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一类分数阶非线性时滞问题的奇摄动*

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摘要: 讨论了一类奇异摄动非线性分数阶时滞问题.首先利用奇异摄动方法求出了问题的外部解.再利用伸展变量法构造了问题在边界附近的两个边界层校正项,得出了所提问题的形式渐近解.最后,在合适的假设条件下,利用微分不等式理论证明了解的一致有效性,并给出了结论及未来的研究方向.

关 键 词: 分数阶微分方程; 非线性; 时滞; 奇摄动

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引 言

奇异摄动问题的研究一直以来是一个热门问题,许多学者在很多领域做了大量的研究工作[1-6].最近几十年来,学者们把奇摄动理论和方法运用到了分数阶微分方程上.莫嘉琪[7]研究了一类奇摄动非线性分数阶微分方程 Cauchy 问题,在适当的条件下,利用奇异摄动理论和方法得到解的形式渐近展开式;Shi 和 Mo[8]又研究了一类奇摄动分数阶微分方程初值问题;林学渊等[9]和莫嘉琪等[10-11]研究了不同分数阶奇异摄动问题,得出了一些结论.所有这些分数阶问题解决了许多常规的导数不能描述和解决的实际问题,例如许多复杂的渗流现象、热传导现象以及神经元和神经网络等,这些问题的研究丰富和发展了奇异摄动理论.同时,带有小参数的时滞奇摄动也是广大学者研究和探讨的问题.Wang 等[12]利用一些奇摄动方法研究了具有边界摄动的非线性时滞反应扩散方程奇摄动问题;Mo 等[13]讨论了一类具有两参数时滞奇摄动非线性问题的冲击波解,在适当的条件下,利用匹配法和微分不等式理论,构造了原问题的冲击波奇摄动解,并讨论了它的渐近性态;Mo[14]和欧阳成[15]分别研究了一些时滞微分问题及微分差分方程在一定条件下的一致有效渐近解.本文把这两类问题进行了综合,研究了一类分数阶时滞非线性问题,利用奇摄动理论和方法构造了一类分数阶时滞微分方程的渐近解,并用微分不等式理论证明了渐近解的一致有效性.

考虑以下的非线性分数阶奇异摄动时滞问题:

$$\varepsilon^2 D_x^\alpha D_x^\alpha y + \varepsilon D_x^\alpha y = f(x, y(x-\varepsilon)), \tag{1}$$

$$y = A(\varepsilon), \qquad -\varepsilon \leqslant x \leqslant 0, \tag{2}$$

$$y = B(\varepsilon), \qquad x = 1, \tag{3}$$

这里 ε 是个很小的正常数.函数 $y(x)$ 的 α 分数阶导数 D_x^α 定义为

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$$D_x^\alpha y = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} y(t) dt,$$

其中 $\Gamma(\cdot)$ 为 Gamma 函数, α 为小于 1 的正分数.

首先做如下假设:

(H1) 函数 $f(x,y)$,$A(\varepsilon)$ 和 $B(\varepsilon)$ 关于其变量在其变化区域内为足够光滑函数.

(H2) 函数 $f(x,y(x-\varepsilon)) \leqslant 0$,且存在正的常数 δ,使得 $f_y'(x,y(x-\varepsilon)) \leqslant -\delta < 0$.

(H3) 退化问题 $f(x,y)=0$ 仅有唯一单调解 $y_0(x)$.

1　外部解

将函数 $y(x-\varepsilon)$ 展开成 ε 的幂级数形式如下:

$$y(x-\varepsilon) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n y(x)}{dx^n} \varepsilon^n, \tag{4}$$

代入式(1),有

$$\varepsilon^2 D_x^\alpha D_x^\alpha y + \varepsilon D_x^\alpha y = f\left(x, \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n y(x)}{dx^n} \varepsilon^n\right), \tag{5}$$

