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二维各向同性多孔介质的弹性动力学通解*

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摘要: 在二维直角坐标系下,从固体位移和流体流速满足的基本方程出发,研究了二维各向同性多孔介质的弹性动力学通解.首先引入 4 个物理量,对固体骨架的运动方程、流体流速运动方程、连续性方程进行整理,将方程组分解成膨胀波和扭转波两部分,并利用 Lur'e 算子矩阵理论,获得由 3 个类调和函数表示的动力学通解,该通解满足全部基本方程.最后将时间项退化获得稳态通解,并证明了稳态通解的完备性.

关 键 词: 多孔介质; 通解; Lur'e 算子矩阵; 完备性; 类调和函数

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引 言

由固相、液相、气相以及其他相相互作用所形成的物质叫做多孔介质.在我们生活中多孔介质普遍存在,例如岩石、煤炭[1]等,所以对多孔介质性能的研究具有重大的意义和实用价值.

多孔介质的固相类似于人体的骨架支撑着整个空间,液相类似于人体的血液,气相以及其他相存在于固液两相之间.通常将除了固相以外的其他物质空间叫做孔隙,而多孔介质中的孔隙尺寸一般都极其微小,并且比表面积很大.多孔介质中的孔隙一般也分 3 类,分别为互相连通的孔隙、部分连通的孔隙和相互不连通的孔隙,在连通的孔隙中流体的运动一般是以渗流[2-3]方式为主.关于多孔介质的理论研究最早见于 Terzaghi 的研究,Biot 继承了 Terzaghi 的思想,把多孔介质的固结理论从一维扩展到了二维和三维,建立了比较完善的固结理论,给出了相应的微分方程组,李向约和李向维[4]研究了饱和多孔介质的热固结理论.

通解是指满足全部基本方程的各种解的集合,是解决具体工程问题的一种有效的数学工具,国内外学者对各种类型基本方程的通解进行了研究.郭时光[5]研究了 Poisson 方程 3 类问题的通解;王敏中和黄克服[6]研究了半空间的热弹性问题弹性通解及其应用;沈惠川[7]研究了弹性动力学的通解;陈伟球和丁皓江[8]研究了横观各向同性三维热弹性力学通解及其势理论法;徐颖等[9]研究了双孔介质弹性动力学通解及其完备性;Unger 和 Aifantis[10]研究了双孔材料通解的完备性;Zhao 和 Lu[11]研究了各向同性多孔介质的热弹性稳态通解.

本文求解了二维多孔介质的通解,第 1 节通过引入 4 个物理量,将基本方程转化成 Cauchy-Riemann 方程组的形式,并将方程分解为膨胀波和扭转波两部分;第 2 节利用 Lur'e 算

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子矩阵理论[12]推导出二维各向同性多孔介质的动力学通解;第 3 节将动态解退化时间项后,获得稳态通解,并证明了稳态通解的完备解;第 4 节对通解进行了讨论.

1 基本方程

在二维直角坐标系下,固体骨架的位移场和流体的速度场有如下形式:

$$\begin{cases} \boldsymbol{u} = \boldsymbol{i}u_x(x,y,t) + \boldsymbol{j}u_y(x,y,t), \\ \boldsymbol{V} = \boldsymbol{i}V_x(x,y,t) + \boldsymbol{j}V_y(x,y,t), \end{cases} \tag{1}$$

其中, $\boldsymbol{i}$, $\boldsymbol{j}$ 是 x,y 方向的单位向量, $\boldsymbol{u}$ 为固体位移, $\boldsymbol{V}$ 表示流体的流速.

根据 Zinkiewicz 等[13]的研究,对饱和多孔介质来说,在低频的情况下,可忽略流体相对于骨架运动的惯性项和流体的可压缩性,其满足下列基本方程:

固体骨架的运动方程

$$\begin{cases} \nabla^2 u_x + \dfrac{1}{1-2\nu}\dfrac{\partial}{\partial x}\left(\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y}\right) - \dfrac{1}{\mu}\dfrac{\partial p}{\partial x} - \dfrac{\rho}{\mu}\dfrac{\partial^2 u_x}{\partial t^2} = 0, \\ \nabla^2 u_y + \dfrac{1}{1-2\nu}\dfrac{\partial}{\partial y}\left(\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y}\right) - \dfrac{1}{\mu}\dfrac{\partial p}{\partial y} - \dfrac{\rho}{\mu}\dfrac{\partial^2 u_y}{\partial t^2} = 0; \end{cases} \tag{2}$$

流体的运动方程

$$\begin{cases} -\dfrac{\partial p}{\partial x} = \rho_{\mathrm{f}}\dfrac{\partial^2 u_x}{\partial t^2} + \dfrac{n}{k}V_x, \\ -\dfrac{\partial p}{\partial y} = \rho_{\mathrm{f}}\dfrac{\partial^2 u_y}{\partial t^2} + \dfrac{n}{k}V_y; \end{cases} \tag{3}$$

流体的连续性方程

$$\frac{\partial e}{\partial t} + \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0, \tag{4}$$

式中

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},\ e = \frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y,$$

ν 为 Poisson 比, p 为液体压力的变化量, ρ 为固体颗粒的密度, μ 为剪切模量, ρ_{f} 为流体的密度, k 为渗透系数,由于本文研究的多孔介质为均质材料,故 k 与坐标无关, n 为流体黏度系数,忽略流体黏度 n 随压力的变化.

