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海洋动力学中二维黏性原始方程组解对热源的收敛性*

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摘要： 考虑了在一个柱形区域上的海洋动力学中二维黏性方程组解的收敛性.在此模型中存在一个关键的参数就是热源,众多周知,它的存在可能会使流体内层之间出现共振从而导致不稳定.因此,通过推导方程组的先验界,得到了方程组的解对热源自身的收敛性.

关 键 词： 海洋原始方程组； 热源； 收敛性； 结构稳定性

中图分类号： O178　　**文献标志码：** A　　DOI：10.21656/1000-0887.400176

引　　言

为了建立以数学物理方法为基础的数值天气预报,文献[1]首先引入了大气原始方程组和海洋原始方程组.空间科学技术(雷达)、气象卫星和计算机技术等一大批近代科学技术的日益成熟,为人们利用大气原始方程组和海洋原始方程组来预报天气和气候提供了强力的技术保证,数值天气预报迎来蓬勃发展的时代.更多关于大气、海洋原始方程组的发展介绍可以参看文献[2].在对基于大气、海洋原始方程组的数值天气预报模式进行研究时,人们主要关心的是这些方程组在数学上是否具有内在的逻辑统一性,即适定性.在这方面的研究已经持续了很长一段时间,出现了大量的成果[3-14].

与上述文献不同,本文研究海洋动力学中二维黏性方程组解对热源的收敛性, 即研究当热源趋近于零时对方程组的解带来的影响.因为在建立数学模型的过程中不可避免地会出现一些微小的误差,我们需要知道这些误差会不会引起方程组解的巨大变化.用数学分析的方法来研究方程组的连续依赖性或收敛性是非常具有实际意义的,并且这种性质已经赢得了一个名字——结构稳定性.结构稳定性的概念最先由 Hirsch 和 Smale[15]提出,有关结构稳定性的本质可参见文献[16].在过去的几十年中,很多文献都在研究各种类型的偏微分方程组的连续依赖性或收敛性,他们的研究主要集中在 Brinkman、Darcy、Forchheimer 方程组和 Navier-Stokes 方程组[17-23].据笔者所知,目前几乎还没有文章关注海洋动力学中二维黏性原始方程组的连续依赖性或收敛性,由于我们研究的模型是高度非线性的,因此本文的分析也是非平凡的,并可为

* **收稿日期：** 2019-05-30；**修订日期：** 2019-07-22

基金项目： 广东省普通高校特色创新类项目(2018KTSCX332);广东省自然科学基金(2017A030313037)

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引用格式： 李远飞. 海洋动力学中二维黏性原始方程组解对热源的收敛性[J]. 应用数学和力学, 2020, **41**(3)：339-352.

其他类型的原始方程组的研究提供借鉴.

二维黏性海洋原始方程组主要由质量、动量、能量守恒方程和盐度守恒方程组成,可以表示为

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu_1 \Delta u + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z} - fv + \dfrac{\partial p}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} - \mu_2 \Delta v + u\dfrac{\partial v}{\partial x} + w\dfrac{\partial v}{\partial z} + fu = 0, \\ \dfrac{\partial p}{\partial z} + \rho g = 0, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial T}{\partial t} - \mu_3 \Delta T + u\dfrac{\partial T}{\partial x} + w\dfrac{\partial T}{\partial z} = Q(x,t), \\ \rho = \rho_0(1 - \beta_T(T - T_0)), \end{cases} \quad \text{in } \Omega, \tag{1}$$

其中 $\Omega = (0,h_1) \times (-h_2,0)$,$h_1,h_2$ 是大于零的常数;未知函数(u,v),w,ρ,p,T 分别表示水平速度场、垂直速度、密度、压强和温度;f 是地球自转的函数,这里取常数;$\mu_i > 0(i=1,2,3)$ 是黏度系数;ρ_0,T_0 是密度和温度的参考值;β_T 是膨胀系数(常数),$\Delta = \partial_x^2 + \partial_z^2$;$Q$ 是给定热源函数.海洋原始方程组(1)中还应包括盐度方程,但由于盐度方程和热量方程类似,并不会为本文带来额外的难度,故方程组(1)中忽略了盐度方程.

区域 Ω 的边界记为 $\partial\Omega$ 并分为 3 个部分:

$$\Gamma_0 = \{(x,z) \in \bar{\Omega}: 0 < x < h_1,\ z = 0\},$$

$$\Gamma_{-h_2} = \{(x,z) \in \bar{\Omega}: 0 < x < h_1,\ z = -h_2\},$$

$$\Gamma_s = \{(x,z) \in \bar{\Omega}: x = 0 \text{ or } x = h_1,\ -h_2 \leqslant z \leqslant 0\}.$$

于是系统(1)的边界条件可以写为

$$\begin{cases} \dfrac{\partial u}{\partial z} = 0,\ \dfrac{\partial v}{\partial z} = 0,\ w = 0,\ \dfrac{\partial T}{\partial z} = -\beta T, & \text{on } \Gamma_0, \\ u = v = w = 0,\ \dfrac{\partial T}{\partial z} = 0, & \text{on } \Gamma_{-h_2}, \\ u = v = 0,\ \dfrac{\partial T}{\partial x} = 0, & \text{on } \Gamma_s, \end{cases} \tag{2}$$

其中 β 是一个大于零的常数.此外,方程组的初始条件为

$$u(x,z,0) = u_0(x,z),\ v(x,z,0) = v_0(x,z),\ T(x,z,0) = T_0(x,z), \quad \text{in } \Omega, \tag{3}$$

其中 u_0,v_0,T_0 是给定的函数.

本文的结构如下:第 1 节给出了一些准备工作,并列举或证明一些本文常用的 Soblev 不等式;第 2 节受文献[24-26]的启发,推导了依赖于方程组的系数、初边值条件以及区域的几何性质的严格先验界;第 3 节推导了方程组对热源的收敛性;最后,第 4 节是全文的总结.

1 准备工作

因为 $w|_{z=-h_2} = 0$,对式(1) 的第四个方程从 $-h_2$ 到 z 积分,可得

$$w(x,z,t)=w(x,-h_2,t)-\int_{-h_2}^{z}\frac{\partial}{\partial x}u(x,\zeta,t)\mathrm{d}\zeta=-\frac{\partial}{\partial x}\int_{-h_2}^{z}u(x,\zeta,t)\mathrm{d}\zeta. \tag{4}$$

又因为 $w|_{z=0}=0$, 则

$$\int_{-h_2}^{0}\frac{\partial}{\partial x}u(x,\zeta,t)\mathrm{d}\zeta=\frac{\partial}{\partial x}\int_{-h_2}^{0}u(x,\zeta,t)\mathrm{d}\zeta=0. \tag{5}$$

所以当 $0\leqslant x\leqslant h_1$ 时,$\int_{-h_2}^{0}u(x,\zeta,t)\mathrm{d}\zeta$ 是一个常数.再注意到式(2)中的第三个条件,有

$$\int_{-h_2}^{0}u(x,\zeta,t)\mathrm{d}\zeta=0,\qquad \forall\, 0\leqslant x\leqslant h_1.$$

对式(1)的第三个方程从 z 到 0 积分并利用式(1)的第六个方程以及边界条件(2),有

$$\frac{\partial}{\partial x}p(x,z,t)=\frac{\partial}{\partial x}p_s-\mu\int_{z}^{0}\frac{\partial}{\partial x}T(x,\zeta,t)\mathrm{d}\zeta, \tag{6}$$