设时滞问题(1)的外部解形如

$$Y(x,\varepsilon) \sim \sum_{i=0}^{\infty} y_i(x) \varepsilon^i. \tag{6}$$

将式(6)代入式(5),并将右端函数按照 ε 的各阶次幂进行展开, 比较等式两边 ε 的各阶次幂系数相等,得

$$f(x,y_0) = 0, \tag{7}$$

$$D_x^\alpha y_0 = f_y'(x,y_0)(y_1 - y_0'), \tag{8}$$

$$D_x^\alpha D_x^\alpha y_0 + D_x^\alpha y_1 = f_y'(x,y_0)\left(y_2 - y_1' + \frac{y_0''}{2!}\right) + F_1, \tag{9}$$

$$D_x^\alpha D_x^\alpha y_1 + D_x^\alpha y_2 = f_y'(x,y_0)\left(y_3 - y_2' + \frac{y_1''}{2!} - \frac{y_0'''}{3!}\right) + F_2, \tag{10}$$

$$\vdots$$

$$D_x^\alpha D_x^\alpha y_{i-2} + D_x^\alpha y_{i-1} = f_y'(x,y_0)\left(y_i - y_{i-1}' + \frac{y_{i-2}''}{2!} - \frac{y_{i-3}'''}{3!} + \cdots + (-1)^i \frac{y_0^{(i)}}{i!}\right) + F_{i-1}, \tag{11}$$

$$\vdots$$

由假设(H3)及以上各式都是代数方程可依次得 $y_i(x)$, 其中

$$F_1 = \frac{f_{yy}''(y_1 - y_0')^2}{2!},$$

$$F_2 = \frac{f_{yyy}'''(y_1 - y_0')^3 + 3!\, f_{yy}''(y_1 - y_0')(y_2 - y_1' + y_0''/2!)}{3!},$$

$$\vdots$$

F_i 的表达式都可依次由前式确定,故其结构省略.将 $y_i(x)$ 代入式(6),便得到了原问题的外部解.显然,由此构造的外部解不一定满足条件(2)、(3),为此需要构造边界层校正项 Z 和 W.

2 构造边界层校正项

首先构造在 $x=0$ 附近的边界层校正项 Z, 为此设分数阶时滞问题(1)~(3)的解为

$$y(x)=Y(x,\varepsilon)+Z(\tau,\varepsilon), \tag{12}$$

其中 $\tau=x/\varepsilon$ 为伸长变量,且

$$Z(\tau,\varepsilon)\sim\sum_{i=0}^{\infty}z_i(\tau)\varepsilon^i. \tag{13}$$

将式(13)、(12)和式(6)代入式(5)和式(2)、(3),按照 ε 的各阶次幂展开式(5)右端的函数及 $A(\varepsilon)$,$B(\varepsilon)$,并比较等式两边 ε 的各阶次幂系数相等,可得

$$\mathrm{D}_\tau^\alpha\mathrm{D}_\tau^\alpha z_0+\mathrm{D}_\tau^\alpha z_0=f(0,y_0+z_0), \tag{14}$$

$$z_0(0)=A(0)-y_0(0), \tag{15}$$

$$z_0(\infty)=B(0)-y_0(1), \tag{16}$$

$$\mathrm{D}_\tau^\alpha\mathrm{D}_\tau^\alpha z_i+\mathrm{D}_\tau^\alpha z_i=f_y'(0,y_0+z_0)z_i-G_i,\qquad i=1,2,\cdots, \tag{17}$$

$$z_i(0)=\frac{1}{i!}\left[\frac{\mathrm{d}^iA}{\mathrm{d}\varepsilon^i}\right]_{\varepsilon=0}-y_i(0),\qquad i=1,2,\cdots, \tag{18}$$

$$z_i(\infty)=\frac{1}{i!}\left[\frac{\mathrm{d}^iB}{\mathrm{d}\varepsilon^i}\right]_{\varepsilon=0}-y_i(1),\qquad i=1,2,\cdots, \tag{19}$$

$$\vdots$$

而 $G_i,i=1,2,\cdots$ 是由 $y_0,y_1,\cdots,y_i$ 和 $z_0,z_1,\cdots,z_{i-1}$ 逐次确定的已知函数,其结构从略.

