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分数阶双参数高阶非线性扰动模型的渐近解*

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摘要: 研究了一类高阶非线性分数阶扰动微分模型.在适当的条件下,首先利用扰动方法求出了原问题的外部解,然后用伸长变量、合成展开和幂级数理论构造出解的第一、第二边界层校正项,并得到了解的形式渐近展开式.最后利用微分不等式理论,研究了问题解的渐近性态,并证明了问题解渐近估计式的一致有效性.

关 键 词: 边界层; 分数阶微分模型; 扰动

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引　　言

分数阶导数理论是整数阶导数概念的延伸.它有广泛而特殊的意义,许多自然现象与分数阶导数有关.譬如在复杂的沙尘流、潜流和导热等现象中,用通常的导数概念不能恰当地描述,用分数阶导数就能加以表述[1].但是分数阶导数非线性微分-积分方程的求解比较困难.本文用奇异扰动理论和方法来构造一类两参数分数阶导数高阶非线性微分-积分方程的渐近解,并用微分不等式理论得到解一致有效的渐近性态的估计.

非线性奇异扰动问题是数学界十分重视的研究对象[2-4].近年来许多近似方法不断被进一步深入地研究,其中包括匹配展开法、边界层法以及多重尺度法等,许多科学工作者,例如 Martinez 等[5]、Tian 等[6]、Kellogg 等[7]、Samusenko[8] 和 Skrynnikov[9] 都做了很多的成果.利用微分不等式、广义变分原理和泛函分析等方法,莫嘉琪和徐建中等也研讨了一些非线性奇扰动问题[10-24].本文是用特殊的方法构造一类两参数奇异扰动分数阶非线性微分方程的渐近解,并证明了解的渐近展开式的一致有效性.

考虑以下一类分数阶双参数高阶非线性微分方程奇异扰动两点边值问题:

$$\varepsilon^{n\alpha}(\mathrm{D}_x^\alpha)^{(n)}y+\mu^\alpha\mathrm{D}_x^\alpha y=f(x,y,\varepsilon,\mu),\qquad a<x<b,\tag{1}$$

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$$(\mathrm{D}_x^\alpha)^{(i)} y(a,\varepsilon,\mu) = g_i(\varepsilon,\mu), \qquad i = 1,2,\cdots,n-1, \tag{2}$$

$$(\mathrm{D}_x^\alpha)^{(i)} y(b,\varepsilon,\mu) = h_i(\varepsilon,\mu), \qquad i = 1,2,\cdots,n-1, \tag{3}$$

其中 ε,μ 为两个正的小参数，$\alpha < 1$ 为正数，$n \geqslant 2$ 为整数，a,b 为常数，$\mathrm{D}_x^\alpha y$ 为 $y(x)$ 的 α 阶分数阶导数，它定义为

$$\mathrm{D}_x^\alpha y \equiv \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x (x-t)^{-\alpha} y(t)\,\mathrm{d}t,$$

这里 Γ 是 Gamma 函数.

首先假设：

[H1] $\lim\limits_{\mu \to 0} \dfrac{\varepsilon}{\mu} = 0$;

[H2] $f(x, y, \varepsilon, \mu)$，$A(\varepsilon, \mu)$ 和 $B(\varepsilon, \mu)$ 在对应的自变量在其定义范围内为充分光滑的函数；

[H3] $\min(f_\varepsilon, f_\mu) \geqslant 0$, $f_y \leqslant -c < 0$，其中 c 为正常数.

1 微分不等式理论

定义两个光滑函数 $\bar{y}$ 和 $\underline{y}$.设 $\bar{y} \geqslant \underline{y}$ 分别满足不等式：

$$\varepsilon^{n\alpha}(\mathrm{D}_x^\alpha)^{(n)}\bar{y} + \mu^\alpha \mathrm{D}_x^\alpha \bar{y} - f(x,\bar{y},\varepsilon,\mu) \leqslant 0,$$

$$(\mathrm{D}_x^\alpha)^{(i)}\bar{y}(a,\varepsilon,\mu) \geqslant g_i(\varepsilon,\mu), \qquad i = 1,2,\cdots,n-1,$$