其中 $p_s=p(x,0,t)$ 表示海洋表面的压强, $\mu=\rho_0\beta_T$.把式(4)和(6)代入到式(1)~(3)中,不失一般性,假设 $\mu=\mu_i=1(i=1,2,3)$, 该问题可以写为

$$\begin{cases}\dfrac{\partial u}{\partial t}-\Delta u+u\dfrac{\partial u}{\partial x}-\left(\displaystyle\int_{-h_2}^{z}\frac{\partial}{\partial x}u(x,\zeta,t)\mathrm{d}\zeta\right)\dfrac{\partial u}{\partial z}-fv+\dfrac{\partial p_s}{\partial x}-\\ \qquad\left(\displaystyle\int_{z}^{0}\frac{\partial}{\partial x}T(x,\zeta,t)\mathrm{d}\zeta\right)=0,\\ \dfrac{\partial v}{\partial t}-\Delta v+u\dfrac{\partial v}{\partial x}-\left(\displaystyle\int_{-h_2}^{z}\frac{\partial}{\partial x}u(x,\zeta,t)\mathrm{d}\zeta\right)\dfrac{\partial v}{\partial z}+fu=0,\\ \dfrac{\partial T}{\partial t}-\Delta T+u\dfrac{\partial T}{\partial x}-\left(\displaystyle\int_{-h_2}^{z}\frac{\partial}{\partial x}u(x,\zeta,t)\mathrm{d}\zeta\right)\dfrac{\partial T}{\partial z}=Q.\end{cases} \tag{7}$$

边界条件为

$$\begin{cases}\dfrac{\partial u}{\partial z}\Big|_{z=0}=\dfrac{\partial v}{\partial z}\Big|_{z=0}=0,\ u|_{z=-h_2}=v|_{z=-h_2}=0,\ (u,v)|_{\Gamma_s}=0,\\ \dfrac{\partial T}{\partial z}\Big|_{z=0}=-\beta T,\ \dfrac{\partial T}{\partial z}\Big|_{z=-h_2}=\dfrac{\partial T}{\partial x}\Big|_{\Gamma_s}=0.\end{cases} \tag{8}$$

初始条件为

$$(u,v,T)|_{t=0}=(u_0,v_0,T_0). \tag{9}$$

接下来我们给出一些本文常用的微分不等式.

引理 1[27-28] 若 $\omega(x)\in C^1(0,h)$ 并且 $\omega(0)=\omega(h)=0$, 则

$$\int_0^h\omega^2\mathrm{d}x\leqslant\frac{h^2}{\pi^2}\int_0^h\left(\frac{\partial\omega}{\partial x}\right)^2\mathrm{d}x. \tag{10}$$

引理 2 若 $\omega(x,z,t)$ 是区域 $\Omega=(0,h_1)\times(-h_2,0)$ 中充分光滑的函数,且 $\omega(0,z,t)=\omega(h_1,z,t)=0$, 则

$$\left(\int_0^t\int_\Omega\omega^4\mathrm{d}A\mathrm{d}\eta\right)^{1/2}\leqslant C\left[\left(\int_0^t\int_\Omega\omega^2\mathrm{d}A\mathrm{d}\eta\right)^{1/2}\left(\int_0^t\int_\Omega|\nabla\omega|^2\mathrm{d}A\mathrm{d}\eta\right)^{1/2}+\left(\int_0^t\int_\Omega\omega^2\mathrm{d}A\mathrm{d}\eta\right)^{1/4}\left(\int_0^t\int_\Omega|\nabla\omega|^2\mathrm{d}A\mathrm{d}\eta\right)^{3/4}\right],$$

即

$$\left(\int_0^t\int_\Omega\omega^4\mathrm{d}A\mathrm{d}\eta\right)^{1/2}\leqslant C\left[\int_0^t\int_\Omega\omega^2\mathrm{d}A\mathrm{d}\eta+\delta\int_0^t\int_\Omega|\nabla\omega|^2\mathrm{d}A\mathrm{d}\eta\right],$$

其中 $\nabla=(\partial_x,\partial_z)$,$C$ 是一个大于零的常数,δ 是一个大于零的任意常数.

证明 应用 Hölder 不等式,可得

$$\int_0^t \|\omega\|_4^4 \mathrm{d}\eta \leqslant \int_{-h_2}^0 \left(\int_0^t\int_0^{h_1}\omega^6 \mathrm{d}x\mathrm{d}\eta\right)^{1/2}\left(\int_0^t\int_0^{h_1}\omega^2 \mathrm{d}x\mathrm{d}\eta\right)^{1/2}\mathrm{d}z. \tag{11}$$

因为

$$\omega(0,z,t)=\omega(h_1,z,t)=0,$$

$$\omega^3(x,z,t)=3\int_0^x \omega^2(\xi,z,t)\frac{\partial\omega(\xi,z,t)}{\partial\xi}\mathrm{d}\xi=-3\int_x^{h_1}\omega^2(\xi,z,t)\frac{\partial\omega(\xi,z,t)}{\partial\xi}\mathrm{d}\xi,$$

所以

$$|\omega|^3 \leqslant \frac{3}{2}\int_0^{h_1}\omega^2(x,z,t)\left|\frac{\partial\omega(x,z,t)}{\partial x}\right|\mathrm{d}x.$$

于是

$$\left(\int_0^t\int_0^{h_1}\omega^6 \mathrm{d}x\mathrm{d}\eta\right)^{1/2} \leqslant \frac{3}{2}\left(\int_0^t\int_0^{h_1}\omega^2\left|\frac{\partial\omega}{\partial x}\right|\mathrm{d}x\mathrm{d}\eta\right). \tag{12}$$

把式(12)代入到式(11),可得

$$\int_0^t\int_\Omega \omega^4 \mathrm{d}A\mathrm{d}\eta \leqslant \frac{3}{2}\int_{-h_2}^0\left(\int_0^t\int_0^{h_1}\omega^2\left|\frac{\partial\omega}{\partial x}\right|\mathrm{d}x\mathrm{d}\eta\right)\left(\int_0^t\int_0^{h_1}\omega^2 \mathrm{d}x\mathrm{d}\eta\right)^{1/2}\mathrm{d}z \leqslant$$

$$\frac{3}{2}\max_{-h_2\leqslant z\leqslant 0}\left[\left(\int_0^t\int_0^{h_1}\omega^2\mathrm{d}x\mathrm{d}\eta\right)^{1/2}\right]\int_0^t\int_\Omega \omega^2\left|\frac{\partial\omega}{\partial x}\right|\mathrm{d}A\mathrm{d}\eta. \tag{13}$$

另一方面,有

$$\omega^2(x,z,t)=2\int_{-h_2}^z \omega(x,\zeta,t)\frac{\partial\omega(x,\zeta,t)}{\partial\zeta}\mathrm{d}\zeta+\omega^2(x,-h_2,t)=$$

$$-2\int_z^0 \omega(x,\zeta,t)\frac{\partial\omega(x,\zeta,t)}{\partial\zeta}\mathrm{d}\zeta+\omega^2(x,0,t), \tag{14}$$

则

$$\omega^2 \leqslant \int_{-h_2}^0 |\omega|\left|\frac{\partial\omega}{\partial z}\right|\mathrm{d}z+\frac{1}{2}[\omega^2(x,0,t)+\omega^2(x,-h_2,t)]. \tag{15}$$

为了控制式(15)的最后一项,定义一个新函数 $f(x_2)$,满足

$$f(0)>0,\ f(-h_2)<0,\ |f'(z)|\leqslant m_1,\ |f(z)|\leqslant m_2,\qquad -h_2\leqslant z\leqslant 0, \tag{16}$$