由分数阶微分方程(14),我们能得到以下 Volterra 积分系统:

$$\int_0^\tau[(\tau-t)^{-\alpha}+\Gamma(1-\alpha)]U_0(t)\mathrm{d}t=\int_0^\tau\Gamma(1-\alpha)f(0,y_0+z_0)\mathrm{d}t+C_1, \tag{20}$$

$$\int_0^\tau(\tau-t)^{-\alpha}z_0(t)\mathrm{d}t=\int_0^\tau\Gamma(1-\alpha)U_0(t)\mathrm{d}t+C_2, \tag{21}$$

其中 C_1,C_2 为任意常数,解 Volterra 积分系统(20)、(21)并结合条件(15)、(16)能够得到 $z_0(\tau)$.类似地,可依次求出 $z_1(\tau)$,$z_2(\tau)$,$\cdots$,将其代入式(13),可得到在 $x=0$ 附近的边界层校正项 $Z(\tau,\varepsilon)$.又由假设(H2)知,函数 $z_i(\tau)$,$i=0,1,\cdots$ 具有性质[16]:

$$z_i(\tau)=O(\exp(-k_i\tau))=O\left(\exp\left(-k_i\frac{x}{\varepsilon}\right)\right),\qquad 0<\varepsilon\ll1,\ i=0,1,\cdots, \tag{22}$$

其中 $k_{i+1}\leqslant k_i$,$i=0,1,\cdots$ 为正常数.

再构造在 $x=1$ 附近的边界层校正项 W, 为此设分数阶时滞问题(1)~(3)的解为

$$y(x)=Y(x,\varepsilon)+Z(\tau,\varepsilon)+W(\eta,\varepsilon), \tag{23}$$

其中 $\eta=(x-1)/\varepsilon$ 为伸长变量,且

$$W(\eta,\varepsilon)\sim\sum_{i=0}^{\infty}w_i(\eta)\varepsilon^i. \tag{24}$$

将式(24)、(13)、(6)和式(23)代入式(5)和式(2)、(3),按照 ε 的各阶次幂展开式(5)右端的函数及 $A(\varepsilon)$,$B(\varepsilon)$,并比较等式两边 ε 的各阶次幂系数相等,得

$$\mathrm{D}_\eta^\alpha\mathrm{D}_\eta^\alpha w_0+\mathrm{D}_\eta^\alpha w_0=f(1,y_0+z_0+w_0)-H_0, \tag{25}$$

$$w_0(1)=B(0)-y_0(1)-z_0(1), \tag{26}$$

$$w_0(\infty)=A(0)-y_0(0)-z_0(0), \tag{27}$$

$$\mathrm{D}_\eta^\alpha\mathrm{D}_\eta^\alpha w_i+\mathrm{D}_\eta^\alpha w_i=f_y'(1,y_0+z_0+w_0)w_i-H_i,\qquad i=1,2,\cdots, \tag{28}$$

$$w_i(1)=\frac{1}{i!}\left[\frac{\mathrm{d}^iB}{\mathrm{d}\varepsilon^i}\right]_{\varepsilon=0}-y_i(1)-z_i(1),\qquad i=1,2,\cdots,\tag{29}$$

$$w_i(\infty)=\frac{1}{i!}\left[\frac{\mathrm{d}^iA}{\mathrm{d}\varepsilon^i}\right]_{\varepsilon=0}-y_i(0)-z_i(0),\qquad i=1,2,\cdots,\tag{30}$$

$$\vdots$$

而 $H_i,i=0,1,\cdots$ 是由 $y_0,y_1,\cdots,y_i,z_0,z_1,\cdots,z_i$ 及 $w_0,w_1,\cdots,w_{i-1}$ 逐次确定的已知函数,其结构从略.

由分数阶微分方程(25),我们能得到以下 Volterra 积分系统:

$$\int_\eta^1[(\eta-t)^{-\alpha}+\Gamma(1-\alpha)]V_0(t)\mathrm{d}t=$$
$$\int_\eta^1\Gamma(1-\alpha)(f(1,y_0+z_0+w_0)-H_0)\mathrm{d}t+D_1,\tag{31}$$

$$\int_\eta^1(\eta-t)^{-\alpha}w_0(t)\mathrm{d}t=\int_\eta^1\Gamma(1-\alpha)V_0(t)\mathrm{d}t+D_2,\tag{32}$$