$$(\mathrm{D}_x^\alpha)^{(i)}\bar{y}(b,\varepsilon,\mu) \geqslant h_i(\varepsilon,\mu), \qquad i = 1,2,\cdots,n-1;$$

$$\varepsilon^{n\alpha}(\mathrm{D}_x^\alpha)^{(n)}\underline{y} + \mu^\alpha \mathrm{D}_x^\alpha \underline{y} - f(x,\underline{y},\varepsilon,\mu) \geqslant 0,$$

$$(\mathrm{D}_x^\alpha)^{(i)}\underline{y}(a,\varepsilon,\mu) \leqslant g_i(\varepsilon,\mu), \qquad i = 1,2,\cdots,n-1,$$

$$(\mathrm{D}_x^\alpha)^{(i)}\underline{y}(b,\varepsilon,\mu) \leqslant h_i(\varepsilon,\mu), \qquad i = 1,2,\cdots,n-1.$$

我们称 $\bar{y}$ 和 $\underline{y}$ 分别为问题(1)~(3)的上解和下解.

现有如下定理.

定理 1 在假设[H1]~[H3]下，$\bar{y}(x,\varepsilon,\mu)$ 和 $\underline{y}(x,\varepsilon,\mu)$ 分别为分数阶双参数高阶非线性微分方程边值问题(1)~(3)的上解和下解，这时边值问题(1)~(3)存在一个解 $y(x,\varepsilon,\mu)$，且 $\underline{y}(x,\varepsilon,\mu) \leqslant y(x,\varepsilon,\mu) \leqslant \bar{y}(x,\varepsilon,\mu)$ 成立.

证明 首先按以下迭代关系式构造一个序列 $\{y_k\}$：

$$\varepsilon^{n\alpha}(\mathrm{D}_x^\alpha)^{(n)} y_{k+1} + \mu^\alpha \mathrm{D}_x^\alpha y_{k+1} = f(x,y_k,\varepsilon,\mu), \qquad k = 0,1,\cdots, \tag{4}$$

$$(\mathrm{D}_x^\alpha)^{(i)} y_{k+1}(a,\varepsilon,\mu) = g_i(\varepsilon,\mu), \qquad i = 1,2,\cdots,n-1,\ k = 0,1,\cdots, \tag{5}$$

$$(\mathrm{D}_x^\alpha)^{(i)} y_{k+1}(b,\varepsilon,\mu) = h_i(\varepsilon,\mu), \qquad i = 1,2,\cdots,n-1,\ k = 0,1,\cdots. \tag{6}$$

分别设 $\bar{y}_0 = \bar{y}$ 和 $\underline{y}_0 = \underline{y}$ 为零次函数，可依次得到 $\bar{y}_k$ 和 $\underline{y}_k, k = 1,2,\cdots$.因此，可得两个函数列 $\{\bar{y}_k\}$ 和 $\{\underline{y}_k\}$.现讨论其收敛性态.

设 $\bar{z}_0 = \bar{y}_0 - \bar{y}_1$，由假设[H1]~[H3]和式(4)~(6)，有

$$\begin{aligned}
&\varepsilon^{n\alpha}(\mathrm{D}_x^\alpha)^{(n)}\bar{z}_0 + \mu^\alpha \mathrm{D}_x^\alpha \bar{z}_0 = \\
&\quad \varepsilon^{n\alpha}(\mathrm{D}_x^\alpha)^{(n)}\bar{y}_0 + \mu^\alpha \mathrm{D}_x^\alpha \bar{y}_0 - \varepsilon^{n\alpha}(\mathrm{D}_x^\alpha)^{(n)}\bar{y}_1 - \mu^\alpha \mathrm{D}_x^\alpha \bar{y}_1 \leqslant \\
&\quad f(x,\bar{y}_0,0,0) - f(x,\bar{y}_0,\varepsilon,\mu) = 0,
\end{aligned}$$

$$(D_x^\alpha)^{(i)}\bar{z}_0(a,\varepsilon,\mu)=(D_x^\alpha)^{(i)}\bar{y}_0|_{x=a}-(D_x^\alpha)^{(i)}\bar{y}_1|_{x=a}=0,\qquad i=1,2,\cdots,n-1,$$
$$(D_x^\alpha)^{(i)}\bar{z}_0(b,\varepsilon,\mu)=(D_x^\alpha)^{(i)}\bar{y}_0|_{x=b}-(D_x^\alpha)^{(i)}\bar{y}_1|_{x=b}=0,\qquad i=1,2,\cdots,n-1.$$

于是由极值原理得到 $\bar{z}_0\geqslant 0$.即 $\bar{y}_0\geqslant\bar{y}_1$.