其中 m_1,m_2 是大于零的常数.例如,$f(z)=(m_1/2)(z+h_2/2)$,$m_1h_2<4m_2$ 满足式(16)中的所有条件.再利用散度定理,可得

$$\min\{f(0),-f(-h_2)\}[\omega^2(x,0,t)+\omega^2(x,-h_2,t)] \leqslant$$

$$f(0)\omega^2(x,0,t)-f(-h_2)\omega^2(x,-h_2,t)=$$

$$\int_{-h_2}^0 \frac{\partial}{\partial z}(f\omega^2)\mathrm{d}z=\int_{-h_2}^0 f'(z)\omega^2\mathrm{d}z+2\int_{-h_2}^0 f\omega\frac{\partial\omega}{\partial z}\mathrm{d}z \leqslant$$

$$m_1\int_{-h_2}^0 \omega^2\mathrm{d}z+2m_2\int_{-h_2}^0|\omega|\left|\frac{\partial\omega}{\partial z}\right|\mathrm{d}z. \tag{17}$$

把式(17)代入到式(15),有

$$\omega^2 \leqslant m_3\int_{-h_2}^0 \omega^2\mathrm{d}z+m_4\int_{-h_2}^0|\omega|\left|\frac{\partial\omega}{\partial z}\right|\mathrm{d}z, \tag{18}$$

其中

$$m_3 = \frac{m_1}{2\min\{f(0), -f(-h_2)\}}, \ m_4 = 1 + \frac{m_2}{\min\{f(0), -f(-h_2)\}}.$$

所以

$$\max_{-h_2 \leqslant z \leqslant 0}\left(\int_0^t\int_0^{h_1}\omega^2 \mathrm{d}x\mathrm{d}\eta\right)^{1/2} \leqslant \left(m_3\int_0^t\int_\Omega \omega^2 \mathrm{d}A\mathrm{d}\eta + m_4\int_0^t\int_\Omega |\omega| \left|\frac{\partial\omega}{\partial z}\right| \mathrm{d}A\mathrm{d}\eta\right)^{1/2}. \tag{19}$$

结合式(13)和(19),并利用 Hölder 不等式,有

$$\int_0^t\int_\Omega \omega^4 \mathrm{d}A\mathrm{d}\eta \leqslant \frac{3}{2}\left[m_3\int_0^t\int_\Omega \omega^2 \mathrm{d}A\mathrm{d}\eta + \right.$$
$$\left. m_4\left(\int_0^t\int_\Omega \omega^2 \mathrm{d}A\mathrm{d}\eta\right)^{1/2}\left(\int_0^t\int_\Omega \left|\frac{\partial\omega}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta\right)^{1/2}\right]^{1/2} \times$$
$$\left(\int_0^t\int_\Omega \omega^4 \mathrm{d}A\mathrm{d}\eta\right)^{1/2}\left(\int_0^t\int_\Omega \left|\frac{\partial\omega}{\partial x}\right|^2 \mathrm{d}A\mathrm{d}\eta\right)^{1/2}.$$

取 $\delta = 1$, 对上式简化之后,可得

$$\left(\int_0^t\int_\Omega \omega^4 \mathrm{d}A\mathrm{d}\eta\right)^{1/2} \leqslant C\left[\left(\int_0^t\int_\Omega \omega^2 \mathrm{d}A\mathrm{d}\eta\right)^{1/2}\left(\int_0^t\int_\Omega |\nabla\omega|^2 \mathrm{d}A\mathrm{d}\eta\right)^{1/2} + \right.$$
$$\left.\left(\int_0^t\int_\Omega \omega^2 \mathrm{d}A\mathrm{d}\eta\right)^{1/4}\left(\int_0^t\int_\Omega |\nabla\omega|^2 \mathrm{d}A\mathrm{d}\eta\right)^{3/4}\right]. \tag{20}$$

应用分部积分,我们易得以下引理.

引理 3 若

$$\omega_1(x,z,t)|_{\Gamma_s} = 0, \ \frac{\partial\omega_1}{\partial z}\bigg|_{z=0} = \frac{\partial\omega_1}{\partial z}\bigg|_{z=-h_2} = 0, \ \int_{-h_2}^0 \frac{\partial}{\partial x}\omega_2(x,\zeta,t)\mathrm{d}\zeta = 0.$$

则

$$\int_\Omega\left[\omega_2\frac{\partial\omega_1}{\partial x} - \left(\int_{-h_2}^z \frac{\partial}{\partial x}\omega_2(x,\zeta,t)\mathrm{d}\zeta\right)\frac{\partial\omega_1}{\partial z}\right]\omega_1 \mathrm{d}A = 0,$$

$$\int_\Omega \frac{\partial g(z,t)}{\partial x}\omega_2 \mathrm{d}A = 0,$$

其中 $g(z,t)$ 是$[-h_2,0]$ 上不依赖于 x 的连续函数.

2 先验界

引理 4 若 $T_0, \ Q \in L_\infty(\Omega)$,则式(7) 第三个方程的解 T 满足

$$\sup_\Omega |T| \leqslant T_m, \tag{21}$$

其中 $T_m = \sup_\Omega\{\|Q\|_\infty, \ \|T_0\|_\infty\}$.

证明 在方程(7)第三个方程的两边乘以 T^{p-1},并在 Ω 上积分可得

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega T^p \mathrm{d}A + \frac{p-1}{p^2}\int_\Omega |\nabla T^{p/2}|^2 \mathrm{d}A = -\beta\int_0^1 T^p \mathrm{d}x + \int_\Omega QT^{p-1}\mathrm{d}A -$$
$$\int_\Omega\left[u\frac{\partial T}{\partial x} - \left(\int_{-h_2}^z \frac{\partial}{\partial x}u(x,\zeta,t)\mathrm{d}\zeta\right)\frac{\partial T}{\partial z}\right]T^{p-1}\mathrm{d}A. \tag{22}$$

应用引理 3 可知式(22)的右端第三项等于零.由 Hölder 不等式和 Cauchy-Schwarz 不等式,有

$$\int_\Omega QT^{p-1}\mathrm{d}A \leqslant \frac{1}{p}\int_\Omega Q^p \mathrm{d}A + \frac{p-1}{p}\int_\Omega T^p \mathrm{d}A. \tag{23}$$

所以

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}T^{p}\mathrm{d}A \leqslant \int_{\Omega}Q^{p}\mathrm{d}A + (p-1)\int_{\Omega}T^{p}\mathrm{d}A. \tag{24}$$

由 Gronwall 不等式,可得

$$\int_{\Omega}T^{p}\mathrm{d}A \leqslant \int_{\Omega}T_{0}^{p}\mathrm{d}A\mathrm{e}^{(p-1)t} + \int_{0}^{t}\int_{\Omega}\mathrm{e}^{(p-1)(t-\eta)}Q^{p}\mathrm{d}A\mathrm{d}\eta.$$

所以

$$\left(\int_{\Omega}T^{p}\mathrm{d}A\right)^{1/p} \leqslant \left\{\int_{\Omega}T_{0}^{p}\mathrm{d}A\mathrm{e}^{(p-1)t} + \int_{0}^{t}\int_{\Omega}\mathrm{e}^{(p-1)(t-\eta)}Q^{p}\mathrm{d}A\mathrm{d}\eta\right\}^{1/p}. \tag{25}$$

在式(25)中令 $p\to\infty$,可得

$$\sup_{\Omega}|T| \leqslant T_{\mathrm{m}}. \tag{26}$$

在式(24)中取 $p=2$,有

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}T^{2}\mathrm{d}A + \frac{1}{2}\int_{\Omega}|\nabla T|^{2}\mathrm{d}A \leqslant \int_{\Omega}T^{2}\mathrm{d}A + \int_{\Omega}Q^{2}\mathrm{d}A. \tag{27}$$

再次利用 Gronwall 不等式我们可以得到以下引理.

