

# 热力耦合问题数学均匀化方法的计算精度\*

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**摘要:** 针对复合材料周期结构热力耦合问题, 推导了数学均匀化方法(MHM)各阶摄动位移的全解耦格式和各阶影响函数控制方程, 并使用加权残量方法将其转化为易于编程计算的有限元列式. 在解耦格式中, 各阶摄动位移是相应阶次的影响函数和宏观场导数的乘积, 即影响函数和宏观场导数的计算精度共同决定摄动项的精度, 其中影响函数的计算精度取决于单胞边界条件选取的适用性. 针对2D复合材料周期结构静力学问题, 使用超单胞边界条件和微分求积有限单元法, 分别提高了影响函数和宏观场导数的求解精度. 在此基础上, 研究了高阶展开项对MHM真实位移精度的影响, 确定了二阶摄动项的必要性. 最后应用最小势能原理评估了各阶摄动MHM的计算精度, 数值比较结果验证了结论的正确性.

**关键词:** 数学均匀化方法; 周期复合材料; 热力耦合; 高阶摄动

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## 引 言

复合材料具有比强度高、比刚度大等优点, 广泛应用于航天、航空工业等领域. 众所周知, 对于很多复合材料的宏观解, 如低阶频率和模态, 可以使用等应变模型或等应力模型<sup>[1]</sup>以及其他均匀化方法<sup>[2]</sup>, 但相对于宏观应力分析, 微观分析要复杂很多. 为了在计算精度和效率之间达到平衡, 各种多尺度方法相继被提出, 例如MHM<sup>[3-5]</sup>、广义有限单元方法(GFEM)<sup>[6-7]</sup>、多尺度有限单元方法(MSFEM)<sup>[8-9]</sup>、异类多尺度方法(HMM)<sup>[10-11]</sup>以及多尺度特征单元方法(MEM)<sup>[12-13]</sup>.

MHM是最具有代表性的多尺度方法之一, 因其具有严格的数学背景, 且可以包含所有的微观结构复合材料信息而广泛应用于分析周期复合材料热力耦合作用下的结构响应问题<sup>[14-19]</sup>. 然而只考虑一阶展开项对于细观结构物理和力学信息的捕捉往往不够<sup>[20]</sup>, 对于很多实际工程问题<sup>[21-26]</sup>, 二阶展开项对于计算精度的影响不可忽略, 这就需要寻找高阶MHM求解复合材料细观结构的物理和力学属性. Han等<sup>[27]</sup>基于二阶渐近展开均匀化方法(AEH)处理功能梯度材料的静态热力耦合问题, 精确地估计了有效弹性模量和应力应变场. Han等<sup>[28]</sup>基于周期均匀化方法推导了二阶展开项用于求解复合材料弹性问题. Guan等<sup>[21]</sup>和Yang等<sup>[23]</sup>研究

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了一种二阶多尺度方法用于预测周期和非周期复合材料的物理和力学属性,并求解一些实际工程问题。Allaire 等<sup>[24]</sup>使用双尺度二阶渐进展开方法研究辐射边界条件下的热传导模型,并证明了均匀化过程的双尺度收敛。显然,通过二阶矫正项可以更加准确地捕捉到材料内部物理和力学行为的细观信息,得到越来越多学者的认可。但从文献检索看出,关于二阶展开项的必要性问题基本上集中在具体物理和力学问题的解决上,鲜有文献针对热力耦合问题 AEH 高阶展开必要性问题以及高阶展开项本身的计算精度进行系统的研究。Xing 等<sup>[29-31]</sup>详细研究了影响线弹性问题 MHM 计算精度与摄动位移阶次之间的关系,指出影响函数和宏观位移导数是决定摄动位移本身精度的两个关键因素,其中影响函数的计算精度主要依赖于单胞边界条件选取的合理性,并构造了超单胞周期边界条件提高一些函数的计算精度。从数学角度看,随着摄动阶次的提高,MHM 的计算精度必然随之提高;从数值计算角度看,随着摄动阶次的提高,宏观场导数的阶次随之提高,精度也愈难保证,从而限制了摄动项对 MHM 精度的积极作用,严重时,高阶摄动项对 MHM 精度起反作用。传统的有限元方法主要采用低阶格式,但为了提高精度,通常需要加密网格或使用大量结点,虽然高阶格式用较少的单元就能达到较高的精度,但存在构造形函数及其导数计算的困难<sup>[32]</sup>;微分求积方法<sup>[33]</sup>可以高精度地近似微分,但在实现单元化过程中遇到了施加边界条件的灵活性和单元矩阵不对称等问题;微分求积有限单元法<sup>[34]</sup>利用微分求积方法和 Gauss-Lobato 积分方法来离散系统的能量泛函,较好地解决了上述有限元方法和微分求积方法中存在的问题。解决宏观场高阶导数的计算精度问题是研究高阶展开项必要性问题的前提条件。

本文在文献[29-31]的基础上,针对更复杂的问题推导各阶摄动项的全解耦格式和影响函数控制方程,并使用加权残量方法将其转化为有限元矩阵列式;使用超单胞边界条件求解各阶弹性影响函数和热影响函数控制方程,同时将微分求积有限单元法和热力耦合问题 MHM 相结合,构造宏观位移场和温度场高阶微分求积有限单元求解影响函数各阶导数,保证高阶摄动位移本身的计算精度;在此基础上通过高阶 MHM 计算结果与精细有限元方法计算结果之间的误差分析,确定了高阶摄动项的必要性依据。

## 1 热力耦合问题数学均匀化方法

### 1.1 基本理论

热力耦合问题结构线性小变形的应力-应变关系为

$$\sigma_{ij} = E_{ijmn}^{\varepsilon} (e_{mn}^{\text{tot}} - e_{mn}^{\text{t}}), \quad (1)$$

其中  $E_{ijmn}^{\varepsilon}$  表示弹性常数张量,  $e_{mn}^{\text{tot}}$  是总应变,  $e_{mn}^{\text{t}}$  表示热应变。总应变-位移和热应变分别表示为

$$e_{mn}^{\text{tot}} = \frac{1}{2} \left( \frac{\partial u_m^{\varepsilon}}{\partial x_n} + \frac{\partial u_n^{\varepsilon}}{\partial x_m} \right), \quad (2)$$

$$e_{mn}^{\text{t}} = a_{mn}^{\varepsilon} T_{mn}, \quad (3)$$

其中  $u^{\varepsilon}$  为真实位移,  $a_{mn}^{\varepsilon}$  为热膨胀系数,  $T_{mn}$  为温差。将式(2)和(3)代入式(1)并考虑弹性常数张量的对称性可以得到

$$\sigma_{ij} = E_{ijmn}^{\varepsilon} \left( \frac{\partial u_m^{\varepsilon}}{\partial x_n} - a_{mn}^{\varepsilon} T_{mn} \right). \quad (4)$$

考虑如图 1 所示的热力耦合作用下的复合材料周期结构平面问题,其平衡方程的数学表达式为

$$\begin{cases} -\frac{\partial}{\partial x_j} \left( E_{ijmn}^\varepsilon \left( \frac{\partial u_m^\varepsilon}{\partial x_n} - a_{mn}^\varepsilon T_{mn} \right) \right) = f_i, & x \in \Omega, \\ u^\varepsilon = 0, & x \in \partial\Omega, \end{cases} \quad (5)$$

其中  $\Omega$  表示域. 由于方程(5)中包含了一个小参数  $\varepsilon$ , 通常会找一个在  $\varepsilon$  上进行幂级数展开的解:

$$u_m^\varepsilon = u_m^0 + \varepsilon u_m^1 + \varepsilon^2 u_m^2 + \varepsilon^3 u_m^3 + \dots \quad (6)$$

多尺度方法背后的基本思想是假设表达式里所有的项依赖于  $x$  和  $y = x/\varepsilon$ , 而且由于微分方程系数是  $x/\varepsilon$  的周期函数, 所以展开式中的所有项也是  $x/\varepsilon$  的周期函数, 从而可以假设  $u_m^\varepsilon$  的形式如下, 其中  $u_m^j (j = 0, 1, \dots)$  在  $y$  上具有周期性, 即

$$u_m^\varepsilon(x, y) = u_m^\varepsilon(x, y + Y), \quad (7)$$

其中  $Y$  为结构的周期.