若 $\bar{y}_{k-1}\geqslant\bar{y}_k,k=1,2,\cdots$, 设 $\bar{z}_k=\bar{y}_k-\bar{y}_{k+1}$, 有

$$\begin{aligned}&\varepsilon^{n\alpha}(D_x^\alpha)^{(n)}\bar{z}_k+\mu^\alpha D_x^\alpha\bar{z}_k=\\&\quad\varepsilon^{n\alpha}(D_x^\alpha)^{(n)}\bar{y}_k+\mu^\alpha D_x^\alpha\bar{y}_k-\varepsilon^{n\alpha}(D_x^\alpha)^{(n)}\bar{y}_{k+1}-\mu^\alpha D_x^\alpha\bar{y}_{k+1}\leqslant\\&\quad f(x,\bar{y}_{k-1},\varepsilon,\mu)-f(x,\bar{y}_k,\varepsilon,\mu)=f_y\leqslant -c<0,\end{aligned}$$
$$(D_x^\alpha)^{(i)}\bar{z}_k(a,\varepsilon,\mu)=(D_x^\alpha)^{(i)}\bar{y}_0|_{x=a}-(D_x^\alpha)^{(i)}\bar{y}_1|_{x=a}=0,\qquad i=1,2,\cdots,n-1,$$
$$(D_x^\alpha)^{(i)}\bar{z}_0(b,\varepsilon,\mu)=(D_x^\alpha)^{(i)}\bar{y}_0|_{x=b}-(D_x^\alpha)^{(i)}\bar{y}_1|_{x=b}=0,\qquad i=1,2,\cdots,n-1.$$

于是 $\bar{z}_k\geqslant 0$.即 $\bar{y}_k\geqslant\bar{y}_{k+1},k=1,2,\cdots$.

于是有

$$\bar{y}=\bar{y}_0\geqslant\bar{y}_1\geqslant\cdots\geqslant\bar{y}_k\geqslant\bar{y}_{k+1}\geqslant\cdots.$$

类似地,有

$$\underline{y}=\underline{y}_0\leqslant\underline{y}_1\leqslant\cdots\leqslant\underline{y}_k\leqslant\underline{y}_{k+1}\leqslant\cdots.$$

同样可得

$$\bar{y}_k\geqslant\underline{y}_k,\qquad k=0,1,\cdots.$$

由上述结果和 Arzela 定理,两参数分数阶高阶非线性微分方程边值问题(1)~(3)存在一个解 $y(x,\varepsilon,\mu)$, 并成立不等式: $\underline{y}(x,\varepsilon,\mu)\leqslant y(x,\varepsilon,\mu)\leqslant\bar{y}(x,\varepsilon,\mu)$.定理 1 证毕.

2 奇异扰动问题的外部解

奇异扰动边值问题(1)~(3)的退化方程为

$$f(x,y,0,0)=0. \tag{7}$$

现构造分数阶两参数奇异扰动边值问题(1)~(3)的外部解 Y.令

$$Y(x,\varepsilon,\mu)\sim\sum_{r,s=0}^{\infty}Y_{rs}(x)\varepsilon^{\alpha r}\mu^{\alpha s}. \tag{8}$$

由假设[H1]~[H3]知,方程(7)有解 $Y_{00}(x)$, 将式(8)代入方程

$$f(x,y,\varepsilon,\mu)=\varepsilon^{n\alpha}(D_x^\alpha)^{(n)}y+\mu^\alpha D_x^\alpha y. \tag{9}$$