引理 5 假设 T 是方程组(7)~(9)的解,且 $T_0,\ Q\in L_2(\Omega)$.则

$$\int_{\Omega}T^{2}\mathrm{d}A + \frac{1}{2}\int_{0}^{t}\int_{\Omega}|\nabla T|^{2}\mathrm{d}A\mathrm{d}\eta \leqslant F_{1}(t),$$

其中 $$F_{1}(t) = \int_{\Omega}T_{0}^{2}\mathrm{d}A\mathrm{e}^{t} + \int_{0}^{t}\mathrm{e}^{t-\eta}Q^{2}\mathrm{d}A\mathrm{d}\eta.$$

引理 6 设 (u,v) 为方程组(7)~(9)的解,且 $u_0,\ v_0,\ T_0\in L_2(\Omega)$.则

$$\int_{\Omega}u^{2}\mathrm{d}A + \int_{\Omega}v^{2}\mathrm{d}A + \int_{0}^{t}\int_{\Omega}|\nabla u|^{2}\mathrm{d}A\mathrm{d}\eta + 2\int_{0}^{t}\int_{\Omega}|\nabla v|^{2}\mathrm{d}A\mathrm{d}\eta \leqslant F_{2}(t),$$

其中 $$F_{2}(t) = \int_{\Omega}u_{0}^{2}\mathrm{d}A + \int_{\Omega}v_{0}^{2}\mathrm{d}A + h_{2}^{2}\int_{0}^{t}F_{1}(\eta)\mathrm{d}\eta.$$

证明 取式(7)中的第一个方程和 u 在 $L^2(\Omega)$ 上的内积,有

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}u^{2}\mathrm{d}A + \int_{\Omega}|\nabla u|^{2}\mathrm{d}A = \\ &\quad f\int_{\Omega}uv\mathrm{d}A - \int_{\Omega}\left[u\frac{\partial u}{\partial x} - \left(\int_{-h_2}^{z}\frac{\partial}{\partial x}u(x,\zeta,t)\mathrm{d}\zeta\right)\frac{\partial u}{\partial z}\right]u\mathrm{d}A - \\ &\quad \int_{\Omega}\frac{\partial p_{\mathrm{s}}}{\partial x}u\mathrm{d}A + \int_{\Omega}\left(\int_{z}^{0}\frac{\partial}{\partial x}T(x,\zeta,t)\mathrm{d}\zeta\right)u\mathrm{d}A.\end{aligned} \tag{28}$$

应用引理 3,并对式(28)的右端第四项实施 Cauchy-Schwarz 不等式,可得

$$\begin{aligned}&\int_{\Omega}\left(\int_{z}^{0}\frac{\partial}{\partial x}T(x,\zeta,t)\mathrm{d}\zeta\right)u\mathrm{d}A = \\ &\quad \int_{\Omega}\left(\int_{z}^{0}T(x,\zeta,t)\mathrm{d}\zeta\right)\frac{\partial u}{\partial x}\mathrm{d}A \leqslant \frac{1}{2}\int_{\Omega}\left(\frac{\partial u}{\partial x}\right)^{2}\mathrm{d}A + \frac{h_{2}^{2}}{2}\int_{\Omega}T^{2}\mathrm{d}A.\end{aligned} \tag{29}$$

于是再由引理 5,有

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}u^{2}\mathrm{d}A + \int_{\Omega}|\nabla u|^{2}\mathrm{d}A \leqslant 2f\int_{\Omega}uv\mathrm{d}A + h_{2}^{2}F_{1}(t). \tag{30}$$

同理由式(7)的第二个方程可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}v^{2}\mathrm{d}A + 2\int_{\Omega}|\nabla v|^{2}\mathrm{d}A \leqslant -2f\int_{\Omega}uv\mathrm{d}A. \tag{31}$$

联合式(30)和(31)并使用式(8),可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^2\mathrm{d}A+\int_\Omega v^2\mathrm{d}A\right)+\int_\Omega|\nabla u|^2\mathrm{d}A+2\int_\Omega|\nabla v|^2\mathrm{d}A\leqslant h_2^2F_1(t).\tag{32}$$

对式(32)从 0 到 t 积分,可得

$$\int_\Omega u^2\mathrm{d}A+\int_\Omega v^2\mathrm{d}A+\int_0^t\int_\Omega|\nabla u|^2\mathrm{d}A\mathrm{d}\eta+2\int_0^t\int_\Omega|\nabla v|^2\mathrm{d}A\mathrm{d}\eta\leqslant F_2(t).\tag{33}$$

引理 7 设 (u,v) 为方程组(7)~(9)的解,且 $\partial_z u_0,\ \partial_z v_0,\ T_0\in L_2(\Omega)$.则

$$\int_\Omega\left(\frac{\partial u}{\partial z}\right)^2\mathrm{d}A+\int_\Omega\left(\frac{\partial v}{\partial z}\right)^2\mathrm{d}A+\int_0^t\int_\Omega\left|\nabla\frac{\partial u}{\partial z}\right|^2\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_\Omega\left|\nabla\frac{\partial v}{\partial z}\right|^2\mathrm{d}A\mathrm{d}\eta\leqslant F_3(t),$$

其中

$$F_3(t)=\int_\Omega\left(\frac{\partial u_0}{\partial z}\right)^2\mathrm{d}A+\int_\Omega\left(\frac{\partial v_0}{\partial z}\right)^2\mathrm{d}A+\sqrt{2F_1(t)F_2(t)}+\frac{6+\sqrt{2}}{4}\sqrt{F_2(t)}F_2(t).$$

证明 将式(7)的第一个方程对 z 求导,并与 $\partial u/\partial z$ 在 $L_2(\Omega)$ 上取内积,有

$$\int_0^t\int_\Omega\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\left(\int_{-h_2}^z\frac{\partial}{\partial x}u(x,\zeta,t)\mathrm{d}\zeta\right)\frac{\partial u}{\partial z}-fv+\frac{\partial p_s}{\partial x}-\right.$$
$$\left.\left(\int_z^0\frac{\partial}{\partial x}T(x,\zeta,t)\mathrm{d}\zeta\right)-\mu_1\Delta u\right]\frac{\partial u}{\partial z}\mathrm{d}A\mathrm{d}\eta=0,\tag{34}$$

即

$$\frac{1}{2}\int_\Omega\left(\frac{\partial u}{\partial z}\right)^2\mathrm{d}A+\int_0^t\int_\Omega\left|\nabla\frac{\partial u}{\partial z}\right|^2\mathrm{d}A\mathrm{d}\eta=$$
$$\frac{1}{2}\int_\Omega\left(\frac{\partial u_0}{\partial z}\right)^2\mathrm{d}A-\int_0^t\int_\Omega\left[u\frac{\partial^2u}{\partial x\partial z}-\left(\int_{-h_2}^z\frac{\partial}{\partial x}u(x,\zeta,\eta)\mathrm{d}\zeta\right)\frac{\partial^2u}{\partial z^2}\right]\frac{\partial u}{\partial z}\mathrm{d}A\mathrm{d}\eta+$$
$$\int_0^t\int_\Omega\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_\Omega\frac{\partial T}{\partial x}\frac{\partial u}{\partial z}\mathrm{d}A\mathrm{d}\eta.\tag{35}$$

