial y}+E_{E_{16}}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^2 \phi_y}{\partial x^2}\right)-C_1H_{11}\left(\frac{\partial^2 \phi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)-\\
&C_1H_{12}\left(\frac{\partial^2 \phi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)-C_1H_{16}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^2 \phi_y}{\partial x^2}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)+E_{16}\frac{\partial^2 u}{\partial x\partial y}+\\
&E_{26}\frac{\partial^2 v}{\partial y^2}+E_{66}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+E_{E_{16}}\frac{\partial^2 \phi_x}{\partial x\partial y}+E_{E_{26}}\frac{\partial^2 \phi_y}{\partial y^2}+\\
&E_{E_{66}}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial x\partial y}\right)-C_1H_{16}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial x^2\partial y}\right)-\\
&\left.C_1H_{26}\left(\frac{\partial^2 \phi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)-C_1H_{66}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial x\partial y}+2\frac{\partial^3 w}{\partial x\partial y^2}\right)\right]=0,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&B_{16}\frac{\partial^2 u}{\partial x^2}+B_{26}\frac{\partial^2 v}{\partial x\partial y}+B_{66}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+D_{16}\frac{\partial^2 \phi_x}{\partial x^2}+D_{26}\frac{\partial^2 \phi_y}{\partial x\partial y}+D_{66}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^2 \phi_y}{\partial x^2}\right)-\\
&C_1E_{E_{16}}\left(\frac{\partial^2 \phi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)-C_1E_{E_{26}}\left(\frac{\partial^2 \phi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)-\\
&C_1E_{E_{66}}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^2 \phi_y}{\partial x^2}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)+B_{12}\frac{\partial^2 u}{\partial x\partial y}+B_{22}\frac{\partial^2 v}{\partial y^2}+B_{26}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+\\
&D_{12}\frac{\partial^2 \phi_x}{\partial x\partial y}+D_{22}\frac{\partial^2 \phi_y}{\partial y^2}+D_{26}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial x\partial y}\right)-C_1E_{E_{12}}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial x^2\partial y}\right)-\\
&C_1E_{E_{22}}\left(\frac{\partial^2 \phi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)-C_1E_{E_{26}}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial x\partial y}+2\frac{\partial^3 w}{\partial x\partial y^2}\right)-
\end{aligned}$$

$$A_{44}\left(\phi_y+\frac{\partial w}{\partial y}\right)-A_{45}\left(\phi_x+\frac{\partial w}{\partial x}\right)+2C_2D_{D_{44}}\left(\phi_y+\frac{\partial w}{\partial y}\right)+2C_2D_{D_{45}}\left(\phi_x+\frac{\partial w}{\partial x}\right)-$$
$$C_2^2F_{44}\left(\phi_y+\frac{\partial w}{\partial y}\right)-C_2^2F_{45}\left(\phi_x+\frac{\partial w}{\partial x}\right)-C_1\left[E_{16}\frac{\partial^2u}{\partial x^2}+E_{26}\frac{\partial^2v}{\partial x\partial y}+\right.$$
$$E_{66}\left(\frac{\partial^2u}{\partial x\partial y}+\frac{\partial^2v}{\partial x^2}\right)+E_{E_{16}}\frac{\partial^2\phi_x}{\partial x^2}+E_{E_{26}}\frac{\partial^2\phi_y}{\partial x\partial y}+E_{E_{66}}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)-$$
$$C_1H_{16}\left(\frac{\partial^2\phi_x}{\partial x^2}+\frac{\partial^3w}{\partial x^3}\right)-C_1H_{26}\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^3w}{\partial x\partial y^2}\right)-$$
$$C_1H_{66}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}+2\frac{\partial^3w}{\partial x^2\partial y}\right)+E_{12}\frac{\partial^2u}{\partial x\partial y}+E_{22}\frac{\partial^2v}{\partial y^2}+E_{26}\left(\frac{\partial^2u}{\partial y^2}+\frac{\partial^2v}{\partial x\partial y}\right)+$$
$$E_{E_{12}}\frac{\partial^2\phi_x}{\partial x\partial y}+E_{E_{22}}\frac{\partial^2\phi_y}{\partial y^2}+E_{E_{26}}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)-C_1H_{12}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^3w}{\partial x^2\partial y}\right)-$$
$$\left.C_1H_{22}\left(\frac{\partial^2\phi_y}{\partial y^2}+\frac{\partial^3w}{\partial y^3}\right)-C_1H_{26}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}+2\frac{\partial^3w}{\partial x\partial y^2}\right)\right]=0. \tag{23}$$