将式(9)按 $\varepsilon^\alpha\mu^\alpha$ 的幂展开,合并 $\varepsilon^{\alpha r}\mu^{\alpha s}$ 同次幂的系数并分别令其为 0,有

$$f_{rs}(x,Y_{00},0,0)Y_{rs}=(D_x^\alpha)^{(n)}Y_{(r-2)s}+D_x^\alpha Y_{r(s-1)}+F_{rs},$$
$$r,s=0,1,\cdots,\ r+s\neq 0,$$

其中 F_{rs},r,s 为依次已知的函数,其结构从略.由上式得

$$Y_{rs}(x)=\frac{(D_x^\alpha)^{(n)}Y_{(r-2)s}+D_x^\alpha Y_{r(s-1)}+F_{rd}}{f_{rs}(x,Y_{00},0,0)},\qquad r,s=0,1,\cdots,\ r+s\neq 0. \tag{10}$$

上面和以下出现负下标的项均设其为 0.将由式(10)得到的 $Y_{rs}(x)(r,s=0,1,\cdots)$ 代入式(8),便得到原问题的外部解 $Y(x,\varepsilon,\mu)$.但它未必满足边界条件(2)和(3),故需在边界附近构造边界层校正项.

3 第一边界层校正项

由扰动理论知,两参数奇异扰动边值问题(1)~(3)的解在 $x=b$ 附近有边界层.为此先在 x

$=b$ 附近构造第一边界层校正项 W. 设边值问题(1)~(3)的解为

$$y \sim Y(x,\varepsilon,\mu)+W(\tau,\sigma,\mu), \tag{11}$$

其中 $\sigma=\dfrac{\varepsilon}{\mu}, \tau=\dfrac{b-x}{\mu^{\alpha}}$ 为伸长变量[2-4],且设

$$W(\tau,\sigma,\mu) \sim \sum_{r,s=0}^{\infty} W_{rs}(\tau)\sigma^{\alpha r}\mu^{\alpha s}. \tag{12}$$

将式(7)、(11)和(12)代入式(1)、(3),按 $\sigma^{\alpha}\mu^{\alpha}$ 的幂展开非线性项,合并 $\sigma^{\alpha r}\mu^{\alpha s}$ 同次幂的系数,并令其为0,得到

$$\mathrm{D}_{\tau}^{\alpha}W_{00}=f(0,Y_{00}+W_{00},0,0), \tag{13}$$

$$W_{00}(0,\sigma,\mu)=h_0(0,0), \tag{14}$$

$$\mathrm{D}_{\tau}^{\alpha}W_{rs}=-\mathrm{D}_{\tau}^{n\alpha}W_{(r-2)s}+G_{rs}, \qquad r,s=0,1,\cdots,\ r+s\neq 0, \tag{15}$$

$$W_{rs}(0)=h_{rs}-Y_{rs}(b), \qquad r,s=0,1,\cdots,\ r+s\neq 0, \tag{16}$$

其中 G_{rs} 为依次已知的函数,其结构也从略.

由分数阶微分方程(13),有

$$\int_0^{\tau}[(b-t)^{-\alpha}W_{00}(t)-\Gamma(1-\alpha)f(0,Y_{00}+W_{00},0,0)]\mathrm{d}t=C_{00}, \tag{17}$$

其中 C_{00} 为任意常数.再由 Volterra 积分方程(17)和边界条件(14),可得到解 W_{00}.

由分数阶微分方程(15),有

$$\int_0^{\tau}[(b-t)^{-\alpha}(W_{rs}(t)-\mathrm{D}_{\tau}^{\alpha}W_{(r-2)s}(t))-\Gamma(1-\alpha)G_{rs}]\mathrm{d}t=C_{rs}, \tag{18}$$

其中 C_{rs} 为任意常数.再由 Volterra 积分方程(18)和边界条件(16),可依次得到解 $W_{rs}, r,s=0,1,\cdots,r+s\neq 0$.

将得到的 $W_{rs}(\tau), r,s=0,1,2,\cdots$ 代入式(12),便得到边值问题(1)~(3)的解 y 的第一边界层校正项 $W(\tau,\sigma,\mu)$.

但由式(11)得到的 $y=Y+W$ 未必满足在 $x=a$ 处的边界条件(2),故尚需在边界 $x=a$ 附近构造第二边界层校正项 Z.