利用引理 3,并对式(35)的最后一项实施 Hölder 不等式以及引理 5 和引理 6,可得

$$-\int_0^t\int_\Omega\frac{\partial T}{\partial x}\frac{\partial u}{\partial z}\mathrm{d}A\mathrm{d}\eta\leqslant$$
$$\sqrt{\int_0^t\int_\Omega\left(\frac{\partial T}{\partial x}\right)^2\mathrm{d}A\mathrm{d}\eta\int_0^t\int_\Omega\left(\frac{\partial u}{\partial z}\right)^2\mathrm{d}A\mathrm{d}\eta}\leqslant\sqrt{\frac{F_1(t)F_2(t)}{2}}.$$

因此

$$\frac{1}{2}\int_\Omega\left(\frac{\partial u}{\partial z}\right)^2\mathrm{d}A+\int_0^t\int_\Omega\left|\nabla\frac{\partial u}{\partial z}\right|^2\mathrm{d}A\mathrm{d}\eta\leqslant$$
$$\frac{1}{2}\int_\Omega\left(\frac{\partial u_0}{\partial z}\right)^2\mathrm{d}A+f\int_0^t\int_\Omega\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}\mathrm{d}A\mathrm{d}\eta+\sqrt{\frac{F_1(t)F_2(t)}{2}}.\tag{36}$$

类似地,重复上述过程可得

$$\frac{1}{2}\int_\Omega\left(\frac{\partial v}{\partial z}\right)^2\mathrm{d}A+\int_0^t\int_\Omega\left|\nabla\frac{\partial v}{\partial z}\right|^2\mathrm{d}A\mathrm{d}\eta=$$
$$\frac{1}{2}\int_\Omega\left(\frac{\partial v_0}{\partial z}\right)^2\mathrm{d}A-f\int_0^t\int_\Omega\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_\Omega\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial z}\mathrm{d}A\mathrm{d}\eta.\tag{37}$$

应用 Cauchy-Schwarz 不等式、引理 2 和引理 6,有

$$-\int_0^t\int_\Omega\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial z}\mathrm{d}A\mathrm{d}\eta\leqslant$$

$$\left[\int_0^t\int_\Omega\left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}A\mathrm{d}\eta\right]^{1/2}\left[\int_0^t\int_\Omega\left(\frac{\partial u}{\partial z}\right)^4 \mathrm{d}A\mathrm{d}\eta\right]^{1/4}\left[\int_0^t\int_\Omega\left(\frac{\partial v}{\partial z}\right)^4 \mathrm{d}A\mathrm{d}\eta\right]^{1/4} +$$

$$\left[\int_0^t\int_\Omega\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}A\mathrm{d}\eta\right]^{1/2}\left[\int_0^t\int_\Omega\left(\frac{\partial v}{\partial z}\right)^4 \mathrm{d}A\mathrm{d}\eta\right]^{1/2} \leqslant$$

$$\sqrt{\frac{F_2(t)}{2}}\left[\int_0^t\int_\Omega\left(\frac{\partial u}{\partial z}\right)^2 \mathrm{d}A\mathrm{d}\eta + \delta_1\int_0^t\int_\Omega\left|\nabla\frac{\partial u}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta\right]^{1/2} \times$$

$$\left[\int_0^t\int_\Omega\left(\frac{\partial v}{\partial z}\right)^2 \mathrm{d}A\mathrm{d}\eta + \delta_2\int_0^t\int_\Omega\left|\nabla\frac{\partial v}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta\right]^{1/2} +$$

$$\sqrt{F_2(t)}\left[\int_0^t\int_\Omega\left(\frac{\partial v}{\partial z}\right)^2 \mathrm{d}A\mathrm{d}\eta + \delta_3\int_0^t\int_\Omega\left|\nabla\frac{\partial v}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta\right] \leqslant$$

$$\frac{1}{2}\sqrt{\frac{F_2(t)}{2}}\left[F_2(t) + \delta_1\int_0^t\int_\Omega\left|\nabla\frac{\partial u}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta\right] +$$

$$\frac{1}{2}\sqrt{\frac{F_2(t)}{2}}\left[\frac{F_2(t)}{2} + \delta_2\int_0^t\int_\Omega\left|\nabla\frac{\partial v}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta\right] +$$

$$\sqrt{F_2(t)}\left[\frac{F_2(t)}{2} + \delta_3\int_0^t\int_\Omega\left|\nabla\frac{\partial v}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta\right] \leqslant$$

$$\frac{6+\sqrt{2}}{4}\sqrt{F_2(t)}F_2(t) + \frac{1}{2}\sqrt{\frac{F_2(t)}{2}}\delta_1\int_0^t\int_\Omega\left|\nabla\frac{\partial u}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta +$$

$$\left[\frac{1}{2}\sqrt{\frac{F_2(t)}{2}}\delta_2 + \sqrt{F_2(t)}\delta_3\right]\int_0^t\int_\Omega\left|\nabla\frac{\partial v}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta, \tag{38}$$

其中 δ_1, δ_2, δ_3 是任意的大于零的常数.把式(38)代入到式(37)然后令式(36)和(37)相加,并取

$$\delta_1 = \sqrt{\frac{2}{F_2(t)}},\ \delta_2 = \frac{1}{2}\sqrt{\frac{2}{F_2(t)}},\ \delta_3 = \frac{1}{4}\sqrt{\frac{1}{F_2(t)}},$$

有

$$\int_\Omega\left(\frac{\partial u}{\partial z}\right)^2 \mathrm{d}A + \int_\Omega\left(\frac{\partial v}{\partial z}\right)^2 \mathrm{d}A + \int_0^t\int_\Omega\left|\nabla\frac{\partial u}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta + \int_0^t\int_\Omega\left|\nabla\frac{\partial v}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta \leqslant F_3(t),$$

其中

$$F_3(t) = \int_\Omega\left(\frac{\partial u_0}{\partial z}\right)^2 \mathrm{d}A + \int_\Omega\left(\frac{\partial v_0}{\partial z}\right)^2 \mathrm{d}A +$$

$$2\sqrt{\frac{F_1(t)F_2(t)}{2}} + \frac{6+\sqrt{2}}{4}\sqrt{F_2(t)}F_2(t). \tag{39}$$

利用引理 3($\delta = 1$)、引理 6 和引理 7,有

$$\left(\int_0^t\int_\Omega\left(\frac{\partial u}{\partial z}\right)^4 \mathrm{d}A\mathrm{d}\eta\right)^{1/2} + \left(\int_0^t\int_\Omega\left(\frac{\partial v}{\partial z}\right)^4 \mathrm{d}A\mathrm{d}\eta\right)^{1/2} \leqslant$$

$$C\left(\int_0^t\int_\Omega\left(\frac{\partial u}{\partial z}\right)^2 \mathrm{d}A\mathrm{d}\eta + \int_0^t\int_\Omega\left(\frac{\partial v}{\partial z}\right)^2 \mathrm{d}A\mathrm{d}\eta +\right.$$

$$\left.\int_0^t\int_\Omega\left|\nabla\frac{\partial u}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta + \int_0^t\int_\Omega\left|\nabla\frac{\partial v}{\partial z}\right|^2 \mathrm{d}A\mathrm{d}\eta\right)^2 \leqslant$$

$$C\left(F_2(t)+F_3(t)\right)^2 \doteq F_4(t). \tag{40}$$

3 对热源的收敛性

设 (u^*, v^*, T^*, p_s^*) 是当 $Q=0$ 时方程组(7)~(9)的一组解.定义

$$\tilde{u}=u-u^*,\ \tilde{v}=v-v^*,\ \tilde{T}=T-T^*,\ \pi_s=p_s-p_s^*, \tag{41}$$

