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一类非线性三阶微分方程边值问题解的存在唯一性*

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摘要: 研究了一类非线性三阶微分方程边值问题解的存在唯一性.首先分析了近年来国内外三阶微分方程边值问题的研究成果,提出了边值条件中含非线性函数的非线性三阶微分方程边值问题.然后寻找相关线性问题的解决途径, 利用 Banach 不动点定理, 证明了提出的边值问题存在唯一解.最后,举例阐述了主要结果的应用.

关 键 词: Banach 不动点定理; 三阶微分方程; 两点边值问题

中图分类号: O175.14; O175.8 **文献标志码:** A DOI: 10.21656/1000-0887.400158

引 言

高阶非线性微分方程在数学物理、工程数学、生态数学、物理化学等学科中都有很广泛的应用.例如,描述控制论中的电气系统、流体力学中的流体流动以及电磁波的传播等一些实际应用领域中都有其重要的研究成果[1-19].非线性三阶微分方程边值问题已成为近年来的研究热点[4-17].

文献[6]研究了非线性两点边值问题:

$$\begin{cases} u'''(t) = \lambda a(t) f(u(t)), & 0 < t < 1, \\ \alpha u'(0) - \beta u''(0) = 0, & u(1) = u'(1) = 0. \end{cases}$$

通过使用 Krasnosel'skii 锥拉伸与压缩不动点定理,确定了该边值问题正解的存在性和非存在性的一些结论.

文献[7]考虑了

$$\begin{cases} u'''(t) + \lambda a(t) f(u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0. \end{cases}$$

通过使用 Krasnosel'skii 锥拉伸与压缩不动点定理,得到了该边值问题单独一个正解和多个正解的存在性.

* **收稿日期:** 2019-05-06; **修订日期:** 2019-05-10
基金项目: 安徽省高校自然科学研究项目(KJ2017A702;KJ2019A1300;KJ2019A1303)
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引用格式: 杨景保, 莫嘉琪. 一类非线性三阶微分方程边值问题解的存在唯一性[J]. 应用数学和力学, 2020, **41**(2): 216-222.

文献[17]研究了非线性三点边值问题：

$$\begin{cases} u'''(t)+f(t,u(t))=0, & 0<t<1, \\ u'(0)=u''(0),\ u(1)=\alpha u(\eta), & \alpha\in(0,1),\ \eta\in(0,1). \end{cases}$$

应用不动点指数理论得到了该边值问题正解的存在性准则.

Mo 和 Yang 等也应用泛函分析、奇摄动、微分不等式讨论了一些大气物理、生物数学、理论物理、传染病传播等实际问题中的非线性微分方程的定性理论和定量方法的数学模型[20-33].

本文讨论如下三阶非线性微分方程：

$$w'''(t)+f(t,w(t))=0,\qquad t\in(0,1). \tag{1}$$

其满足边值条件

$$w(0)=w'(0)=0,\ w''(1)=g(w(1)) \tag{2}$$

的非线性边界值问题,其中 $f\in C((0,1)\times\mathbf{R}\to\mathbf{R})$,$g$ 是关于其变量的光滑函数.利用 Banach 不动点定理证明了边值问题(1)和(2)解的存在唯一性.

1 引 理

设 $E=C[0,1]$ 是一个 Banach 空间,定义其范数是 $\|w\|=\max_{t\in[0,1]}|w(t)|$.

引理 1 设 $w,h\in E,\alpha\in\mathbf{R}$, 则边值问题

$$\begin{cases} w'''(t)+h(t)=0, & 0<t<1, \\ w(0)=w'(0)=0, & w''(1)=\alpha \end{cases} \tag{3}$$

有唯一解：

$$w(t)=\int_0^1 G(t,s)h(s)\mathrm{d}s+\frac{1}{2}\alpha t^2,\qquad t\in[0,1],$$

其中

$$G(t,s)=\begin{cases} \dfrac{1}{2}(2ts-s^2), & 0\leqslant s\leqslant t\leqslant 1, \\ \dfrac{1}{2}t^2, & 0\leqslant t\leqslant s\leqslant 1. \end{cases}$$