为了更好地对基本方程进行分析,此处引入 4 个与 x,y,t 相关的函数 v_x,v_y,w_x,w_y, 令

$$\begin{cases} u_x = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y},\ u_y = \dfrac{\partial v_x}{\partial y} - \dfrac{\partial v_y}{\partial x}, \\ V_x = \dfrac{\partial w_x}{\partial x} + \dfrac{\partial w_y}{\partial y},\ V_y = \dfrac{\partial w_x}{\partial y} - \dfrac{\partial w_y}{\partial x}. \end{cases} \tag{5}$$

由方程(2)~(4)可得

$$\begin{cases} \dfrac{\partial}{\partial x}\left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 v_x - \dfrac{\rho}{\mu}\dfrac{\partial^2 v_x}{\partial t^2} - \dfrac{1}{\mu}p\right] + \dfrac{\partial}{\partial y}\left(\nabla^2 - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right)v_y = 0, \\ \dfrac{\partial}{\partial y}\left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 v_x - \dfrac{\rho}{\mu}\dfrac{\partial^2 v_x}{\partial t^2} - \dfrac{1}{\mu}p\right] - \dfrac{\partial}{\partial x}\left(\nabla^2 - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right)v_y = 0, \end{cases} \tag{6}$$

$$\begin{cases}\dfrac{\partial}{\partial x}\left(\rho_{\mathrm{f}}\dfrac{\partial^2 v_x}{\partial t^2}+p+\dfrac{n}{k}w_x\right)+\dfrac{\partial}{\partial y}\left(\rho_{\mathrm{f}}\dfrac{\partial^2 v_y}{\partial t^2}+\dfrac{n}{k}w_y\right)=0,\\ \dfrac{\partial}{\partial y}\left(\rho_{\mathrm{f}}\dfrac{\partial^2 v_x}{\partial t^2}+p+\dfrac{n}{k}w_x\right)-\dfrac{\partial}{\partial x}\left(\rho_{\mathrm{f}}\dfrac{\partial^2 v_y}{\partial t^2}+\dfrac{n}{k}w_y\right)=0,\end{cases}\tag{7}$$

$$\nabla^2\frac{\partial v_x}{\partial t}+\nabla^2 w_x=0.\tag{8}$$

由于式(6)和(7)均为 Cauchy-Riemann 方程组,则其可改写为

$$\begin{cases}\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 v_x-\dfrac{\rho}{\mu}\dfrac{\partial^2 v_x}{\partial t^2}-\dfrac{1}{\mu}p=f_1(x,y,t),\\ \left(\nabla^2-\dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right)v_y=f_2(x,y,t),\end{cases}\tag{9}$$

$$\begin{cases}\rho_{\mathrm{f}}\dfrac{\partial^2 v_x}{\partial t^2}+p+\dfrac{n}{k}w_x=g_1(x,y,t),\\ \rho_{\mathrm{f}}\dfrac{\partial^2 v_y}{\partial t^2}+\dfrac{n}{k}w_y=g_2(x,y,t),\end{cases}\tag{10}$$

其中$f_1(x,y,t)$,$f_2(x,y,t)$,$g_1(x,y,t)$,$g_2(x,y,t)$ 满足

$$\begin{cases}\dfrac{\partial f_1}{\partial x}+\dfrac{\partial f_2}{\partial y}=0,\ \dfrac{\partial f_1}{\partial y}-\dfrac{\partial f_2}{\partial x}=0,\ \dfrac{\partial g_1}{\partial x}+\dfrac{\partial g_2}{\partial y}=0,\ \dfrac{\partial g_1}{\partial y}-\dfrac{\partial g_2}{\partial x}=0,\\ \nabla^2 f_1=\nabla^2 f_2=0,\ \nabla^2 g_1=\nabla^2 g_2=0.\end{cases}\tag{11}$$

方程(9)和(10)有如下特解:

$$\begin{cases}v_x=-\dfrac{\mu}{\rho}\displaystyle\int_0^t\int_0^t f_1(x,y,t)\,\mathrm{d}t\mathrm{d}t,\\ v_y=-\dfrac{\mu}{\rho}\displaystyle\int_0^t\int_0^t f_2(x,y,t)\,\mathrm{d}t\mathrm{d}t,\\ w_x=\dfrac{k}{n}g_1(x,y,t)+\dfrac{\mu\rho_{\mathrm{f}}k}{\rho n}f_1(x,y,t),\\ w_y=\dfrac{k}{n}g_2(x,y,t)+\dfrac{\mu\rho_{\mathrm{f}}k}{n}f_2(x,y,t),\\ p=0.\end{cases}\tag{12}$$