简支边界条件如下：

$$\begin{cases}x=0,\ a:\ v=0,\ \phi_y=0,\ w=0,\ M_x-C_1P_x=0,\ N_x=0,\\ y=0,\ b:\ u=0,\ \phi_x=0,\ w=0,\ M_y-C_1P_y=0,\ N_y=0.\end{cases} \tag{24}$$

2 求解方法

$$\begin{cases}u=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}U_{rs}\cos(\alpha x)\sin(\beta y),\\ v=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}V_{rs}\sin(\alpha x)\cos(\beta y),\\ w=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}W_{rs}\sin(\alpha x)\sin(\beta y),\\ \phi_x=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}X_{rs}\cos(\alpha x)\sin(\beta y),\\ \phi_y=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}Y_{rs}\sin(\alpha x)\cos(\beta y),\end{cases} \tag{25}$$

式中，$\alpha=r\pi/a$，$\beta=s\pi/b$，r 和 s 是 x，y 方向的半波数.

将式(25)代入式(19)~(23)，可以得到如下的方程：

$$(\boldsymbol{K}-\bar{N}\boldsymbol{M})\boldsymbol{\Delta}=\boldsymbol{0},\qquad \boldsymbol{\Delta}^{\mathrm{T}}=\{U_{rs},V_{rs},W_{rs},X_{rs},Y_{rs}\}, \tag{26}$$

式中，$\boldsymbol{K}$ 为刚度矩阵，$\bar{N}$ 是临界屈曲载荷，$\boldsymbol{M}$ 为质量矩阵.方程(26)有非零解的充分必要条件是($\boldsymbol{K}-\bar{N}\boldsymbol{M}$)的行列式为零，通过求解方程 $\det[\boldsymbol{K}-\bar{N}\boldsymbol{M}]=0$，临界屈曲载荷可以被求出.

3 数值算例

复合材料层合板每层的材料属性如下：

$$\frac{E_1}{E_2} = 10,20,30,40;\ G_{12} = G_{13} = 0.6E_2;\ G_{23} = 0.5E_2;\ \nu_{21} = \nu_{12}\frac{E_1}{E_2},\ \nu_{12} = 0.25.$$

下式被用来对屈曲载荷进行无量纲处理：

$$\bar{N} = \frac{\bar{N}_{xx}a^2}{E_2h^3}. \tag{27}$$

表 1 列出了简支复合材料层合板(0°/90°/0°, $a/h = 10$) 在单轴载荷 ($\bar{N}_{xy} = 0, \bar{N}_{yy} = 0$) 作用下的临界屈曲载荷.表 2 列出了简支复合材料层合板(0°/90°/0°/90°/0°, $a/h = 10$) 在单轴载荷作用下的临界屈曲载荷.表 3 列出了简支复合材料层合板(0°/90°/0°/90°/0°/90°/0°/90°/0°, $a/h = 10$) 在单轴载荷作用下的临界屈曲载荷.通过表 1~3 可以看出,本文结果与参考文献中的结果具有较好的一致性.

表 1 简支复合材料层合板(0°/90°/0°, $a/h = 10$) 在单轴载荷作用下的临界屈曲载荷

Table 1 The non-dimensional buckling load of the simply supported composite laminate plate (0°/90°/0°, $a/h = 10$) under uniaxial load

method	$E_1/E_2 = 3$	$E_1/E_2 = 10$	$E_1/E_2 = 20$	$E_1/E_2 = 30$	$E_1/E_2 = 40$
ref. [14]	5.387	9.833	14.898	18.892	22.153
ref. [15]	5.393	9.940	15.298	19.674	23.340
this paper, $n = 3$	5.389	9.832	14.889	18.877	22.120
this paper, $n = 5$	5.399	9.877	15.000	19.055	22.361
this paper, $n = 7$	5.408	9.918	15.104	19.228	22.601

表 2 简支复合材料层合板(0°/90°/0°/90°/0°, $a/h = 10$) 在单轴载荷作用下的临界屈曲载荷

Table 2 The non-dimensional buckling load of the simply supported composite laminate plate (0°/90°/0°/90°/0°, $a/h = 10$) under uniaxial load

method	$E_1/E_2 = 3$	$E_1/E_2 = 10$	$E_1/E_2 = 20$	$E_1/E_2 = 30$	$E_1/E_2 = 40$
ref. [14]	5.404	10.088	15.791	20.591	24.69
ref. [15]	5.409	10.15	16.008	20.999	25.308
this paper, $n = 3$	5.406	10.089	15.787	20.578	24.675
this paper, $n = 5$	5.412	10.1	15.804	20.602	24.712
this paper, $n = 7$	5.42	10.127	15.869	20.713	24.873