证明 设

$$w(t)=-\int_0^t \frac{1}{2}(t-s)^2h(s)\mathrm{d}s+At^2, \tag{4}$$

其中 A 表示一个常数.对式(4)两边分别求导, 得到

$$w'(t)=-\int_0^t (t-s)h(s)\mathrm{d}s+2At. \tag{5}$$

由式(4)和(5)可知, 显然 $w(0)=w'(0)=0$ 成立.对式(5)两边求导得

$$\begin{cases} w''(t)=-\int_0^t h(s)\mathrm{d}s+2A, \\ w''(1)=-\int_0^1 h(s)\mathrm{d}s+2A. \end{cases}$$

于是

$$A=\frac{1}{2}\alpha+\frac{1}{2}\int_0^1 h(s)\mathrm{d}s.$$

因此得到

$$w(t)=\int_0^t \frac{1}{2}(2ts-s^2)h(s)\mathrm{d}s+\int_t^1 \frac{1}{2}t^2 h(s)\mathrm{d}s+\frac{1}{2}\alpha t^2,\qquad t\in[0,1]. \tag{6}$$

引理证毕.

在式(6)中,用 $f(s,w(s))$ 代替 $h(s)$,用 $g(w(1))$ 代替 α ,可以得到

$$w(t)=\int_0^1 G(t,s)f(s,w(s))\mathrm{d}s+\frac{1}{2}g(w(1))t^2,\qquad t\in[0,1],\ w\in E.$$

定义积分算子 T:

$$(\mathrm{T}w)(t)=\int_0^1 G(t,s)f(s,w(s))\mathrm{d}s+\frac{1}{2}g(w(1))t^2,\qquad t\in[0,1],\ w\in E, \tag{7}$$

其中

$$G(t,s)=\begin{cases}\frac{1}{2}(2ts-s^2), & 0\leqslant s\leqslant t\leqslant 1,\\ \frac{1}{2}t^2, & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

显然求边值问题(1)和(2)的解等价于寻求算子 T 在空间 E 中的不动点 w_0, 使得

$$\mathrm{T}w_0=w_0,\qquad w_0\in E.$$

现有如下引理:

引理 2[19](Banach 不动点定理) 设 X 是一个非空的度量空间,若 X 是完备的度量空间,且 $X\to X$ 是一个压缩映像,则 T 在 X 中有一个不动点.

2 主要结论

定理 1 设函数 $f(t,w)$ 是一个满足 Lipschitz 条件的函数, 即存在正常数 L_1, 使得

$$|f(t,w)-f(t,v)|\leqslant L_1|w-v|,\qquad w,v\in E, \tag{8}$$

且函数 g 也是一个 Lipschitz 函数,即存在正常数 L_2, 使得

$$|g(w)-g(v)|\leqslant L_2|w-v|,\qquad w,v\in E. \tag{9}$$

设

$$\beta:=\frac{1}{3}L_1+\frac{1}{2}L_2<1, \tag{10}$$

则边值问题(1)和(2)存在唯一解.

证明 显然 Banach 空间 E 是一个完备的度量空间.由式(7)和引理 1 知,T: $E\to E$.仅需证明 T 是一个压缩映像.事实上,不难得到

$$\int_0^1 G(t,s)\mathrm{d}s\leqslant\frac{1}{3}. \tag{11}$$

对于 $w,v\in E,t\in[0,1]$, 由式(7)~(9)和(11)可得

$$\begin{aligned}&|(\mathrm{T}w-\mathrm{T}v)(t)|\leqslant\\ &\quad\int_0^1 G(t,s)|f(s,w(s))-f(s,v(s))|\mathrm{d}s+\frac{t^2}{2}|g(w(1))-g(v(1))|\leqslant\\ &\quad\int_0^1 G(t,s)L_1|w(s)-v(s)|\mathrm{d}s+\frac{1}{2}L_2|w(1)-v(1)|\leqslant\\ &\quad L_1\|w-v\|\int_0^1 G(t,s)\mathrm{d}s+\frac{1}{2}L_2\|w-v\|\leqslant\end{aligned}$$

$$\left(\frac{1}{3}L_1+\frac{1}{2}L_2\right)\|w-v\|.$$

再由假设(10)有

$$\|\mathrm{T}w-\mathrm{T}v\|\leqslant\beta\|w-v\|,$$

其中$0<\beta<1$.于是算子T是一个压缩映像.由引理2可知,算子T在空间E中有唯一不动点,即边值问题(1)和(2)在空间E中有唯一解.