真实位移  $u_m^\varepsilon$  对宏观尺度  $x$  的导数可以根据链式法则得到

$$\frac{\partial}{\partial x_j} [u_m^\varepsilon(x, y)] = \frac{\partial u_m^\varepsilon}{\partial x_j} + \frac{1}{\varepsilon} \frac{\partial u_m^\varepsilon}{\partial y_j}, \quad (8)$$

变量  $x$  和  $y = x/\varepsilon$  分别代表周期结构的慢(宏观)尺度和快(微观)尺度. 当  $\varepsilon \ll 1$  时, 变量  $y$  比  $x$  变化得更快, 并且在微观尺度下考虑问题时, 可将  $x$  看作常数, 这就应用了尺度分离, 即将  $x$  和  $y$  看作独立变量.

将式(6)代入式(5), 可以得到渐进展开形式的控制方程:

$$-\frac{\partial}{\partial x_j} \left( E_{ijmn}^\varepsilon \left( \frac{\partial (u_m^0 + \varepsilon u_m^1 + \varepsilon^2 u_m^2 + \varepsilon^3 u_m^3 + \dots)}{\partial x_n} - a_{mn}^\varepsilon T_{mn} \right) \right) = f_i. \quad (9)$$

值得注意的是, 此时热应变已经耦合进入方程(9), 联合式(8), 对其真实位移进行求导并展开. 假设当  $\varepsilon \rightarrow 0^+$  时方程(9)的极限存在, 方程式可以满足, 所以基于  $\varepsilon$  的阶数可以将方程(9)排列为三级分层方程:

$$\varepsilon^{-2}: \quad \frac{1}{\varepsilon^2} \frac{\partial}{\partial y_j} \left( E_{ijmn}^\varepsilon \frac{\partial u_m^0}{\partial y_n} \right) = 0, \quad (10)$$

$$\varepsilon^{-1}: \quad \frac{1}{\varepsilon} \frac{\partial}{\partial y_j} \left[ E_{ijmn}^\varepsilon \left( \frac{\partial u_m^0}{\partial x_n} + \frac{\partial u_m^1}{\partial y_n} - a_{mn}^\varepsilon T_{mn} \right) \right] = 0, \quad (11)$$

$$\varepsilon^0: \quad -\frac{\partial}{\partial x_j} \left[ E_{ijmn}^\varepsilon \left( \frac{\partial u_m^0}{\partial x_n} + \frac{\partial u_m^1}{\partial y_n} - a_{mn}^\varepsilon T_{mn} \right) \right] - \frac{\partial}{\partial y_j} \left[ E_{ijmn}^\varepsilon \left( \frac{\partial u_m^1}{\partial x_n} + \frac{\partial u_m^2}{\partial y_n} \right) \right] = f_i, \quad (12)$$

$$\varepsilon^1: \quad \varepsilon \left\{ \frac{\partial}{\partial x_j} \left[ E_{ijmn}^\varepsilon \left( \frac{\partial u_m^1}{\partial x_n} + \frac{\partial u_m^2}{\partial y_n} \right) \right] + \frac{\partial}{\partial y_j} \left[ E_{ijmn}^\varepsilon \left( \frac{\partial u_m^2}{\partial x_n} + \frac{\partial u_m^3}{\partial y_n} \right) \right] \right\} = 0. \quad (13)$$

## 1.2 求解分级方程

分别对方程(10)~(12)进行求解, 可得到如下方程.

### 1.2.1 $\varepsilon^{-2}$

通过在方程(10)两端同时乘以  $\varepsilon^2$ , 并考虑  $\varepsilon \rightarrow 0^+$  的极限情况, 可以得到如下方程:

$$\frac{\partial}{\partial y_j} \left( E_{ijmn}^\varepsilon \frac{\partial u_m^0}{\partial y_n} \right) = 0. \quad (14)$$

方程(14)成立的前提是  $\partial u_m^0 / \partial y_n = 0$ , 所以  $u_m^0 = u_m^0(x)$ , 即均匀化位移是宏观尺度  $x$  的方程, 与微观尺度  $y$  无关.

### 1.2.2 $\varepsilon^{-1}$

通过在方程(11)两端同时乘以  $\varepsilon$ , 并考虑  $\varepsilon \rightarrow 0^+$  的极限情况, 可以得到如下方程:

$$\frac{\partial}{\partial y_j} \left[ E_{ijmn}^\varepsilon \left( \frac{\partial u_m^0}{\partial x_n} + \frac{\partial u_m^1}{\partial y_n} - a_{mn}^\varepsilon T \right) \right] = 0. \quad (15)$$

$u_m^1$  为一阶摄动位移,使用变量分离得到方程(15)的解为

$$u_m^1(x, y) = -\chi_{1m}^{kl}(y) \frac{\partial u_k^0(x)}{\partial x_l} - \psi_{1m}(y) T(x), \quad (16)$$

其中  $\chi_{1m}^{kl}$  和  $\psi_{1m}$  分别表示一阶弹性影响函数和一阶热影响函数,  $u_k^0$  为结构均匀化位移(宏观位移场),  $T$  为结构的温差(宏观温度场的一阶导数).

将式(16)代入方程(15)可得

$$\frac{\partial}{\partial y_j} \left[ \left( E_{ijkl}^\varepsilon - E_{ijmn}^\varepsilon \frac{\partial \chi_{1m}^{kl}}{\partial y_n} \right) \frac{\partial u_k^0(x)}{\partial x_l} - \left( E_{ijkl}^\varepsilon a_{kl}^\varepsilon + E_{ijmn}^\varepsilon \frac{\partial \psi_{1m}^{kl}}{\partial y_n} \right) T_{kl}(x) \right] = 0. \quad (17)$$

方程(17)可以解耦成两个分别用于求解  $\chi_1$  和  $\psi_1$  的控制方程:

一阶弹性影响函数控制方程

$$\frac{\partial}{\partial y_j} \left( E_{ijkl}^\varepsilon - E_{ijmn}^\varepsilon \frac{\partial \chi_{1m}^{kl}}{\partial y_n} \right) = 0; \quad (18)$$

一阶热问题影响函数控制方程

$$\frac{\partial}{\partial y_j} \left( E_{ijkl}^\varepsilon a_{kl}^\varepsilon + E_{ijmn}^\varepsilon \frac{\partial \psi_{1m}^{kl}}{\partial y_n} \right) = 0. \quad (19)$$

由方程(18)和(19)可以得到

$$\frac{\partial \psi_{1m}^{kl}}{\partial y_n} = -a_{mn}^\varepsilon \frac{\partial \chi_{1m}^{kl}}{\partial y_n}.$$

### 1.2.3 $\varepsilon^0$

将方程(12)转化为如下形式:

$$-\frac{\partial}{\partial y_j} \left( E_{ijmn}^\varepsilon \frac{\partial u_m^2}{\partial y_n} \right) = \frac{\partial}{\partial x_j} \left[ E_{ijmn}^\varepsilon \left( \frac{\partial u_m^0}{\partial x_n} + \frac{\partial u_m^1}{\partial y_n} - a_{mn}^\varepsilon T_{mn} \right) \right] + \frac{\partial}{\partial y_j} \left( E_{ijmn}^\varepsilon \frac{\partial u_m^1}{\partial x_n} \right) + f_i. \quad (20)$$

为了使方程(20)具有适定性,方程右端在单胞域内的均值必须为0,即

$$\frac{1}{|Y|} \int_Y \left\{ \frac{\partial}{\partial x_j} \left[ E_{ijmn}^\varepsilon \left( \frac{\partial u_m^0}{\partial x_n} + \frac{\partial u_m^1}{\partial y_n} - a_{mn}^\varepsilon T_{mn} \right) \right] + \frac{\partial}{\partial y_j} \left( E_{ijmn}^\varepsilon \frac{\partial u_m^1}{\partial x_n} \right) + f_i \right\} dY = 0. \quad (21)$$

将方程(16)代入方程(21)可以得到方程如下:

$$-\frac{1}{|Y|} \int_Y \left[ \left( E_{ijkl}^\varepsilon - E_{ijmn}^\varepsilon \frac{\partial \chi_{1m}^{kl}}{\partial y_n} \right) \frac{\partial^2 u_k^0(x)}{\partial x_l \partial x_j} - \left( E_{ijkl}^\varepsilon a_{kl}^\varepsilon + E_{ijmn}^\varepsilon \frac{\partial \psi_{1m}^{kl}}{\partial y_n} \right) \frac{\partial T_{kl}}{\partial x_j} \right] dY = f_i. \quad (22)$$