表 3 简支复合材料层合板(0°/90°/0°/90°/0°/90°/0°/90°/0°, $a/h = 10$) 在单轴载荷作用下的临界屈曲载荷

Table 3 The non-dimensional buckling load of the simply supported composite laminate plate (0°/90°/0°/90°/0°/90°/0°/90°/0°, $a/h = 10$) under uniaxial load

method	$E_1/E_2 = 3$	$E_1/E_2 = 10$	$E_1/E_2 = 20$	$E_1/E_2 = 30$	$E_1/E_2 = 40$
ref. [14]	5.409	10.176	16.106	21.191	25.608
ref. [15]	5.431	10.197	16.172	21.315	25.79
this paper, $n = 3$	5.412	10.177	16.1	21.178	25.584
this paper, $n = 5$	5.418	10.195	16.143	21.248	25.683
this paper, $n = 7$	5.425	10.221	16.206	21.356	25.842

图 1、2 所示为三层和四层板的临界屈曲载荷随模量比的变化.从图 1、2 可以看出,随着模量比的增加,临界屈曲载荷逐渐增加, n 值越大,临界屈曲载荷越大.

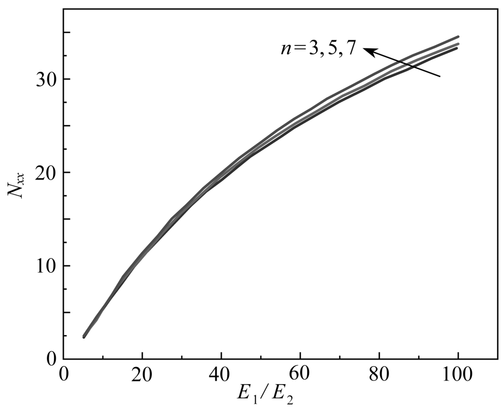

图 1　简支复合材料层合板(0°/90°/0°, a/h = 10)在单轴载荷作用下的临界屈曲载荷随 E_1/E_2 的变化

Fig. 1　Variations of the non-dimensional buckling load of the simply supported composite laminate plate (0°/90°/0°, a/h = 10) under uniaxial load with E_1/E_2

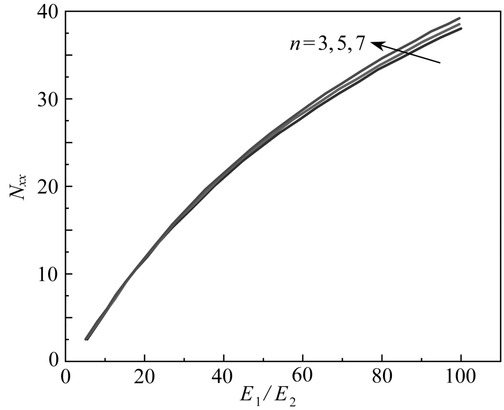

图 2　简支复合材料层合板(0°/90°/90°/0°, a/h = 10)在单轴载荷作用下的临界屈曲载荷随 E_1/E_2 的变化

Fig. 2　Variations of the non-dimensional buckling load of the simply supported composite laminate plate (0°/90°/90°/0°, a/h = 10) under uniaxial load with E_1/E_2

4 结　　论

本文采用 n 阶剪切变形理论分析了复合材料层合板的屈曲问题,采用虚功原理,推导出了复合材料层合板在面内载荷作用下临界屈曲的控制微分方程.结合数值算例,对简支复合材料层合板在单轴载荷作用下的临界屈曲载荷进行了分析和计算.数值计算结果表明,本文的 n 阶理论在屈曲载荷计算方面具有较高的精度.

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Buckling Analysis of Composite Laminate Plates Based on the *n*th-Order Shear Deformation Theory

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Abstract: The buckling problem of composite laminate plates was analyzed with the *n*th-order shear deformation theory. The virtual work principle was used to derive the governing equations for the critical buckling of laminate pates under in-plane loads. The comparison between the present results and the results of previous literatures shows good agreement and high calculation accuracy.

Key words: composite laminate plate; buckling; analytical method; in-plane load; *n*th-order shear deformation theory