则 $(\tilde{u},\tilde{v},\tilde{T},\pi_s)$ 满足

$$\begin{cases}\dfrac{\partial\tilde{u}}{\partial t}-\Delta\tilde{u}+\tilde{u}\dfrac{\partial u}{\partial x}-\left(\displaystyle\int_{-h_2}^{z}\dfrac{\partial}{\partial x}\tilde{u}(x,\zeta,t)\mathrm{d}\zeta\right)\dfrac{\partial u}{\partial z}+u^*\dfrac{\partial\tilde{u}}{\partial x}-\\ \quad\left(\displaystyle\int_{-h_2}^{z}\dfrac{\partial}{\partial x}u^*(x,\zeta,t)\mathrm{d}\zeta\right)\dfrac{\partial\tilde{u}}{\partial z}-f\tilde{v}+\dfrac{\partial\pi_s}{\partial x}-\left(\displaystyle\int_{z}^{0}\dfrac{\partial}{\partial x}\tilde{T}(x,\zeta,t)\mathrm{d}\zeta\right)=0,\\ \dfrac{\partial\tilde{v}}{\partial t}-\Delta\tilde{v}+\tilde{u}\dfrac{\partial v}{\partial x}-\left(\displaystyle\int_{-h_2}^{z}\dfrac{\partial}{\partial x}\tilde{u}(x,\zeta,t)\mathrm{d}\zeta\right)\dfrac{\partial v}{\partial z}+u^*\dfrac{\partial\tilde{v}}{\partial x}-\\ \quad\left(\displaystyle\int_{-h_2}^{z}\dfrac{\partial}{\partial x}u^*(x,\zeta,t)\mathrm{d}\zeta\right)\dfrac{\partial\tilde{v}}{\partial z}-f\tilde{u}=0,\\ \dfrac{\partial\tilde{T}}{\partial t}-\Delta\tilde{T}+\tilde{u}\dfrac{\partial T}{\partial x}-\left(\displaystyle\int_{-h_2}^{z}\dfrac{\partial}{\partial x}\tilde{u}(x,\zeta,t)\mathrm{d}\zeta\right)\dfrac{\partial T}{\partial z}+u^*\dfrac{\partial\tilde{T}}{\partial x}-\\ \quad\left(\displaystyle\int_{-h_2}^{z}\dfrac{\partial}{\partial x}u^*(x,\zeta,t)\mathrm{d}\zeta\right)\dfrac{\partial\tilde{T}}{\partial z}=Q,\end{cases} \tag{42}$$

边界条件可以写为

$$\begin{cases}\left.\dfrac{\partial\tilde{u}}{\partial z}\right|_{z=0}=0,\ \left.\dfrac{\partial\tilde{v}}{\partial z}\right|_{z=0}=0,\ \tilde{u}\big|_{z=-h_2}=\tilde{v}\big|_{z=-h_2}=0,\ (\tilde{u},\tilde{v})\big|_{\Gamma_s}=0,\\ \left.\dfrac{\partial\tilde{T}}{\partial z}\right|_{z=0}=-\beta\tilde{T},\ \left.\dfrac{\partial\tilde{T}}{\partial z}\right|_{z=-h_2}=0,\ \left.\dfrac{\partial\tilde{T}}{\partial x}\right|_{\Gamma_s}=0,\end{cases} \tag{43}$$

初始条件为

$$(\tilde{u},\tilde{v},\tilde{T})\big|_{t=0}=(0,0,0). \tag{44}$$

定理 1 设 $(\tilde{u},\tilde{v},\tilde{T})$ 是方程组(42)~(44)的解,且 $T_0\in L_\infty(\Omega)$ 以及 $T_0,u_0,v_0\in L_2(\Omega)$.则$(\tilde{u},\tilde{v},\tilde{T})$ 对 $\gamma(t)>0$ 满足

$$\begin{aligned}&\int_\Omega\left[\frac{2h_2^2}{\pi^2}T_m^2(\tilde{u}^2+\tilde{v}^2)+\tilde{T}^2\right]\mathrm{d}A+\\ &\quad\int_0^t\int_\Omega\left[\frac{h_2^2}{\pi^2}T_m^2(|\nabla\tilde{u}|^2+2|\nabla\tilde{v}|^2)+|\nabla\tilde{T}|^2\right]\mathrm{d}A\mathrm{d}\eta\leqslant\\ &\quad\gamma(t)\int_0^t\int_0^s\int_\Omega \mathrm{e}^{\int_s^t\gamma(\eta)\mathrm{d}\eta}Q^2\mathrm{d}A\mathrm{d}\eta\mathrm{d}s+\int_0^t\int_\Omega Q^2\mathrm{d}A\mathrm{d}\eta,\end{aligned} \tag{45}$$

此式表明了方程组(7)~(9)的解对热源 Q 的收敛性.

证明 取式(42)中的第二个方程与 $\tilde{u}$ 在 $L_2(\Omega)$ 上的内积,有

$$\begin{aligned}&\frac{1}{2}\int_\Omega\tilde{u}^2\mathrm{d}A+\int_0^t\int_\Omega|\nabla\tilde{u}|^2\mathrm{d}A\mathrm{d}\eta=\\ &\quad f\int_0^t\int_\Omega\tilde{u}\tilde{v}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_\Omega\frac{\partial\pi_s}{\partial x}\tilde{u}\mathrm{d}A\mathrm{d}\eta+\end{aligned}$$

$$
\int_0^t\int_\Omega\left(\int_z^0\frac{\partial}{\partial x}\tilde{T}(x,\zeta,\eta)\mathrm{d}\zeta\right)\tilde{u}\mathrm{d}A\mathrm{d}\eta-
$$

$$
\int_0^t\int_\Omega\left[\tilde{u}\frac{\partial u}{\partial x}-\left(\int_{-h_2}^z\frac{\partial}{\partial x}\tilde{u}(x,\zeta,\eta)\mathrm{d}\zeta\right)\frac{\partial u}{\partial z}\right]\tilde{u}\mathrm{d}A\mathrm{d}\eta-
$$

$$
\int_0^t\int_\Omega\left[u^*\frac{\partial\tilde{u}}{\partial x}-\left(\int_{-h_2}^z\frac{\partial}{\partial x}u^*(x,\zeta,\eta)\mathrm{d}\zeta\right)\frac{\partial\tilde{u}}{\partial z}\right]\tilde{u}\mathrm{d}A\mathrm{d}\eta. \tag{46}
$$

对式(46)的右端第二项和第五项实施引理 3,对右端第三项实施分部积分和 Cauchy-Schwarz 不等式,有

$$
\begin{aligned}
&\int_0^t\int_\Omega\left(\int_z^0\frac{\partial}{\partial x}\tilde{T}(x,\zeta,\eta)\mathrm{d}\zeta\right)\tilde{u}\mathrm{d}A\mathrm{d}\eta=\\
&\quad-\int_0^t\int_\Omega\left(\int_z^0\tilde{T}(x,\zeta,\eta)\mathrm{d}\zeta\right)\frac{\partial\tilde{u}}{\partial x}\mathrm{d}A\mathrm{d}\eta\leqslant\\
&\quad h_2^2\int_0^t\int_\Omega\tilde{T}^2\mathrm{d}A\mathrm{d}\eta+\frac{1}{4}\int_0^t\int_\Omega\left(\frac{\partial\tilde{u}}{\partial x}\right)^2\mathrm{d}A\mathrm{d}\eta.
\end{aligned}\tag{47}
$$