均匀化弹性常数张量和均匀化热弹性常数张量分别由如下公式计算:

$$E_{ijmn}^H = \frac{1}{|Y|} \int_Y \left( E_{ijkl}^\varepsilon - E_{ijmn}^\varepsilon \frac{\partial \chi_{1m}^{kl}}{\partial y_n} \right) dY, \quad (23)$$

$$\beta^H = \frac{1}{|Y|} \int_Y \left( E_{ijkl}^\varepsilon a_{kl}^\varepsilon + E_{ijmn}^\varepsilon \frac{\partial \psi_{1m}^{kl}}{\partial y_n} \right) dY = \frac{1}{|Y|} \int_Y \left( E_{ijkl}^\varepsilon a_{kl}^\varepsilon - E_{ijmn}^\varepsilon a_{mn}^\varepsilon \frac{\partial \chi_{1m}^{kl}}{\partial y_n} \right) dY. \quad (24)$$

将方程(23)和(24)代入方程(5),得到均匀化结构平衡方程:

$$-\frac{\partial}{\partial x_j} \left( E_{ijmn}^H \left( \frac{\partial u_m^0}{\partial x_n} - a_{mn}^H T_{mn} \right) \right) = f_i, \quad (25)$$

其中

$$a_{mn}^H = [E_{ijmn}^H]^{-1} \beta_{mn}^H,$$

$\alpha_{mn}^H$  为均匀化热膨胀常数张量。

$u_k^2$  为二阶摄动位移, 与一阶摄动位移相同可以变量分离为

$$u_m^2(x, y) = -\chi_{2m}^{klp}(y) \frac{\partial^2 u_k^0(x)}{\partial x_l \partial x_p} - \psi_{2m}^p(y) \frac{\partial T(x)}{\partial x_p}. \quad (26)$$

将方程(16)、(25)和(26)代入方程(20)可得

$$\begin{aligned} \frac{\partial}{\partial y_j} \left( E_{ijmn}^\varepsilon \frac{\partial \chi_{2m}^{klp}}{\partial y_n} \right) \frac{\partial^2 u_k^0}{\partial x_l \partial x_p} + \frac{\partial}{\partial y_j} \left( E_{ijmn}^\varepsilon \frac{\partial \psi_{2m}^{klp}}{\partial y_n} \right) \frac{\partial T_{kl}}{\partial x_p} = \\ \left( E_{ipkl}^\varepsilon - E_{ijkl}^H - \frac{\partial}{\partial y_j} (E_{ijmp}^\varepsilon \chi_{1k}^{kl}) - E_{ipmn}^\varepsilon \frac{\partial \chi_{1m}^{kl}}{\partial y_n} \right) \frac{\partial^2 u_k^0}{\partial x_l \partial x_p} - E_{ipmn}^\varepsilon \frac{\partial \psi_{1m}}{\partial y_n} \frac{\partial T_{kl}}{\partial x_p} - \\ E_{ipkl}^\varepsilon \alpha_{mn}^\varepsilon \frac{\partial T_{kl}}{\partial x_p} - \frac{\partial}{\partial y_j} (E_{ijmp}^\varepsilon \psi_{1m}^{kl}) \frac{\partial T_{kl}}{\partial x_p} + E_{ijkl}^H \alpha_{kl}^H \frac{\partial T_{kl}}{\partial x_p}. \end{aligned} \quad (27)$$

方程(27)可以解耦为分别求解  $\chi_2$  和  $\psi_2$  的控制方程:

二阶弹性影响函数控制方程

$$\frac{\partial}{\partial y_j} \left( E_{ijmn}^\varepsilon \frac{\partial \chi_{2m}^{klp}}{\partial y_n} \right) = -E_{ipmn}^\varepsilon \frac{\partial \chi_{1m}^{kl}}{\partial y_n} - \frac{\partial}{\partial y_j} (E_{ijmp}^\varepsilon \chi_{1m}^{kl}) + E_{ipkl}^\varepsilon - E_{ipkl}^H; \quad (28)$$

二阶热影响函数控制方程

$$\frac{\partial}{\partial y_j} \left( E_{ijmn}^\varepsilon \frac{\partial \psi_{2m}^{klp}}{\partial y_n} \right) = -E_{ipmn}^\varepsilon \frac{\partial \psi_{1m}^{kl}}{\partial y_n} - \frac{\partial}{\partial y_j} (E_{ijmp}^\varepsilon \psi_{1m}^{kl}) - E_{ipkl}^\varepsilon \alpha_{kl}^\varepsilon + E_{ipkl}^H \alpha_{kl}^H. \quad (29)$$

方程(13)中的  $u_m^3$  为三阶摄动位移, 可以解耦为与一阶、二阶摄动位移相同的三阶弹性影响函数和宏观位移导数以及三阶热影响函数和温差导数的乘积:

$$u_m^3(x, y) = -\chi_{3m}^{klpq}(y) \frac{\partial^3 u_k^0(x)}{\partial x_l \partial x_p \partial x_q} - \psi_{3m}^{klpq}(y) \frac{\partial^2 T_{kl}(x)}{\partial x_p \partial x_q}. \quad (30)$$

将方程(16)、(26)以及(30)代入方程(13)可得

$$\begin{aligned} \left[ \left( E_{iqmp}^\varepsilon \chi_{1m}^{kl} + E_{iqmn}^\varepsilon \frac{\partial \chi_{2m}^{klp}}{\partial y_n} \right) + \frac{\partial}{\partial y_j} \left( E_{ijmp}^\varepsilon \chi_{2m}^{klp} + E_{ijmn}^\varepsilon \frac{\partial \chi_{3m}^{klpq}}{\partial y_n} \right) \right] \frac{\partial^3 u_k^0}{\partial x_l \partial x_p \partial x_q} + \\ \left[ \left( E_{iqmp}^\varepsilon \psi_{1m}^{kl} + E_{iqmn}^\varepsilon \frac{\partial \psi_{2m}^{klp}}{\partial y_n} \right) + \frac{\partial}{\partial y_j} \left( E_{ijmp}^\varepsilon \psi_{2m}^{klp} + E_{ijmn}^\varepsilon \frac{\partial \psi_{3m}^{klpq}}{\partial y_n} \right) \right] \frac{\partial^2 T_{kl}}{\partial x_p \partial x_q} = 0. \end{aligned} \quad (31)$$

方程(31)可以解耦为分别求解  $\chi_3$  和  $\psi_3$  的控制方程:

三阶弹性影响函数控制方程

$$\left( E_{iqmp}^\varepsilon \chi_{1m}^{kl} + E_{iqmn}^\varepsilon \frac{\partial \chi_{2m}^{klp}}{\partial y_n} \right) + \frac{\partial}{\partial y_j} \left( E_{ijmp}^\varepsilon \chi_{2m}^{klp} + E_{ijmn}^\varepsilon \frac{\partial \chi_{3m}^{klpq}}{\partial y_n} \right) = 0; \quad (32)$$

三阶热影响函数控制方程

$$\left( E_{iqmp}^\varepsilon \psi_{1m}^{kl} + E_{iqmn}^\varepsilon \frac{\partial \psi_{2m}^{klp}}{\partial y_n} \right) + \frac{\partial}{\partial y_j} \left( E_{ijmp}^\varepsilon \psi_{2m}^{klp} + E_{ijmn}^\varepsilon \frac{\partial \psi_{3m}^{klpq}}{\partial y_n} \right) = 0. \quad (33)$$

使用相同的分析方法可以得到摄动位移和影响函数控制方程的递推形式 ( $s \geq 3$ ) 为

$$u_m^s(x, y) = -\chi_{sm}^{klpq\dots}(y) \frac{\partial^s u_k^0(x)}{\partial x_l \partial x_p \partial x_q \dots} - \psi_{sm}^{klpq\dots}(y) \frac{\partial^{s-1} T_{kl}(x)}{\partial x_p \partial x_q \dots}, \quad (34)$$