其中 D_1,D_2 为任意常数,解 Volterra 积分系统(31)、(32)并结合条件(26)、(27)能够得到 $w_0(\eta)$.类似地,可依次求出 $w_1(\eta),w_2(\eta),\cdots$,将其代入式(24),可得到在 $x=1$ 附近的边界层校正项 $W(\eta,\varepsilon)$.又由假设(H2)知,函数 $w_i(\eta),i=0,1,\cdots$ 具有性质[16]:

$$w_i(\eta)=O(\exp(-\bar{k}_i\eta))=O\left(\exp\left(-\bar{k}_i\frac{x-1}{\varepsilon}\right)\right),\qquad 0<\varepsilon\ll1,\ i=0,1,\cdots,\tag{33}$$

其中 $\bar{k}_{i+1}\leqslant\bar{k}_i,i=0,1,\cdots$ 为正常数.

由式(6)、(13)、(24)和(23)及以上讨论所确定的 $y_i(x),z_i(\tau)$ 及 $w_i(\eta),i=0,1,\cdots$,我们得到分数阶非线性时滞奇摄动问题(1)~(3)的形式渐近展开式为

$$y(x)\sim\sum_{i=0}^{\infty}[y_i(x)+z_i(\tau)+w_i(\eta)]\varepsilon^i,\qquad 0<\varepsilon\ll1.\tag{34}$$

3　渐近展开式的一致有效性

下面证明以上的展开式为关于 ε 一致有效的展开式.首先给出如下定义.

定义 1　设存在两个光滑函数 $\bar{y}$ 和 $\underline{y},\bar{y}\geqslant\underline{y}$,且分别满足

$$\varepsilon^2\mathrm{D}_x^\alpha\mathrm{D}_x^\alpha\bar{y}+\varepsilon\mathrm{D}_x^\alpha\bar{y}-f(x,\bar{y}(x-\varepsilon))\leqslant0,$$
$$0<x<1,\ \bar{y}(0,\varepsilon)\geqslant A(\varepsilon),\ \bar{y}(1,\varepsilon)\geqslant B(\varepsilon),\tag{35}$$

$$\varepsilon^2\mathrm{D}_x^\alpha\mathrm{D}_x^\alpha\underline{y}+\varepsilon\mathrm{D}_x^\alpha\underline{y}-f(x,\underline{y}(x-\varepsilon))\geqslant0,$$
$$0<x<1,\ \underline{y}(0,\varepsilon)\leqslant A(\varepsilon),\ \underline{y}(1,\varepsilon)\leqslant B(\varepsilon),\tag{36}$$

则分别称 $\bar{y}$ 和 $\underline{y}$ 为问题(1)~(3)的上解和下解.

定理 1　在假设(H1)~(H3)的条件下,$\bar{y}(x,\varepsilon)$ 和 $\underline{y}(x,\varepsilon)$ 分别为分数阶奇摄动边值问题的上解和下解,则问题(1)~(3)存在一个解 $y(x,\varepsilon)$,且有关系式:

$$\underline{y}(x,\varepsilon)\leqslant y(x,\varepsilon)\leqslant\bar{y}(x,\varepsilon).$$

证明　利用数学归纳法证明定理,先按以下迭代关系式构造函数序列:

$$\varepsilon^2\mathrm{D}_x^\alpha\mathrm{D}_x^\alpha y_{n+1}+\varepsilon\mathrm{D}_x^\alpha y_{n+1}=f(x,y_n(x-\varepsilon)),\qquad n=0,1,\cdots,\tag{37}$$

$$y_{n+1}(0,\varepsilon)=A(\varepsilon),\ y_{n+1}(1,\varepsilon)=B(\varepsilon),\qquad n=0,1,\cdots,\tag{38}$$

分别以 $\bar{y}_0=\bar{y}$ 和 $\underline{y}_0=\underline{y}$ 为式(37)、(38)的初始迭代函数，可分别依次构造 $\bar{y}_n,\underline{y}_n,n=0,1,\cdots$.于是可得到两个函数列 $\{\bar{y}_n\}$ 和 $\{\underline{y}_n\}$，现讨论它们的收敛性态.