对右端第四项实施 Hölder 不等式、引理 2、引理 6 和式(40),有

$$
\begin{aligned}
&-\int_0^t\int_\Omega\left[\tilde{u}\frac{\partial u}{\partial x}-\left(\int_{-h_2}^z\frac{\partial}{\partial x}\tilde{u}(x,\zeta,\eta)\mathrm{d}\zeta\right)\frac{\partial u}{\partial z}\right]\tilde{u}\mathrm{d}A\mathrm{d}\eta\leqslant\\
&\quad\left[\int_0^t\int_\Omega\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}A\mathrm{d}\eta\right]^{1/2}\left(\int_0^t\int_\Omega\tilde{u}^4\mathrm{d}A\mathrm{d}\eta\right)^{1/2}+\\
&\quad\left[\int_0^t\int_\Omega\left(\int_{-h_2}^z\frac{\partial}{\partial x}\tilde{u}(x,\zeta,\eta)\mathrm{d}\zeta\right)^2\mathrm{d}A\mathrm{d}\eta\right]^{1/2}\times\\
&\quad\left[\int_0^t\int_\Omega\left(\frac{\partial u}{\partial z}\right)^4\mathrm{d}A\mathrm{d}\eta\right]^{1/4}\left(\int_0^t\int_\Omega\tilde{u}^4\mathrm{d}A\mathrm{d}\eta\right)^{1/4}\leqslant\\
&\quad\sqrt{F_2(t)}\,C\left(\int_0^t\int_\Omega\tilde{u}^2\mathrm{d}A\mathrm{d}\eta+\delta_4\int_0^t\int_\Omega|\nabla\tilde{u}|^2\mathrm{d}A\mathrm{d}\eta\right)+\\
&\quad\frac{\sqrt{C}h_2}{\pi}\sqrt[4]{F_4(t)}\left[\int_0^t\int_\Omega\left(\frac{\partial\tilde{u}}{\partial x}\right)^2\mathrm{d}A\mathrm{d}\eta\right]^{1/2}\left(\int_0^t\int_\Omega\tilde{u}^2\mathrm{d}A\mathrm{d}\eta+\delta_4\int_0^t\int_\Omega|\nabla\tilde{u}|^2\mathrm{d}A\mathrm{d}\eta\right)^{1/2}\leqslant\\
&\quad b_1(t)\int_0^t\int_\Omega\tilde{u}^2\mathrm{d}A\mathrm{d}\eta+b_2(t)\delta_4\int_0^t\int_\Omega|\nabla\tilde{u}|^2\mathrm{d}A\mathrm{d}\eta,
\end{aligned}\tag{48}
$$

其中 $b_1(t)$, $b_2(t)$ 是大于零的可加性函数, δ_4 是一个大于零的任意常数.基于上述结果,取 $\delta_4=1/(8b_2(t))$, 则式(46)可以写为

$$
\begin{aligned}
&\int_\Omega\tilde{u}^2\mathrm{d}A+\frac{5}{4}\int_0^t\int_\Omega|\nabla\tilde{u}|^2\mathrm{d}A\mathrm{d}\eta\leqslant\\
&\quad 2f\int_0^t\int_\Omega\tilde{u}\tilde{v}\mathrm{d}A\mathrm{d}\eta+2h_2^2\int_0^t\int_\Omega\tilde{T}^2\mathrm{d}A\mathrm{d}\eta+2b_1(t)\int_0^t\int_\Omega\tilde{u}^2\mathrm{d}A\mathrm{d}\eta.
\end{aligned}\tag{49}
$$

现在取式(42)中的第二个方程与 $\tilde{v}$ 在 $L_2(\Omega)$ 上的内积,有

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega\tilde{v}^2\mathrm{d}A+\mu_2\int_0^t\int_\Omega|\nabla\tilde{v}|^2\mathrm{d}A\mathrm{d}\eta=\\
&\quad-f\int_0^t\int_\Omega\tilde{u}\tilde{v}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_\Omega\left[\tilde{u}\frac{\partial v}{\partial x}-\left(\int_{-h_2}^z\frac{\partial}{\partial x}\tilde{u}(x,\zeta,\eta)\mathrm{d}\zeta\right)\frac{\partial v}{\partial z}\right]\tilde{v}\mathrm{d}A\mathrm{d}\eta-\\
&\quad\int_0^t\int_\Omega\left[u^*\frac{\partial\tilde{v}}{\partial x}-\left(\int_{-h_2}^z\frac{\partial}{\partial x}u^*(x,\zeta,\eta)\mathrm{d}\zeta\right)\frac{\partial\tilde{v}}{\partial z}\right]\tilde{v}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}\tag{50}
$$

经过与式(49)同样的计算,易得

$$\int_\Omega \tilde{v}^2 \mathrm{d}A + \int_0^t\int_\Omega |\nabla\tilde{v}|^2 \mathrm{d}A\mathrm{d}\eta \leqslant$$
$$-2f\int_0^t\int_\Omega \tilde{u}\tilde{v}\mathrm{d}A\mathrm{d}\eta + 2b_3(t)\int_0^t\int_\Omega \tilde{v}^2\mathrm{d}A\mathrm{d}\eta +$$
$$b_4(t)\delta_5\int_0^t\int_\Omega |\nabla\tilde{u}|^2\mathrm{d}A\mathrm{d}\eta, \tag{51}$$

其中 $b_3(t)$, $b_4(t)$ 是一个大于零的函数,δ_5 是一个大于零的任意常数.联合式(49)和(51)并取 $\delta_5 = 1/(8b_4(t))$,可得

$$\int_\Omega \tilde{u}^2 \mathrm{d}A + \int_\Omega \tilde{v}^2 \mathrm{d}A + \int_0^t\int_\Omega |\nabla\tilde{u}|^2 \mathrm{d}A\mathrm{d}\eta + \int_0^t\int_\Omega |\nabla\tilde{v}|^2 \mathrm{d}A\mathrm{d}\eta \leqslant$$
$$2h_2^2\int_0^t\int_\Omega \tilde{T}^2\mathrm{d}A\mathrm{d}\eta + 2b_1(t)\int_0^t\int_\Omega \tilde{u}^2\mathrm{d}A\mathrm{d}\eta + 2b_3(t)\int_0^t\int_\Omega \tilde{v}^2\mathrm{d}A\mathrm{d}\eta. \tag{52}$$

取式(42)的第三个方程与 $\tilde{T}$ 在 $L_2(\Omega)$ 上的内积,有

$$\frac{1}{2}\int_\Omega \tilde{T}^2 \mathrm{d}A + \int_0^t\int_\Omega |\nabla\tilde{T}|^2 \mathrm{d}A\mathrm{d}\eta =$$
$$-\int_0^t\int_\Omega \left[\tilde{u}\frac{\partial T}{\partial x} - \left(\int_{-h_2}^z \frac{\partial}{\partial x}\tilde{u}(x,\zeta,\eta)\mathrm{d}\zeta\right)\frac{\partial T}{\partial z}\right]\tilde{T}\mathrm{d}A\mathrm{d}\eta -$$
$$\int_0^t\int_\Omega \left[u^*\frac{\partial \tilde{T}}{\partial x} - \left(\int_{-h_2}^z \frac{\partial}{\partial x}u^*(x,\zeta,\eta)\mathrm{d}\zeta\right)\frac{\partial \tilde{T}}{\partial z}\right]\tilde{T}\mathrm{d}A\mathrm{d}\eta + \int_0^t\int_\Omega Q\tilde{T}\mathrm{d}A\mathrm{d}\eta. \tag{53}$$