其中, 省略号“...”表示上标字母的个数与分母的求导阶次相对应:

$$u_m^s(x, y) = -\chi_{sm}^{klpqa\dots}(y) \frac{\partial^s u_k^0(x)}{\partial x_l \partial x_p \partial x_q \partial x_a \dots} - \psi_{sm}^{klpqa\dots}(y) \frac{\partial^{s-1} T_{kl}(x)}{\partial x_p \partial x_q \partial x_a \dots}.$$

s 阶弹性影响函数控制方程为

$$\left( E^\varepsilon \chi_{s-2} + E^\varepsilon \frac{\partial \chi_{s-1}}{\partial y_n} \right) + \frac{\partial}{\partial y_j} \left( E^\varepsilon \chi_{s-1} + E^\varepsilon \frac{\partial \chi_s}{\partial y_n} \right) = 0; \quad (35)$$

s 阶热影响函数控制方程为

$$\left( E^\varepsilon \psi_{s-2} + E^\varepsilon \frac{\partial \psi_{s-1}}{\partial y_n} \right) + \frac{\partial}{\partial y_j} \left( E^\varepsilon \psi_{s-1} + E^\varepsilon \frac{\partial \psi_s}{\partial y_n} \right) = 0. \quad (36)$$

## 2 有限元列式

使用加权残量方法将影响函数控制方程转化为易于编程计算的矩阵形式, 高阶 MMH 热-力耦合问题有限元计算过程和公式如下:

1) 单胞一阶影响函数控制方程

一阶弹性影响函数控制方程

$$\sum_{e=1} \left( \int_{Y^e} \mathbf{B}^T \mathbf{E}^\varepsilon \mathbf{B} dY^e \right) \chi_1^e = \sum_{e=1} \int_{Y^e} \mathbf{B}^T \mathbf{E}^\varepsilon dY^e; \quad (37)$$

一阶热影响函数控制方程

$$\sum_{e=1} \left( \int_{Y^e} \mathbf{B}^T \mathbf{E}^\varepsilon \mathbf{B} dY^e \right) \{ \psi_1^e \} = - \sum_{e=1} \int_{Y^e} \mathbf{B}^T \mathbf{E}^\varepsilon \{ a_{kl}^\varepsilon \} dY^e. \quad (38)$$

2) 均匀化参数矩阵列式

$$\begin{cases} \mathbf{E}^H = \sum_{e=1} \frac{1}{|Y|} \int_{Y^e} (\mathbf{E}^\varepsilon - \mathbf{E}^\varepsilon \mathbf{B} \chi_1^e) dY^e, \\ \{ \boldsymbol{\beta}_{kl}^H \} = \sum_{e=1} \frac{1}{|Y|} \int_{Y^e} (\mathbf{E}^\varepsilon \{ a_{kl}^\varepsilon \} + \mathbf{E}^\varepsilon \mathbf{B} \{ \psi_1^e \}) dY^e, \\ \{ a_{kl}^H \} = [\mathbf{E}^H]^{-1} \{ \boldsymbol{\beta}_{kl}^H \} = [a^H \quad a^H \quad 0]^T. \end{cases} \quad (39)$$

3) 均匀化结构控制方程

$$\sum_{e=1} \left( \int_{Y^e} \mathbf{B}^T \mathbf{E}^H \mathbf{B} dY^e \right) \mathbf{u}^0 = \sum_{e=1} \int_{Y^e} \mathbf{N}^T f dY^e + \sum_{e=1} \int_{Y^e} \mathbf{B}^T \mathbf{E}^H \boldsymbol{\varepsilon}_0 dY^e, \quad (40)$$

$$\boldsymbol{\varepsilon}_0 = a^H \sum_{i=1}^4 N_i T_i [1 \quad 1 \quad 0]^T.$$

4) 周期结构控制方程

$$\sum_{e=1} \left( \int_{Y^e} \mathbf{B}^T \mathbf{E}^\varepsilon \mathbf{B} dY^e \right) \mathbf{u}^\varepsilon = \sum_{e=1} \int_{Y^e} \mathbf{N}^T f dY^e + \sum_{e=1} \int_{Y^e} \mathbf{B}^T \mathbf{E}^\varepsilon \boldsymbol{\varepsilon}_0^\varepsilon dY^e, \quad (41)$$

$$\boldsymbol{\varepsilon}_0^\varepsilon = a^\varepsilon \sum_{i=1}^4 N_i T_i [1 \quad 1 \quad 0]^T.$$

5) 单胞二阶影响函数控制方程

二阶弹性影响函数控制方程

$$\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} \mathbf{B}^T \mathbf{E}^\varepsilon \mathbf{B} dY^\varepsilon \right) \chi_2^\varepsilon = \sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} \mathbf{N}^T [\mathbf{E}_1^H \quad \mathbf{E}_2^H] dY^\varepsilon \right) - \sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} \mathbf{N}^T [\mathbf{E}_1^\varepsilon \quad \mathbf{E}_2^\varepsilon] dY^\varepsilon \right) + \sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} \mathbf{N}^T [\mathbf{E}_1^\varepsilon \mathbf{B} \chi_1 \quad \mathbf{E}_2^\varepsilon \mathbf{B} \chi_1] dY^\varepsilon \right) - \sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} \mathbf{B}^T [\mathbf{E}_3^\varepsilon \mathbf{N} \chi_1 \quad \mathbf{E}_4^\varepsilon \mathbf{N} \chi_1] dY^\varepsilon \right); \quad (42)$$

二阶热影响函数控制方程

$$\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} \mathbf{B}^T \mathbf{E}^\varepsilon \mathbf{B} dY^\varepsilon \right) \boldsymbol{\psi}_2^\varepsilon =$$

$$\begin{aligned}
& \sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} N^T [E_1^\varepsilon \{a_{kl}^\varepsilon\} - E_1^H \{a_{kl}^H\} \quad E_2^\varepsilon \{a_{kl}^\varepsilon\} - E_2^H \{a_{kl}^H\}] dY^\varepsilon \right) + \\
& \sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} N^T [E_1^\varepsilon B \{\psi_1^\varepsilon\} \quad E_2^\varepsilon B \{\psi_1^\varepsilon\}] dY^\varepsilon \right) - \\
& \sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} B^T [E_3^\varepsilon N \{\psi_1^\varepsilon\} \quad E_4^\varepsilon N \{\psi_1^\varepsilon\}] dY^\varepsilon \right). \tag{43}
\end{aligned}$$

6) 单胞三阶影响函数控制方程

三阶弹性影响函数控制方程

$$\begin{aligned}
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} B^T E^\varepsilon B dY^\varepsilon \right) \chi_3^\varepsilon &= \sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} N^T [E_5^\varepsilon N \chi_1 \quad E_6^\varepsilon N \chi_1 \quad E_7^\varepsilon N \chi_1 \quad E_8^\varepsilon N \chi_1] dY^\varepsilon \right) + \\
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} N^T [E_1^\varepsilon B \chi_2 \quad E_2^\varepsilon B \chi_2] dY^\varepsilon \right) &- \sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} B^T [E_3^\varepsilon N \chi_2 \quad E_4^\varepsilon N \chi_2] dY^\varepsilon \right); \tag{44}
\end{aligned}$$

三阶热影响函数控制方程

$$\begin{aligned}
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} B^T E^\varepsilon B dY^\varepsilon \right) \psi_3^\varepsilon &= \\
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} N^T [E_5^\varepsilon N \{\psi_1\} \quad E_6^\varepsilon N \{\psi_1\} \quad E_7^\varepsilon N \{\psi_1\} \quad E_8^\varepsilon N \{\psi_1\}] dY^\varepsilon \right) &+ \\
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} N^T [E_1^\varepsilon B \psi_2 \quad E_2^\varepsilon B \psi_2] dY^\varepsilon \right) &- \sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} B^T [E_3^\varepsilon N \chi_2 \quad E_4^\varepsilon N \chi_2] dY^\varepsilon \right). \tag{45}
\end{aligned}$$

7)  $s$  单胞高阶影响函数控制方程

$s$  阶弹性影响函数控制方程

$$\begin{aligned}
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} B^T E^\varepsilon B dY^\varepsilon \right) \chi_n^\varepsilon &= \\
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} N^T [E_5^\varepsilon N \chi_{n-2} \quad E_6^\varepsilon N \chi_{n-2} \quad E_7^\varepsilon N \chi_{n-2} \quad E_8^\varepsilon N \chi_{n-2}] dY^\varepsilon \right) &+ \\
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} N^T [E_1^\varepsilon B \chi_{n-1} \quad E_2^\varepsilon B \chi_{n-1}] dY^\varepsilon \right) &- \\
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} B^T [E_3^\varepsilon N \chi_{n-1} \quad E_4^\varepsilon N \chi_{n-1}] dY^\varepsilon \right); \tag{46}
\end{aligned}$$

$s$  阶热影响函数控制方程

$$\begin{aligned}
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} B^T E^\varepsilon B dY^\varepsilon \right) \psi_n^\varepsilon &= \\
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} N^T [E_5^\varepsilon N \{\psi_{n-2}\} \quad E_6^\varepsilon N \{\psi_{n-2}\} \quad E_7^\varepsilon N \{\psi_{n-2}\} \quad E_8^\varepsilon N \{\psi_{n-2}\}] dY^\varepsilon \right) &+ \\
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} N^T [E_1^\varepsilon B \psi_{n-1} \quad E_2^\varepsilon B \psi_{n-1}] dY^\varepsilon \right) &- \\
\sum_{\varepsilon=1} \left( \int_{Y^\varepsilon} B^T [E_3^\varepsilon N \chi_{n-1} \quad E_4^\varepsilon N \chi_{n-1}] dY^\varepsilon \right), \tag{47}
\end{aligned}$$

其中, 矩阵  $E_1 \sim E_8$  如附录所示.