设 $\bar{u}_i=\bar{y}_i-\bar{y}_{i+1},i=0,1,\cdots$.因此，当 $i=0$ 时，$\bar{u}_0=\bar{y}_0-\bar{y}_1$，由假设(H2)，则有

$$\begin{aligned}&\varepsilon^2 D_x^\alpha D_x^\alpha \bar{u}_0+\varepsilon D_x^\alpha \bar{u}_0=\varepsilon^2 D_x^\alpha D_x^\alpha \bar{y}_0+\varepsilon D_x^\alpha \bar{y}_0-\varepsilon^2 D_x^\alpha D_x^\alpha \bar{y}_1-\varepsilon D_x^\alpha \bar{y}_1\leqslant\\&\qquad f(x,\bar{y}_0(x-\varepsilon))-f(x,\bar{y}_0(x-\varepsilon))=0,\\&\bar{u}_0|_{x=0}=\bar{y}_0|_{x=0}-\bar{y}_1|_{x=0}=0,\ \bar{u}_0|_{x=1}=\bar{y}_0|_{x=1}-\bar{y}_1|_{x=1}=0.\end{aligned}$$

于是由极值原理[17]得 $\bar{u}_0\geqslant 0$，即 $\bar{y}_0\geqslant\bar{y}_1$.

假设当 $i=n-1$ 时，有 $\bar{y}_{n-1}\geqslant\bar{y}_n$，则

$$\begin{aligned}&\varepsilon^2 D_x^\alpha D_x^\alpha \bar{u}_n+\varepsilon D_x^\alpha \bar{u}_n=\varepsilon^2 D_x^\alpha D_x^\alpha \bar{y}_n+\varepsilon D_x^\alpha \bar{y}_n-\varepsilon^2 D_x^\alpha D_x^\alpha \bar{y}_{n+1}-\varepsilon D_x^\alpha \bar{y}_{n+1}=\\&\qquad f(x,\bar{y}_{n-1}(x-\varepsilon))-f(x,\bar{y}_n(x-\varepsilon))\leqslant\\&\qquad -c(\bar{y}_{n-1}(x-\varepsilon)-\bar{y}_n(x-\varepsilon))\leqslant 0,\\&\bar{u}_n|_{x=0}=\bar{y}_n|_{x=0}-\bar{y}_{n+1}|_{x=0}=0,\ \bar{u}_n|_{x=1}=\bar{y}_n|_{x=1}-\bar{y}_{n+1}|_{x=1}=0.\end{aligned}$$

于是由极值原理得 $\bar{u}_n\geqslant 0$，即 $\bar{y}_n\geqslant\bar{y}_{n+1}$.因此，由数学归纳法知

$$\bar{y}=\bar{y}_0\geqslant\bar{y}_1\geqslant\cdots\geqslant\bar{y}_n\geqslant\bar{y}_{n+1}\geqslant\cdots.$$

同样的方法，类似可得

$$\underline{y}=\underline{y}_0\leqslant\underline{y}_1\leqslant\cdots\leqslant\underline{y}_n\leqslant\underline{y}_{n+1}\leqslant\cdots,$$

以及

$$\bar{y}_n\geqslant\underline{y}_n,\qquad n=0,1,\cdots.$$

通过以上讨论结合 Arzela-Ascoli 定理，分数阶时滞奇摄动边值问题(1)~(3)有一个解 $y(x,\varepsilon)$，使得

$$\underline{y}(x,\varepsilon)\leqslant y(x,\varepsilon)\leqslant\bar{y}(x,\varepsilon).$$

定理 1 证毕.

定理 2　在假设(H1)~(H3)下，分数阶时滞奇摄动边值问题(1)~(3)有一个解 $y(x,\varepsilon)$，且具有如下关于 ε 的一致有效的渐近展开式：

$$y(x)=\sum_{i=0}^{n}\left[y_i(x)+z_i\left(\frac{x}{\varepsilon}\right)+w_i\left(\frac{x-1}{\varepsilon}\right)\right]\varepsilon^i+O(\varepsilon^{n+1}),\quad 0<\varepsilon\ll 1,\ 0\leqslant x\leqslant 1.\tag{39}$$