利用分部积分、Hölder 不等式、引理 4、引理 5 和算术几何平均不等式,有

$$-\int_0^t\int_\Omega \left[\tilde{u}\frac{\partial T}{\partial x} - \left(\int_{-h_2}^z \frac{\partial}{\partial x}\tilde{u}(x,\zeta,\eta)\mathrm{d}\zeta\right)\frac{\partial T}{\partial z}\right]\tilde{T}\mathrm{d}A\mathrm{d}\eta =$$
$$\int_0^t\int_\Omega \tilde{u}T\frac{\partial \tilde{T}}{\partial x}\mathrm{d}A\mathrm{d}\eta - \int_0^t\int_\Omega \left(\int_{-h_2}^z \frac{\partial}{\partial x}\tilde{u}(x,\zeta,\eta)\mathrm{d}\zeta\right)T\frac{\partial \tilde{T}}{\partial z}\mathrm{d}A\mathrm{d}\eta \leqslant$$
$$T_{\mathrm{m}}\left(\int_0^t\int_\Omega\left(\frac{\partial \tilde{T}}{\partial x}\right)^2\mathrm{d}A\mathrm{d}\eta\right)^{1/2}\left(\int_0^t\int_\Omega \tilde{u}^2\mathrm{d}A\mathrm{d}\eta\right)^{1/2} +$$
$$T_{\mathrm{m}}\left(\int_0^t\int_\Omega\left(\int_{-h_2}^z \frac{\partial}{\partial x}\tilde{u}(x,\zeta,\eta)\mathrm{d}\zeta\right)^2\mathrm{d}A\mathrm{d}\eta\right)^{1/2}\left(\int_0^t\int_\Omega\left(\frac{\partial \tilde{T}}{\partial z}\right)^2\mathrm{d}A\mathrm{d}\eta\right)^{1/2} \leqslant$$
$$\frac{1}{2}\int_0^t\int_\Omega\left(\frac{\partial \tilde{T}}{\partial x}\right)^2\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}T_{\mathrm{m}}^2\int_0^t\int_\Omega \tilde{u}^2\mathrm{d}A\mathrm{d}\eta +$$
$$\frac{h_2^2}{2\pi^2}T_{\mathrm{m}}^2\int_0^t\int_\Omega\left(\frac{\partial \tilde{u}}{\partial x}\right)^2\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_\Omega\left(\frac{\partial \tilde{T}}{\partial z}\right)^2\mathrm{d}A\mathrm{d}\eta. \tag{54}$$

再由 Hölder 不等式和算术几何平均不等式,可得

$$\int_0^t\int_\Omega Q\tilde{T}\mathrm{d}A\mathrm{d}\eta \leqslant \frac{1}{2}\int_0^t\int_\Omega Q^2\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_\Omega \tilde{T}^2\mathrm{d}A\mathrm{d}\eta. \tag{55}$$

对式(53)中的右端第二项实施引理 3,然后把式(54)和(55)代入到式(53),可得

$$\int_\Omega \tilde{T}^2 \mathrm{d}A + \int_0^t\int_\Omega |\nabla\tilde{T}|^2 \mathrm{d}A\mathrm{d}\eta \leqslant$$
$$T_{\mathrm{m}}^2\int_0^t\int_\Omega \tilde{u}^2\mathrm{d}A\mathrm{d}\eta + \frac{h_2^2}{\pi^2}T_{\mathrm{m}}^2\int_0^t\int_\Omega\left(\frac{\partial \tilde{u}}{\partial x}\right)^2\mathrm{d}A\mathrm{d}\eta + \int_0^t\int_\Omega \tilde{T}^2\mathrm{d}A\mathrm{d}\eta + \int_0^t\int_\Omega Q^2\mathrm{d}A\mathrm{d}\eta. \tag{56}$$

接下来,在式(52)的两边乘以 $(2h_2^2/\pi^2)T_{\mathrm{m}}^2$ 再和式(56)相加,可得

$$\int_\Omega\left[\frac{2h_2^2}{\pi^2}T_{\mathrm m}^2(\tilde u^2+\tilde v^2)+\tilde T^2\right]\mathrm dA+$$

$$\int_0^t\int_\Omega\left[\frac{h_2^2}{\pi^2}T_{\mathrm m}^2(|\nabla\tilde u|^2+2|\nabla\tilde v|^2)+|\nabla\tilde T|^2\right]\mathrm dA\mathrm d\eta\leqslant$$

$$\gamma(t)\int_0^t\int_\Omega\left[\frac{2h_2^2}{\pi^2}T_{\mathrm m}^2(\tilde u^2+\tilde v^2)+\tilde T^2\right]\mathrm dA\mathrm d\eta+\int_0^t\int_\Omega Q^2\mathrm dA\mathrm d\eta,\tag{57}$$

其中

$$\gamma(t)=\max\left\{1+\frac{2h_2{}^4}{\pi^2}T_{\mathrm m}^2,\ 2b_1(t)+\frac{\pi^2}{2h_2^2},\ 2b_3(t)\right\}.$$

所以

$$\frac{\mathrm d}{\mathrm dt}\left\{\int_0^t\int_\Omega\left[\frac{2h_2^2}{\pi^2}T_{\mathrm m}^2(\tilde u^2+\tilde v^2)+\tilde T^2\right]\mathrm dA\mathrm d\eta\mathrm e^{-\int_0^t\gamma(\eta)\mathrm d\eta}\right\}\leqslant$$

$$\int_0^t\int_\Omega \mathrm e^{-\int_0^t\gamma(\eta)\mathrm d\eta}Q^2\mathrm dA\mathrm d\eta.\tag{58}$$

对式(58)从 0 到 t 积分

$$\int_0^t\int_\Omega\left[\frac{2h_2^2}{\pi^2}T_{\mathrm m}^2(\tilde u^2+\tilde v^2)+\tilde T^2\right]\mathrm dA\mathrm d\eta\leqslant\int_0^t\int_0^s\int_\Omega \mathrm e^{\int_s^t\gamma(\eta)\mathrm d\eta}Q^2\mathrm dA\mathrm d\eta\mathrm ds.\tag{59}$$

再将式(59)代入到式(57)即可完成定理 1 的证明.

4 总　　结

本文对海洋动力学中原始方程组中的热源进行了收敛性分析.通过推导方程组的先验界,引入辅助函数,证明了方程组的解对热源具有收敛性.大多文献主要关注原始方程组的适定性,本文的研究是对文献的一个有益补充,而且这方面的研究还可以持续下去.比如下一步可以继续研究方程组对黏性系数的收敛性, 在“能量” 函数中必然会缺少$\|\nabla\tilde u\|$, 这会对式(48)的推导带来一定的困难,这将是接下来研究的一个方向.

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Convergence Results on Heat Source for 2D Viscous Primitive Equations of Ocean Dynamics

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Abstract: The convergence of solutions to 2D viscous primitive equations of ocean dynamics in a cylindrical region was considered. A key parameter in this model is heat source, which is known to cause resonance between the inner layers of fluid and in turn trigger instability. Therefore, through derivation of the priori bounds of the equations, the convergence of solutions to the equations on the heat source itself was obtained.

Key words: primitive equations of ocean dynamics; heat source; convergence; structural stability