4 第二边界层校正项

由奇异扰动理论,设边值问题(1)~(3)的解为

$$y \sim Y(x,\varepsilon,\mu)+W(\tau,\sigma,\mu)+Z(\xi,\sigma,\mu), \tag{19}$$

其中 $\xi=(x-a)/\sigma$ 为伸长变量,且设

$$Z(\xi,\sigma,\mu) \sim \sum_{r,s=0}^{\infty} Z_{rs}(\xi)\sigma^{\alpha r}\mu^{\alpha s}. \tag{20}$$

将式(7)、(11)、(12)、(19)和(20)代入式(1)、(3),按 $\sigma^{\alpha}\mu^{\alpha}$ 的幂展开非线性项,合并 $\sigma^{\alpha r}\mu^{\alpha s}$ 同次幂的系数,并令其为0,得到

$$(\mathrm{D}_{\xi}^{\alpha})^{(n)}Z_{00}=f(0,Y_{00}+W_{00}+Z_{00},0,0), \tag{21}$$

$$(\mathrm{D}_{\xi}^{\alpha})^{(i)}Z_{00}(0)=g_i(0,0), \qquad i=1,2,\cdots,n-1, \tag{22}$$

$$(\mathrm{D}_{\xi}^{\alpha})^{(n)}Z_{rs}=-(\mathrm{D}_{\xi}^{\alpha})^{(n)}Z_{(r-2)j}+\bar{G}_{rs}, \qquad r,s=0,1,\cdots,\ r+s\neq 0, \tag{23}$$

$$(\mathrm{D}_{\xi}^{\alpha})^{(i)}Z_{rs}(0)=g_{irs}-Y_{rs}(a)-W_{rs}(a), \quad r,s=0,1,\cdots,\ r+s\neq 0,\ i=1,2,\cdots,n-1, \tag{24}$$

其中 $$g_{irs}=\frac{1}{r!\ s!}\left[\frac{\partial^{r+s}g_i}{\partial\sigma^{\alpha r}\partial\mu^{\alpha s}}\right]_{\sigma=\mu=0},\qquad i=1,2,\cdots,n-1,$$

而 $\bar{G}_{rs}$ 为依次已知的函数,其结构也从略.

同样,由分数阶线性微分方程初值问题(21)~(24),可以依次得到解 Z_{irs},$r,s=0,1,\cdots,i=0,1,\cdots$.

将得到的 $Z_{irs}(\xi)$,$r,s=0,1,2,\cdots,i=0,1,\cdots$ 代入式(19),便得到原边值问题(1)~(3)的解 y 的第二边界层校正项 $Z(\xi,\sigma,\mu)$.

由假设 $\lim_{\mu\to0}(\varepsilon/\mu)=0$ 可知,两参数分数阶高阶非线性微分方程边值问题(1)~(3)的解的第二边界层的厚度 $\sigma=\varepsilon/\mu$ 比第一边界层的厚度 μ 更薄.构成边界邻域内的“层中层”现象.

由上述对问题解的构造,我们便得到了分数阶奇异扰动高阶非线性微分方程边值问题(1)~(3)的解的渐近展开式:

$$y\sim\sum_{r,s=0}^{\infty}[Y_{rs}\varepsilon^{\alpha r}\mu^{\alpha s}+(W_{rs}+Z_{rs})\sigma^{\alpha r}\mu^{\alpha s}],\qquad 0<\varepsilon,\ \sigma=\frac{\varepsilon}{\mu},\ \mu\ll1. \tag{25}$$

5 渐近解的一致有效性

现证上述关于小参数 ε,μ 的渐近展开式(25)的一致有效性.

定理 2 在假设[H1]~[H3]下,奇异扰动分数阶高阶非线性奇扰动微分方程边值问题(1)~(3)存在一个解 y,在 $x\in[a,b]$ 上具有关于两个小参数 ε,μ 一致有效的渐近展开式(25).