证明　首先构造两个辅助函数 $\alpha(x,\varepsilon),\beta(x,\varepsilon)$，定义如下：

$$\alpha(x,\varepsilon)=\sum_{i=0}^{n}\left[y_i(x)+z_i\left(\frac{x}{\varepsilon}\right)+w_i\left(\frac{x-1}{\varepsilon}\right)\right]\varepsilon^i-r\varepsilon^{n+1},\tag{40}$$

$$\beta(x,\varepsilon)=\sum_{i=0}^{n}\left[y_i(x)+z_i\left(\frac{x}{\varepsilon}\right)+w_i\left(\frac{x-1}{\varepsilon}\right)\right]\varepsilon^i+r\varepsilon^{n+1},\tag{41}$$

其中 r 为适当大的正常数，将在随后的证明中给出.显然

$$\alpha(x,\varepsilon)\leqslant\beta(x,\varepsilon),\tag{42}$$

且对于足够小的 ε，存在一个正常数 δ_1，使得

$$\begin{aligned}\beta(0,\varepsilon)&=\sum_{i=0}^{n}\left[y_i(x)+z_i\left(\frac{x}{\varepsilon}\right)+w_i\left(\frac{x-1}{\varepsilon}\right)\right]_{x=0}\varepsilon^i+r\varepsilon^{n+1}\geqslant\\&\quad A(\varepsilon)-\delta_1\varepsilon^{n+1}+r\varepsilon^{n+1}=A(\varepsilon)+(r-\delta_1)\varepsilon^{n+1}.\end{aligned}$$

故选取 $r > \delta_1$，就可得到

$$\beta(0,\varepsilon) \geqslant A(\varepsilon). \tag{43}$$

同理，类似可得

$$\alpha(0,\varepsilon) \leqslant A(\varepsilon),\ \alpha(1,\varepsilon) \leqslant B(\varepsilon) \leqslant \beta(1,\varepsilon). \tag{44}$$

下面证明

$$\varepsilon^2 D_x^\alpha D_x^\alpha \beta + \varepsilon D_x^\alpha \beta - f(x,\beta(x-\varepsilon)) \leqslant 0, \tag{45}$$

$$\varepsilon^2 D_x^\alpha D_x^\alpha \alpha + \varepsilon D_x^\alpha \alpha - f(x,\alpha(x-\varepsilon)) \geqslant 0. \tag{46}$$