### 3 数学均匀化方法计算精度的关键影响因素

文献[31]给出了影响MHM计算精度的两个关键影响因素: 影响函数和宏观位移导数的计算精度. 其中影响函数的计算精度依赖于单胞边界条件施加的合理性, 常用的四边固支周期边界条件对于一维复合材料周期结构单胞问题是精确的, 但用于求解2D周期结构单胞问题影响函数的精度较低, 提出高精度的超单胞周期边界条件有效解决了该问题. 将微分求积有限单元与MHM结合形成微分求积MHM, 可有效提高宏观位移场和温度场各阶导数的求解精度.

### 3.1 超单胞周期边界条件

考虑 9 个单胞(3×3)组成的一个超单胞施加四边固支边界条件,如图 1(a)所示,具体实现过程如下:

- 1) 使用式(4)计算超单胞的一阶、二阶以及三阶虚拟位移;
- 2) 将超单胞包含的 9 个单胞根据其在超单胞中的位置分为 9 种单胞,分别为①~⑨,如图 1(b)所示,每个单胞均有独立的一阶、二阶以及三阶虚拟位移;
- 3) 将单胞⑤的一阶、二阶以及三阶虚拟位移复制到结构内部的每一个单胞上,①、②、③、④、⑥、⑦、⑧、⑨这 8 个单胞的一阶、二阶以及三阶虚拟位移根据其在超单胞中的位置分别对应复制给结构边界上的单胞,如图 1(c)所示。

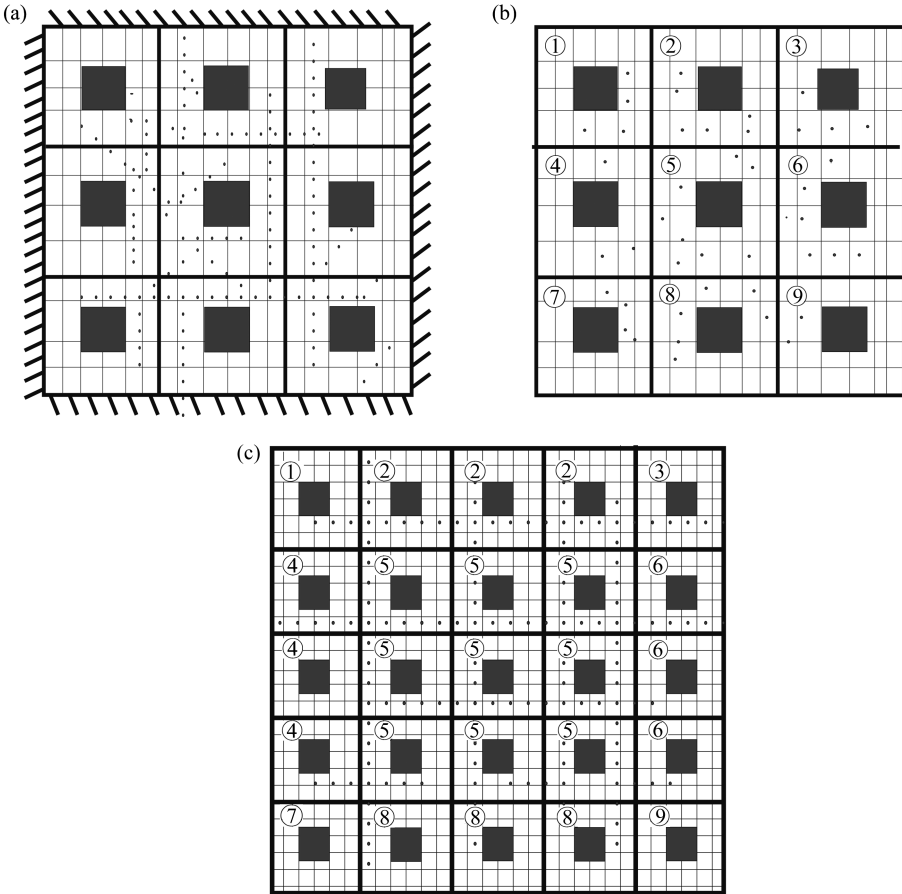


图 1 超单胞边界条件实现过程

Fig. 1 The procedure of the super unit cell boundary condition realization

### 3.2 微分求积 MHM

微分求积 MHM 首先用超单胞周期边界条件计算单胞问题各阶弹性影响函数和热影响函数,再构造高阶微分求解有限单元(如图 2 所示)求解宏观场及其各阶导数,最后得到 MHM 真实位移的方法.微分求积 MHM 的计算步骤如下:

- 1) 使用超单胞周期边界条件求解单胞问题,得到各阶弹性影响函数、热影响函数和均匀化弹性常数张量;



- 2) 分别构造宏观位移场和温度场高阶微分求积有限单元求解结构宏观场;
- 3) 使用微分求积法计算宏观位移场和温度场在高阶微分求积有限单元内部节点的各阶导数;
- 4) 使用 Lagrange 插值函数计算高阶微分求积有限单元内部与单胞内部节点相对应位置的各阶导数;
- 5) 计算结构的二阶 MHM 的真实位移.

微分求积单元平衡方程为

$$\mathbf{k}\mathbf{u}^0 = \mathbf{f}, \quad (48)$$

其中  $\mathbf{k}$  为微分求积单元刚度矩阵,  $\mathbf{f}$  为施加在结构上的外载荷和热载荷之和,  $\mathbf{u}^0$  为结构的均匀化位移. 均匀化位移的一阶导数、二阶以及三阶导数如下:

$$\begin{cases} \left. \frac{\partial u^0}{\partial x} \right|_{ij} = p_i^{(1)} q_j = \sum_{m=1}^{15} A_{im}^{(1)} p_m q_j = \sum_{m=1}^{15} A_{im}^{(1)} u_{mj}^0, \\ \left. \frac{\partial u^0}{\partial y} \right|_{ij} = p_i q_j^{(1)} = \sum_{m=1}^{15} B_{jn}^{(1)} q_n p_i = \sum_{n=1}^{15} B_{jn}^{(1)} u_{in}^0; \end{cases} \quad (49)$$

$$\begin{cases} \left. \frac{\partial^2 u^0}{\partial x^2} \right|_{ij} = p_i^{(2)} q_j = \sum_{m=1}^{15} A_{im}^{(2)} p_m q_j = \sum_{m=1}^{15} A_{im}^{(2)} u_{mj}^0, \\ \left. \frac{\partial^2 u^0}{\partial y^2} \right|_{ij} = p_i q_j^{(2)} = \sum_{n=1}^{15} B_{jn}^{(2)} q_n p_i = \sum_{n=1}^{15} B_{jn}^{(2)} u_{in}^0, \end{cases} \quad (50)$$