证明 首先构造辅助函数 $\bar{\alpha}(x,\sigma,\mu)$,$\bar{\beta}(x,\sigma,\mu)$:

$$\bar{\alpha}(x,\sigma,\mu)=\sum_{r=0}^{m}\sum_{s=0}^{n}[Y_{rs}\varepsilon^{\alpha r}\mu^{\alpha s}+(W_{rs}+Z_{rs})\sigma^{\alpha r}\mu^{\alpha s}]-\bar{r}\zeta,\qquad x\in[a,b], \tag{26}$$

$$\bar{\beta}(x,\sigma,\mu)=\sum_{r=0}^{m}\sum_{s=0}^{n}[Y_{rs}\varepsilon^{\alpha r}\mu^{\alpha s}+(W_{rs}+V_{rs})\sigma^{\alpha r}\mu^{\alpha s}]+\bar{r}\zeta,\qquad x\in[a,b], \tag{27}$$

其中 $\zeta=\max(\sigma^{\alpha(m+1)}\mu^{\alpha n},\sigma^{\alpha m}\mu^{\alpha(n+1)})$,$m,n$ 为任意的确定的正整数,而 $\bar{r}$ 为足够大的正常数,它将在下面选定.

显然

$$\bar{\alpha}(x,\varepsilon,\mu)\leqslant\bar{\beta}(x,\varepsilon,\mu), \tag{28}$$

$$(\mathrm{D}_x^{\alpha})^{(i)}\bar{\alpha}(a,\varepsilon,\mu)\leqslant A\leqslant(\mathrm{D}_x^{\alpha})^{(i)}\bar{\beta}(a,\varepsilon,\mu),\qquad i=1,2,\cdots,n-1, \tag{29}$$

$$(\mathrm{D}_x^{\alpha})^{(i)}\bar{\alpha}(b,\varepsilon,\mu)\leqslant B\leqslant(\mathrm{D}_x^{\alpha})^{(i)}\bar{\beta}(b,\varepsilon,\mu),\qquad i=1,2,\cdots,n-1. \tag{30}$$

现证

$$\varepsilon^{n\alpha}(\mathrm{D}_x^{\alpha})^{(n)}\bar{\alpha}+\mu^{\alpha}\mathrm{D}_x^{\alpha}\bar{\alpha}-f(x,\bar{\alpha},\varepsilon,\mu)\geqslant0,\qquad 0<x<b, \tag{31}$$

$$\varepsilon^{n\alpha}(\mathrm{D}_x^{\alpha})^{(n)}\bar{\beta}+\mu^{\alpha}\mathrm{D}_x^{\alpha}\bar{\beta}-f(x,\bar{\beta},\varepsilon,\mu)\leqslant0,\qquad 0<x<b. \tag{32}$$

事实上由假设[H1]~[H3],对于足够小的 ε,μ, 存在一个正常数 M, 使得

$$\begin{aligned}&\varepsilon^{n\alpha}(\mathrm{D}_x^{\alpha})^{(n)}\bar{\beta}+\mu^{\alpha}\mathrm{D}_x^{\alpha}\bar{\beta}-f(x,\bar{\beta},\varepsilon,\mu)=\\&\quad\varepsilon^{n\alpha}(\mathrm{D}_x^{\alpha})^{(n)}\sum_{r=0}^{m}\sum_{s=0}^{n}[Y_{rs}\varepsilon^{\alpha r}\mu^{\alpha s}+(W_{rs}+Z_{rs})\sigma^{\alpha r}\mu^{\alpha s}]+\\&\quad\mu^{\alpha}\mathrm{D}_x^{\alpha}\Big(\sum_{r=0}^{m}\sum_{s=0}^{n}[Y_{rs}\varepsilon^{\alpha r}\mu^{\alpha s}+(W_{rs}+Z_{rs})\sigma^{\alpha r}\mu^{\alpha s}]\Big)-\end{aligned}$$