由式(7)和定理1,可确定一个迭代程序:

$$w_{k+1}(t)=\int_0^1(t,s)f(s,w_k(s))\mathrm{d}s+\frac{1}{2}g(w_k(1))t^2,\qquad t\in[0,1],\ w_k,w_{k+1}\in E,$$

从而可求非线性边值问题(1)和(2)的唯一解.

定理2 设函数$f(s,w)$是一个连续函数,关于w的偏导存在且有界,且函数g满足Lipschitz条件,即存在常数L_3使得

$$|g(w)-g(v)|\leqslant L_3|w-v|,$$

取

$$\gamma:=\frac{D_f}{3}+\frac{L_3}{2}<1,\tag{12}$$

其中

$$D_f:=\max_{[0,1]\times\mathbf{R}}\left|\frac{\partial f}{\partial w}\right|,\tag{13}$$

则边值问题(1)和(2)存在唯一解.

证明 类似于定理1的证明,仅需证明算子T是一个压缩映像.根据式(7)、(11)~(13),对于$w,v\in E,t\in[0,1]$,可得

$$\begin{aligned}&|(\mathrm{T}w-\mathrm{T}v)(t)|\leqslant\\&\quad\int_0^1G(t,s)|f(s,w(s))-f(s,v(s))|\mathrm{d}s+\frac{t^2}{2}|g(w(1))-g(v(1))|\leqslant\\&\quad\int_0^1G(t,s)D_f|w(s)-v(s)|\mathrm{d}s+\frac{1}{2}L_3|w(1)-v(1)|\leqslant\\&\quad D_f\|w-v\|\int_0^1G(t,s)\mathrm{d}s+\frac{1}{2}L_3\|w-v\|\leqslant\\&\quad\left(\frac{1}{3}D_f+\frac{1}{2}L_3\right)\|w-v\|\leqslant\gamma\|w-v\|.\end{aligned}$$

故

$$\|\mathrm{T}w-\mathrm{T}v\|\leqslant\gamma\|w-v\|,$$

其中$0<\gamma<1$.即算子T是一个压缩映像.由引理2可知,算子T在空间E中有唯一不动点.即边值问题(1)和(2)在空间E中有唯一解.

3 举 例

例 设$w(t)$表示流体随时间变化的流速,$w'(t)$,$w''(t)$和$w'''(t)$分别表示流体在时间t时的隅角、弯矩和剪切力.函数$w(t)$在$t=1$处的弯矩为$\sin w$.这时边值问题(1)和(2)为如下三阶非线性微分方程边值问题:

$$\begin{cases} w'''(t) + \dfrac{1}{5}t^2\cos w(t) = 0, \quad 0 < t < 1, \\ w(0) = w'(0) = 0,\ w''(1) = \dfrac{1}{8}\sin w(1). \end{cases} \tag{14}$$

由定理 1,边值问题(14)存在唯一解.通过迭代方法, 可以得到唯一流速解满足

$$w(t) = -\frac{1}{300}t^5\cos w(t) + \frac{1}{30}t^2\cos w(t) + \frac{1}{16}t^2\sin(w(t)), \quad t \in [0,1].$$

4 总　　结

本文研究了非线性三阶微分方程边值问题,提出了适当的假设条件,利用 Banach 不动点定理确定了边值问题存在唯一解的两个充分条件,并给出了求解的迭代程序式.对于更弱的条件将有待进一步深入探讨.

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Existence and Uniqueness of Solutions to Boundary Value Problems of a Class of Nonlinear 3rd-Order Differential Equations

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Abstract: The existence and uniqueness of solutions to the boundary value problem of a class of nonlinear 3rd-order differential equations were studied. Firstly, the results of the research on the boundary values of 3rd-order differential equations at home and abroad in recent years were combed. Then the boundary value problem of nonlinear 3rd-order differential equations with nonlinear boundary value conditions was put forth, and the solution to the related linear problem was explored. Finally, the Banach fixed point theorem was used to prove that the proposed boundary value problem has a unique solution. An example illustrates the applicability of the main results.

Key words: Banach fixed point theorem; 3rd-order differential equation; 2-point boundary value problem