$$\begin{cases} \left. \frac{\partial^2 u^0}{\partial x \partial y} \right|_{ij} = p_i^{(1)} q_j^{(1)} = \sum_{m=1}^{15} A_{im}^{(1)} p_m \sum_{n=1}^{15} B_{jn}^{(1)} q_n = \sum_{m=1}^{15} A_{im}^{(1)} \sum_{n=1}^{15} B_{jn}^{(1)} u_{mn}^0; \\ \left. \frac{\partial^3 u^0}{\partial x^3} \right|_{ij} = p_i^{(3)} q_j = \sum_{m=1}^{15} A_{im}^{(3)} p_m q_j = \sum_{m=1}^{15} A_{im}^{(3)} u_{mj}^0, \\ \left. \frac{\partial^3 u^0}{\partial y^3} \right|_{ij} = p_i q_j^{(3)} = \sum_{n=1}^{15} B_{jn}^{(3)} q_n p_i = \sum_{n=1}^{15} B_{jn}^{(3)} u_{in}^0, \\ \left. \frac{\partial^3 u^0}{\partial x^2 \partial y} \right|_{ij} = p_i^{(2)} q_j^{(1)} = \sum_{m=1}^{15} A_{im}^{(2)} p_m \sum_{n=1}^{15} B_{jn}^{(1)} q_n = \sum_{m=1}^{15} A_{im}^{(2)} \sum_{n=1}^{15} B_{jn}^{(1)} u_{mn}^0, \\ \left. \frac{\partial^3 u^0}{\partial x \partial y^2} \right|_{ij} = p_i^{(1)} q_j^{(2)} = \sum_{m=1}^{15} A_{im}^{(1)} p_m \sum_{n=1}^{15} B_{jn}^{(2)} q_n = \sum_{m=1}^{15} A_{im}^{(1)} \sum_{n=1}^{15} B_{jn}^{(2)} u_{mn}^0. \end{cases} \quad (51)$$

使用 Lagrange 插值结构内任意一点的位移及其导数计算公式为

$$\begin{cases} [u(x, y), v(x, y)] = \sum_{i=1}^M \sum_{j=1}^N l_i(x) l_j(y) [u_{ij}, v_{ij}], \\ [u^{(1)}(x, y), v^{(1)}(x, y)] = \sum_{i=1}^M \sum_{j=1}^N l_i(x) l_j(y) [u_{ij}^{(1)}, v_{ij}^{(1)}], \\ \left[ u^{(2)}(x, y), v^{(2)}(x, y), \frac{\partial^2 u}{\partial x \partial y} \right] = \sum_{i=1}^M \sum_{j=1}^N l_i(x) l_j(y) \left[ u_{ij}^{(2)}, v_{ij}^{(2)}, \frac{\partial^2 u^0}{\partial x \partial y} \right], \\ \left[ u^{(3)}(x, y), v^{(3)}(x, y), \frac{\partial^3 u}{\partial x^2 \partial y}, \frac{\partial^3 u}{\partial x \partial y^2} \right] = \\ \sum_{i=1}^M \sum_{j=1}^N l_i(x) l_j(y) \left[ u_{ij}^{(3)}, v_{ij}^{(3)}, \frac{\partial^3 u}{\partial x^2 \partial y}, \frac{\partial^3 u}{\partial x \partial y^2} \right]. \end{cases} \quad (52)$$

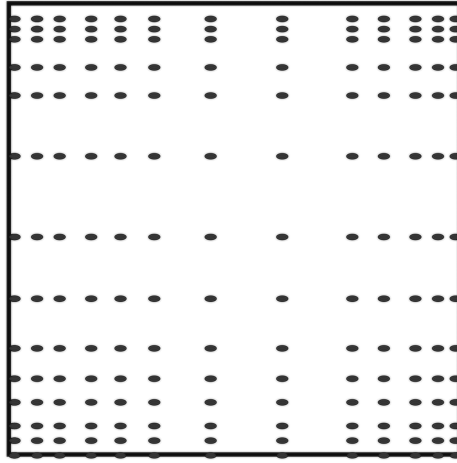


图2 微分求积有限单元

Fig. 2 The differential quadrature finite element

### 3.3 势能泛函

势能泛函越小说明计算结果越精确,为了准确评估各阶摄动的求解精度,计算周期复合材料在温度载荷作用下  $MHM1(u^0 + u^1)$ 、 $MHM2(u^0 + u^1 + u^2)$ 、 $MHM3(u^0 + u^1 + u^2 + u^3)$  和 FEM 的总势能泛函  $\Pi$  的计算公式为

$$\Pi = U + W, U = \frac{1}{2}(\mathbf{u}^e)^T \mathbf{K} \mathbf{u}^e, W = -(\mathbf{u}^e)^T \mathbf{f}, \quad (53)$$

其中  $U$  和  $W$  分别为应变能和外力势能,  $\mathbf{K}$  为结构刚度矩阵,  $\mathbf{f}$  为外载荷,本文算例中没有在结构中施加外载荷,把温度应变作为结构的初始应变,初始应变作为外载荷作用下的应变,即

$$\begin{cases} \mathbf{e} = \mathbf{a}_{mn}^e T, \\ \mathbf{f} = E_{ijmn} \mathbf{e}. \end{cases} \quad (54)$$

## 4 算例分析

### 4.1 二维周期复合材料单胞结构

结构大小为  $30 \text{ mm} \times 30 \text{ mm}$ ,单胞内部含有 4 块夹杂,如图 3 所示,基体的弹性模量和热膨胀系数分别为  $E_1 = 2 \times 10^9 \text{ Pa}$ ,  $a_1 = 3 \times 10^{-6} \text{ } ^\circ\text{C}^{-1}$ ,夹杂的弹性模量和热膨胀系数分别为  $E_2 = 6 \times 10^{10} \text{ Pa}$ ,  $a_2 = 1 \times 10^{-6} \text{ } ^\circ\text{C}^{-1}$ ,基体和夹杂的 Poisson 比均为  $\mu = 0.2$ ,结构温差  $T = 10^6(x^2 + y^2) \text{ } ^\circ\text{C}$ .

使用式(36)分别计算均匀化弹性模量和均匀化热膨胀系数为  $E^H = 2.67 \times 10^9 \text{ Pa}$ ,  $a^H = 2.5 \times 10^{-6} \text{ } ^\circ\text{C}^{-1}$ . 图 4(a)~(d) 分别给出结构沿 A、B、C、D 线上节点在  $x_1$  方向的位移曲线,表 1 给出了结构总势能泛函计算结果,其中 FEM 为有限元解,作为标准检验 MHM 的求解精度,MHM1 表示包含一阶摄动项的 MHM 计算结果,MHM2 表示包含二阶摄动项的 MHM 计算结果,

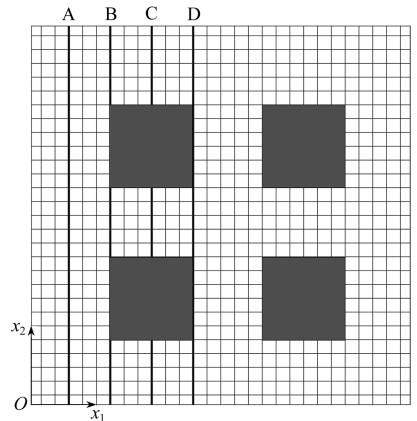


图3 二维复合材料单胞结构

Fig. 3 The unit cell of the 2D composite structure

MHM3 表示包含三阶摄动项的 MHM 计算结果。

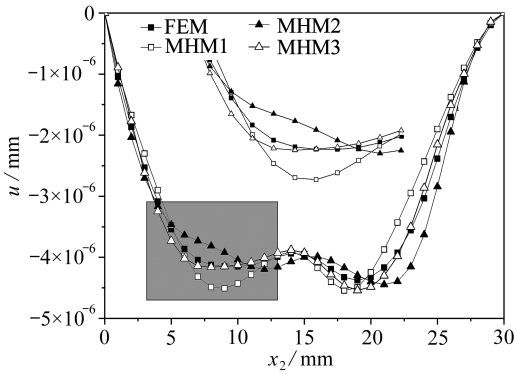
结合图 4 和表 1 可以得到：

- 1) 包含一阶摄动位移的计算结果 MHM1 误差较大；
- 2) MHM2 和 MHM3 的精度明显高于 MHM1；
- 3) MHM3 精度高于 MHM2。