$$f\Big(x,\sum_{r=0}^{m}\sum_{s=0}^{n}[Y_{rs}\varepsilon^{\alpha r}\mu^{\alpha s}+(W_{rs}+Z_{rs})\sigma^{\alpha r}\mu^{\alpha s}],\varepsilon,\mu\Big)+$$
$$f\Big(x,\sum_{r=0}^{m}\sum_{s=0}^{n}[Y_{rs}\varepsilon^{\alpha r}\mu^{\alpha s}+(W_{rs}+Z_{rs})\sigma^{\alpha r}\mu^{\alpha s}]+\bar{r}\zeta,\varepsilon,\mu\Big)\leqslant$$
$$f(x,Y_{00},0,0)+$$
$$\sum_{\substack{r=0\\ r+s\neq 0}}^{m}\sum_{s=0}^{n}[f_y(x,Y_{00},0,0)Y_{rs}-(\mathrm{D}_x^{\alpha})^{(n)}Y_{(r-2)j}-\mathrm{D}_x^{\alpha}Y_{r(s-1)}-F_{rs}]\sigma^{\alpha r}\mu^{\alpha s}+$$
$$[\mathrm{D}_\tau^{\alpha}W_{00}-f(0,Y_{00}+W_{00},0,0)]+$$
$$\sum_{\substack{r=0\\ r+s\neq 0}}^{m}\sum_{s=0}^{n}[\mathrm{D}_\tau^{\alpha}W_{rs}+\mathrm{D}_\tau^{\alpha}\mathrm{D}_\tau^{\alpha}W_{(r-2)s}-G_{rs}]\sigma^{\alpha r}\mu^{\alpha s}+$$
$$[\mathrm{D}_\xi^{\alpha}Z_{00}-f(0,Y_{00}+W_{00}+Z_{00},0,0)]+$$
$$\sum_{\substack{r=0\\ r+s\neq 0}}^{m}\sum_{s=0}^{n}[\mathrm{D}_\xi^{\alpha}Z_{rs}+\mathrm{D}_\xi^{\alpha}\mathrm{D}_\xi^{\alpha}Z_{(ir2)s}-\bar{G}_{ij}]\sigma^{\alpha r}\mu^{\alpha s}+$$
$$f_y\Big(x,\sum_{r=0}^{m}\sum_{s=0}^{n}[Y_{rs}\varepsilon^{\alpha r}\mu^{\alpha s}+(W_{rs}+Z_{rs})\sigma^{\alpha r}\mu^{\alpha s}]\Big)\bar{r}\zeta+M\zeta\leqslant$$
$$(M-\bar{r}c)\zeta.$$

选取 $\bar{r}\geqslant M/c$，这时不等式(32)成立.类似地，我们能证明不等式(31).由式(28)~(32)，$\bar{\alpha}$ 和 $\bar{\beta}$ 分别为问题(1)~(3)的上解和下解.由定理 1，奇扰动分数阶高阶非线性微分方程边值问题(1)~(3)存在一个解 $y(x,\varepsilon,\mu)$，并成立 $\bar{\alpha}(x,\varepsilon,\mu)\leqslant y(x,\varepsilon)\leqslant\bar{\beta}(x,\varepsilon,\mu)$.再由式(26)、(27)，关于两个小参数 ε,μ 一致有效的渐近展开式(25)成立.定理 2 证毕.

6 结 论

当前，对奇异摄动非线性方程定解问题，改进了很多渐近求解的解析方法.本文讨论了一类分数阶双参数高阶非线性扰动模型的渐近解，利用近代微分不等式等理论和方法，得到了所讨论的问题渐近解的解析表示式，并证明了其解的一致有效性.本文所述的内容、思路和方法清楚明了，并能由得到的渐近解表示式，继续施行微分积分等解析运算，进一步得到其他相关物理量的渐近表示式，而这是运用一般单纯的数值模拟方法的计算所不及的.因此，本文对一类分数阶双参数高阶非线性扰动模型的渐近解的研讨，具有较大的应用前景.

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Asymptotic Solution for Fractional-Order 2-Parameter High-Order Nonlinear Perturbed Models

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Abstract: A class of nonlinear fractional-order perturbed higher-order differential models was considered. Firstly, under suitable conditions, the outer solution to the original problem was obtained with the perturbation method. Then by means of the stretched variable, the composite expansion method and the theory of power series, the first and second boundary layer correction terms were constructed and the formal asymptotic expansion was obtained. Finally, with the theory of differential inequalities the asymptotic behavior of the solution to the problem was studied and the uniform validity of the asymptotic estimate expression was proved.

Key words: boundary layer; fractional-order differential model; perturbation

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