仅证明式(45)，类似的方法证明式(46)。事实上，由假设和式(22)、(33)，存在一个正常数 δ_2，使得

$$\begin{aligned}
&\varepsilon^2 D_x^\alpha D_x^\alpha \beta + \varepsilon D_x^\alpha \beta - f(x,\beta(x-\varepsilon)) = \\
&\quad \varepsilon^2 D_x^\alpha D_x^\alpha \left(\sum_{i=0}^n \left[y_i(x) + z_i\left(\frac{x}{\varepsilon}\right) + w_i\left(\frac{x-1}{\varepsilon}\right) \right] \varepsilon^i \right) + \\
&\quad \varepsilon D_x^\alpha \left(\sum_{i=0}^n \left[y_i(x) + z_i\left(\frac{x}{\varepsilon}\right) + w_i\left(\frac{x-1}{\varepsilon}\right) \right] \varepsilon^i \right) - \\
&\quad f\left(x, \sum_{i=0}^n \left[y_i(x-\varepsilon) + z_i\left(\frac{x-\varepsilon}{\varepsilon}\right) + w_i\left(\frac{x-\varepsilon-1}{\varepsilon}\right) \right] \varepsilon^i \right) + \\
&\quad f\left(x, \sum_{i=0}^n \left[y_i(x-\varepsilon) + z_i\left(\frac{x-\varepsilon}{\varepsilon}\right) + w_i\left(\frac{x-\varepsilon-1}{\varepsilon}\right) \right] \varepsilon^i \right) - \\
&\quad f\left(x, \sum_{i=0}^n \left[y_i(x-\varepsilon) + z_i\left(\frac{x-\varepsilon}{\varepsilon}\right) + w_i\left(\frac{x-\varepsilon-1}{\varepsilon}\right) \right] \varepsilon^i + r\varepsilon^{n+1} \right) \leqslant \\
&\quad f(x,y_0) + [D_x^\alpha y_0 - f_y'(x,y_0)(y_1 - y_0')] + \\
&\quad \sum_{i=2}^n \left[D_x^\alpha D_x^\alpha y_{i-2} + D_x^\alpha y_{i-1} - f_y'(x,y_0)\left(y_i - y_{i-1}' + \frac{y_{i-2}''}{2!} - \frac{y_{i-3}'''}{3!} + \cdots + \right.\right. \\
&\quad \left.\left. (-1)^i \frac{y_0^{(i)}}{1!} \right) - F_{i-1} \right] \varepsilon^i + [D_\tau^\alpha D_\tau^\alpha z_0 + D_\tau^\alpha z_0 - f(0,y_0+z_0)] + \\
&\quad \sum_{i=1}^n [D_\tau^\alpha D_\tau^\alpha z_i + D_\tau^\alpha z_i - f_y'(0,y_0+z_0)z_i + G_i]\varepsilon^i + \\
&\quad [D_\eta^\alpha D_\eta^\alpha w_0 + D_\eta^\alpha w_0 - f(1,y_0+z_0+w_0) + H_0] + \\
&\quad \sum_{i=1}^n [D_\eta^\alpha D_\eta^\alpha w_i + D_\eta^\alpha w_i - f_y'(1,y_0+z_0+w_0)w_i + H_i]\varepsilon^i + \\
&\quad \left(f\left(x, \sum_{i=0}^n \left[y_i(x-\varepsilon) + z_i\left(\frac{x-\varepsilon}{\varepsilon}\right) + w_i\left(\frac{x-\varepsilon-1}{\varepsilon}\right) \right] \varepsilon^i \right) - \right. \\
&\quad \left. f\left(x, \sum_{i=0}^n \left[y_i(x-\varepsilon) + z_i\left(\frac{x-\varepsilon}{\varepsilon}\right) + w_i\left(\frac{x-\varepsilon-1}{\varepsilon}\right) \right] \varepsilon^i + r\varepsilon^{n+1} \right) \right) + \delta_2\varepsilon^{n+1} \leqslant \\
&\quad -cr\varepsilon^{n+1} + \delta_2\varepsilon^{n+1} = (\delta_2 - cr)\varepsilon^{n+1} \leqslant 0.
\end{aligned}$$

上式成立只需要选择足够的 $r > \delta_2/c$ 即可，所以式(45)成立，同理可证式(46)成立。故函数 $\beta(x,\varepsilon)$ 和 $\alpha(x,\varepsilon)$ 分别为问题(1)~(3)的上解和下解。由定理1知，分数阶非线性时滞奇摄动边值问题(1)~(3)有一个解 $y(x,\varepsilon)$，且满足关系式：

$$\alpha(x,\varepsilon) \leqslant y(x,\varepsilon) \leqslant \beta(x,\varepsilon).$$

再由式(40)、(41)得式(39)成立，定理2证毕。

4 结　论

对分数阶时滞奇摄动边值问题(1)~(3),在假设(H1)~(H3)下,构造了问题的外部解和边界层解,得出了如下的一致有效的形式渐近解:

$$y(x) = \sum_{i=0}^{n}\left[y_i(x) + z_i\left(\frac{x}{\varepsilon}\right) + w_i\left(\frac{x-1}{\varepsilon}\right)\right]\varepsilon^i + O(\varepsilon^{n+1}),$$

$$0 < \varepsilon \ll 1,\ 0 \leqslant x \leqslant 1.$$

本文讨论的分数阶时滞奇摄动问题,其时滞变量和小参数都是同一个 ε.笔者可以继续推广,研究时滞变量和小参数不同时的问题,也即双参数的分数阶时滞奇摄动问题,这将是笔者今后努力研究的方向.

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A Class of Fractional Nonlinear Singularly Perturbed Problems With Time Delays

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Abstract: A class of fractional nonlinear singularly perturbed problems with time delays were considered. Firstly, the outer solution was constructed by means of the singular perturbation method. Then, a stretched variable was introduced to obtain 2 boundary layer correction items for the solution, and the asymptotic analytic expansion solution to the problem was also acquired. Finally, under suitable conditions, the theory of differential inequalities was applied to prove the uniformly valid asymptotic expansion of the solution to the original problem, and the conclusion with the future research directions was given.

Key words: fractional differential equation; nonlinear; time delay; singularly perturbed

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