表 1 2D 单胞结构的势能泛函

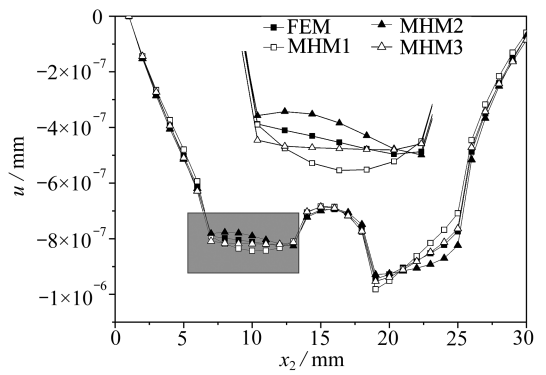
Table 1 The potential energy functional for the 2D unit-cell structure

$u^e$	$\Pi/J$		$((\Pi_{FEM} - \Pi_{MHM})/\Pi_{FEM} \times 100\%) / \%$
	MHM	FEM	
MHM1	$-2.475 \times 10^{-6}$	$-2.939 \times 10^{-6}$	15.8
MHM2	$-2.916 \times 10^{-6}$	$-2.939 \times 10^{-6}$	0.78
MHM3	$-2.937 \times 10^{-6}$	$-2.939 \times 10^{-6}$	0.068



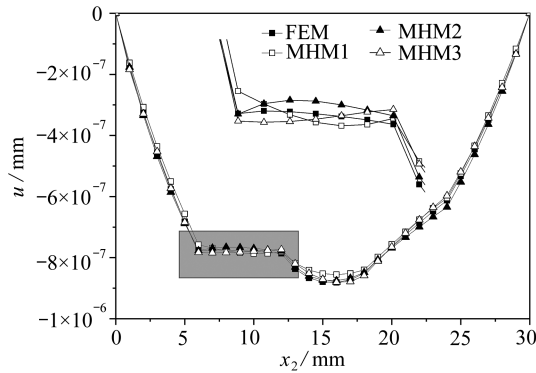
(a) A 线节点沿  $x_1$  方向位移曲线

(a) Displacement curves along longitudinal lines A



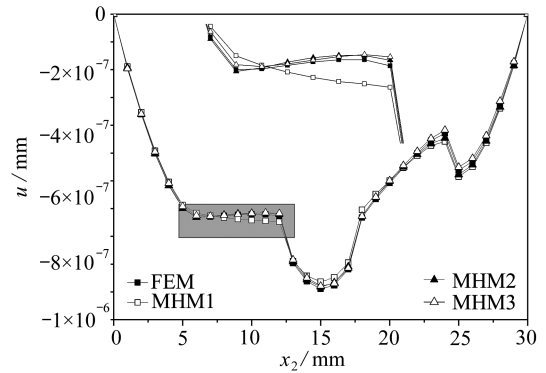
(b) B 线节点沿  $x_1$  方向位移曲线

(b) Displacement curves along longitudinal lines B



(c) C 线节点沿  $x_1$  方向位移曲线

(c) Displacement curves along longitudinal lines C



(d) D 线节点沿  $x_1$  方向位移曲线

(d) Displacement curves along longitudinal lines D

图 4 沿纵线 A、B、C、D 线上节点位移曲线

Fig. 4 Displacement curves along longitudinal lines A, B, C and D

## 4.2 二维周期复合材料多胞结构

结构大小为  $45 \text{ mm} \times 45 \text{ mm}$ ，包含  $5 \times 5$  个单胞，每个单胞内部含有一块夹杂，如图 5 所示，材料参数以及温差和算例 1 相同。影响函数和宏观位移导数分别使用超单胞周期边界条件和微分求积有限单元法计算得到，均匀化弹性模量和均匀化热膨胀系数分别为  $E^H = 2.42 \times 10^9 \text{ Pa}$ ，

$a^H = 2.63 \times 10^{-6} \text{ } ^\circ\text{C}^{-1}$ .图 6(a) ~ (d) 分别给出结构沿 A、B、C、D 线上节点  $x_1$  方向的位移曲线.表 2 给出了结构的总势能泛函计算结果,FEM、MHM1、MHM2、MHM3 与算例 1 相同.

结合图 6 和表 2 可以得到:

- 1) 包含一阶摄动位移的计算结果 MHM1 误差较大;
- 2) MHM2 和 MHM3 的计算精度明显好于 MHM1;
- 3) MHM2 和 MHM3 的计算精度较高,且两者相差不大;
- 4) 二阶摄动项对计算精度的影响较大,而三阶摄动项对计算精度的影响可以忽略.

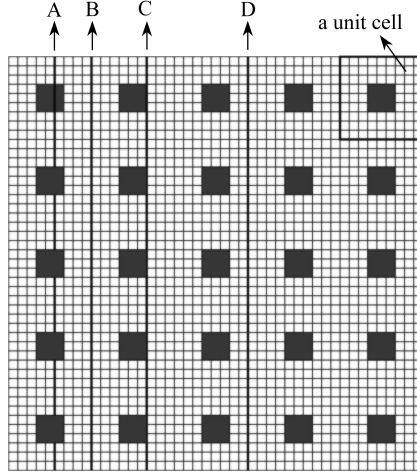


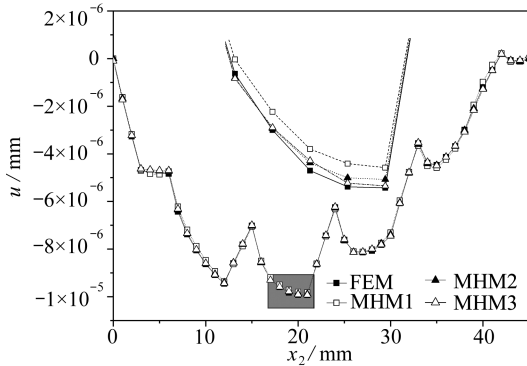
图 5 二维周期复合材料结构

Fig. 5 The 2D periodical composite structure

表 2 2D 多胞结构的势能泛函

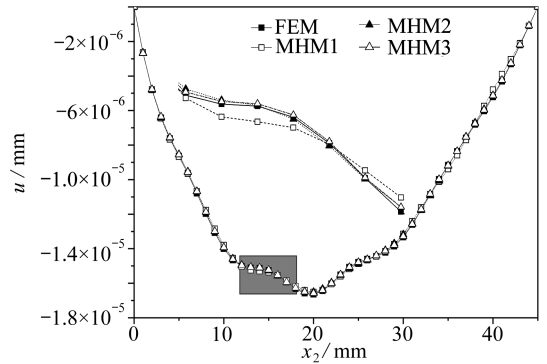
Table 2 The potential energy functional for the 2D multi-cell structure

$u^e$	$\Pi / J$		$((\Pi_{\text{FEM}} - \Pi_{\text{MHM}}) / \Pi_{\text{FEM}} \times 100\%) / \%$
	MHM	FEM	
MHM1	$-3.5477 \times 10^{-6}$	$-4.4361 \times 10^{-6}$	20
MHM2	$-4.3356 \times 10^{-6}$	$-4.4361 \times 10^{-6}$	2.27
MHM3	$-4.3968 \times 10^{-6}$	$-4.4361 \times 10^{-6}$	0.89



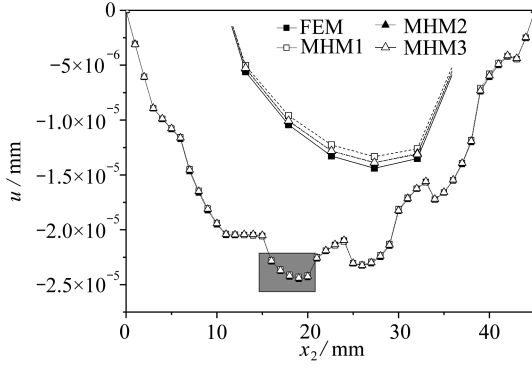
(a) A 线节点沿  $x_1$  方向位移曲线

(a) Displacement curves along longitudinal lines A

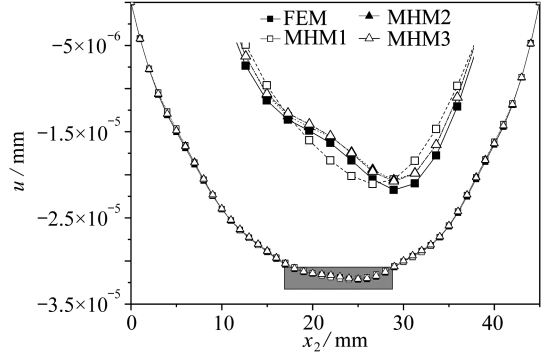


(b) B 线节点沿  $x_1$  方向位移曲线

(b) Displacement curves along longitudinal lines B

(c) C线节点沿  $x_1$  方向位移曲线

(c) Displacement curves along longitudinal lines C

(d) D线节点沿  $x_1$  方向位移曲线

(d) Displacement curves along longitudinal lines D

图6 沿纵线 A、B、C、D 上节点位移曲线

Fig. 6 Displacement curves along longitudinal lines A, B, C and D

## 5 结 论

本文推导了热力耦合问题高阶 MHM 通用计算公式,并将其转化为有限元列式,结合超单胞周期边界条件提出了微分求积 MHM,通过提高影响函数和宏观场导数计算精度保证了高阶展开本身的计算精度;数值算例和最小总势能泛函结果验证了公式的正确性和微分求积 MHM 的有效性.本文得到的结论如下:

- 1) 弹性影响系数和热影响系数的导数是线性关系,斜率为材料热膨胀系数;
- 2) 高阶(三阶及以上)影响函数控制方程具有相同的形式;
- 3) 二阶摄动项对计算精度作用较大,不可忽略.

## 附 录

对于平面线性单元来说,  $\chi_1$  为  $8 \times 3$  阶矩阵,  $\chi_2$  为  $8 \times 6$  阶矩阵,  $\mathbf{E}_i^H (i = 1, 2)$ ,  $\mathbf{E}_i^e (i = 1, 2)$  为  $2 \times 3$  阶矩阵,  $\mathbf{E}_i^e (i = 3, 4)$  为  $3 \times 2$  阶矩阵,其中各矩阵元素均是按照式(23)和(27)中各指标的求和关系生产.  $\mathbf{E}_i^H (i = 1, 2)$  和  $\mathbf{E}_i^e (i = 1, 2, 3, 4, 5, 6, 7, 8)$  分别为

$$\mathbf{E}_1^H = \begin{bmatrix} E_{1111}^H & E_{1122}^H & E_{1112}^H \\ E_{1211}^H & E_{1222}^H & E_{1212}^H \end{bmatrix}, \mathbf{E}_2^H = \begin{bmatrix} E_{1211}^H & E_{1222}^H & E_{1212}^H \\ E_{2211}^H & E_{2222}^H & E_{2212}^H \end{bmatrix},$$

$$\mathbf{E}_1^e = \begin{bmatrix} E_{1111}^e & E_{1122}^e & E_{1112}^e \\ E_{1211}^e & E_{1222}^e & E_{1212}^e \end{bmatrix}, \mathbf{E}_2^e = \begin{bmatrix} E_{1211}^e & E_{1222}^e & E_{1212}^e \\ E_{2211}^e & E_{2222}^e & E_{2212}^e \end{bmatrix},$$

$$\mathbf{E}_3^e = \begin{bmatrix} E_{1111}^e & E_{1112}^e \\ E_{2211}^e & E_{2212}^e \end{bmatrix}, \mathbf{E}_4^e = \begin{bmatrix} E_{1112}^e & E_{1122}^e \\ E_{2212}^e & E_{2222}^e \end{bmatrix},$$

$$\mathbf{E}_5^e = \begin{bmatrix} E_{1111}^e & E_{1112}^e \\ E_{1211}^e & E_{1212}^e \end{bmatrix}, \mathbf{E}_6^e = \begin{bmatrix} E_{1112}^e & E_{1122}^e \\ E_{1212}^e & E_{1222}^e \end{bmatrix}, \mathbf{E}_7^e = \begin{bmatrix} E_{1211}^e & E_{1212}^e \\ E_{2211}^e & E_{2212}^e \end{bmatrix}, \mathbf{E}_8^e = \begin{bmatrix} E_{1212}^e & E_{1222}^e \\ E_{2212}^e & E_{2222}^e \end{bmatrix}.$$

由弹性矩阵和弹性张量分量的对应关系,可以将弹性矩阵  $\mathbf{E}^H$  和  $\mathbf{E}^e$  写成

$$\mathbf{E}^H = \begin{bmatrix} E_{1111}^H & E_{1122}^H & E_{1112}^H \\ E_{2211}^H & E_{2222}^H & E_{2212}^H \\ E_{1211}^H & E_{1222}^H & E_{1212}^H \end{bmatrix}, \mathbf{E}^e = \begin{bmatrix} E_{1111}^e & E_{1122}^e & E_{1112}^e \\ E_{2211}^e & E_{2222}^e & E_{2212}^e \\ E_{1211}^e & E_{1222}^e & E_{1212}^e \end{bmatrix}.$$

所以  $\mathbf{E}_i^H (i = 1, 2)$  和  $\mathbf{E}_i^e (i = 1, 2, \dots, 8)$  可以用对应的弹性矩阵元素分别表示为

$$\begin{aligned} \mathbf{E}_1^H &= \begin{bmatrix} E_{(1,1)}^H & E_{(1,2)}^H & E_{(1,3)}^H \\ E_{(3,1)}^H & E_{(3,2)}^H & E_{(3,3)}^H \end{bmatrix}, \mathbf{E}_2^H = \begin{bmatrix} E_{(3,1)}^H & E_{(3,2)}^H & E_{(3,3)}^H \\ E_{(2,1)}^H & E_{(2,2)}^H & E_{(2,3)}^H \end{bmatrix}, \\ \mathbf{E}_1^e &= \begin{bmatrix} E_{(1,1)}^e & E_{(1,2)}^e & E_{(1,3)}^e \\ E_{(3,1)}^e & E_{(3,2)}^e & E_{(3,3)}^e \end{bmatrix}, \mathbf{E}_2^e = \begin{bmatrix} E_{(3,1)}^e & E_{(3,2)}^e & E_{(3,3)}^e \\ E_{(2,1)}^e & E_{(2,2)}^e & E_{(2,3)}^e \end{bmatrix}, \\ \mathbf{E}_3^e &= \begin{bmatrix} E_{(1,1)}^e & E_{(1,3)}^e \\ E_{(2,1)}^e & E_{(2,3)}^e \\ E_{(3,1)}^e & E_{(3,3)}^e \end{bmatrix}, \mathbf{E}_4^e = \begin{bmatrix} E_{(1,3)}^e & E_{(1,2)}^e \\ E_{(2,3)}^e & E_{(2,2)}^e \\ E_{(3,3)}^e & E_{(3,2)}^e \end{bmatrix}, \\ \mathbf{E}_5^e &= \begin{bmatrix} E_{(1,1)}^e & E_{(1,3)}^e \\ E_{(3,1)}^e & E_{(3,3)}^e \end{bmatrix}, \mathbf{E}_6^e = \begin{bmatrix} E_{(1,3)}^e & E_{(1,2)}^e \\ E_{(3,3)}^e & E_{(3,2)}^e \end{bmatrix}, \\ \mathbf{E}_7^e &= \begin{bmatrix} E_{(3,1)}^e & E_{(3,3)}^e \\ E_{(2,1)}^e & E_{(2,3)}^e \end{bmatrix}, \mathbf{E}_8^e = \begin{bmatrix} E_{(3,3)}^e & E_{(3,2)}^e \\ E_{(2,3)}^e & E_{(2,2)}^e \end{bmatrix}. \end{aligned}$$

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## Accuracy of the Mathematical Homogenization Method for Thermomechanical Problems

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**Abstract:** For thermo-mechanical problems of periodical composite structures, the full decoupled scheme of each order perturbation and the governing equation of each order influence function for the mathematical homogenization method (MHM) were derived, then the weighted residual method was utilized to transform them into the conveniently programmable finite element matrix form. The perturbation displacements in the uncoupled form were defined as the products of influence functions and the macro field derivatives, and the calculating accuracy of the perturbation displacements were determined by the accuracy of influence functions and the macro field derivatives, in turn the accuracy of influence functions depended mainly on the applicability of unit cell boundary conditions. For the static problems of 2D periodical composite structures, the super unit cell periodical boundary condition and the differential quadrature finite element method were applied to guarantee the calculating accuracy of the influence function and the macro field derivatives respectively. On this basis, the influence of the high-order perturbations on the true displacement of the MHM was studied, and the necessity of the 2nd-order perturbation was emphasized. Finally, the potential energy functional was used to evaluate the accuracy of the MHM. Numerical comparisons validate the conclusions.

**Key words:** mathematical homogenization method; periodical composite; thermo-mechanical; high-order perturbation