把特解(12)代入式(2)可知,特解(12)不影响流场和位移场同时满足流体的连续性方程,故可舍去.即以下方程组成立:

$$\begin{cases}\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 v_x-\dfrac{\rho}{\mu}\dfrac{\partial^2 v_x}{\partial t^2}-\dfrac{1}{\mu}p=0,\\ \left(\nabla^2-\dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right)v_y=0,\end{cases}\tag{13}$$

$$\begin{cases}\rho_{\mathrm{f}}\dfrac{\partial^2 v_x}{\partial t^2}+p+\dfrac{n}{k}w_x=0,\\ \rho_{\mathrm{f}}\dfrac{\partial^2 v_y}{\partial t^2}+\dfrac{n}{k}w_y=0.\end{cases}\tag{14}$$

根据式(5)可以看出 v_x,w_x 和 p 与膨胀有关,其波动属于膨胀波,而 v_y 和 w_y 与扭转有关,其波动属于扭转波,两部分方程相互解耦,下面分别进行求解.

2 动力学通解

方程(13)第一式、方程(14)第一式、方程(8)属于膨胀波方程,可写成如下形式:

$$\boldsymbol{A}\boldsymbol{\eta} = \boldsymbol{0}, \tag{15}$$

其中

$$\boldsymbol{A} = \begin{pmatrix} \dfrac{2(1-\nu)}{1-2\nu}\nabla^2 - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2} & 0 & -\dfrac{1}{\mu} \\ \rho_{\mathrm{f}}\dfrac{\partial^2}{\partial t^2} & \dfrac{n}{k} & 1 \\ \nabla^2\dfrac{\partial}{\partial t} & \nabla^2 & 0 \end{pmatrix}, \tag{16}$$

$$\boldsymbol{\eta} = (v_x, w_x, p)^{\mathrm{T}}. \tag{17}$$

根据 Lur'e 算子理论[12]可知方程的解有如下形式:

$$\boldsymbol{\eta} = \boldsymbol{B}\boldsymbol{\varphi}, \tag{18}$$

其中, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)^{\mathrm{T}}$, 满足方程

$$|\boldsymbol{A}|\varphi_i = 0, \qquad i = 1,2,3, \tag{19}$$

其中, $\boldsymbol{B}$ 是算子矩阵 $\boldsymbol{A}$ 的伴随矩阵, $|\boldsymbol{A}|$ 为算子矩阵 $\boldsymbol{A}$ 的行列式,满足

$$|\boldsymbol{A}| = -\nabla^2\left[\frac{2(1-\nu)}{1-2\nu}\nabla^2 + \frac{\rho_{\mathrm{f}} - \rho}{\mu}\frac{\partial^2}{\partial t^2} - \frac{n}{\mu k}\frac{\partial}{\partial t}\right], \tag{20}$$

伴随矩阵 $\boldsymbol{B}$ 的各分量为

$$\begin{cases} B_{11} = -\nabla^2,\ B_{12} = -\nabla^2\dfrac{1}{\mu},\ B_{13} = \dfrac{n}{k\mu},\ B_{21} = \nabla^2\dfrac{\partial}{\partial t},\ B_{22} = \nabla^2\dfrac{1}{\mu}\dfrac{\partial}{\partial t}, \\ B_{23} = -\left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2} + \dfrac{\rho_{\mathrm{f}}}{\mu}\dfrac{\partial^2}{\partial t^2}\right],\ B_{31} = \nabla^2\left(\rho_{\mathrm{f}}\dfrac{\partial^2}{\partial t^2} - \dfrac{n}{k}\dfrac{\partial}{\partial t}\right), \\ B_{32} = -\nabla^2\left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right],\ B_{33} = \dfrac{n}{k}\left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right]. \end{cases} \tag{21}$$

将式(21)代入式(18)可得

$$\begin{cases} v_x = -\nabla^2\varphi_1 - \dfrac{1}{\mu}\nabla^2\varphi_2 + \dfrac{n}{k\mu}\varphi_3, \\ w_x = \nabla^2\dfrac{\partial\varphi_1}{\partial t} + \nabla^2\dfrac{1}{\mu}\dfrac{\partial\varphi_2}{\partial t} - \left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2} + \dfrac{\rho_{\mathrm{f}}}{\mu}\dfrac{\partial^2}{\partial t^2}\right]\varphi_3, \\ p = \nabla^2\left(\rho_{\mathrm{f}}\dfrac{\partial^2}{\partial t^2} - \dfrac{n}{k}\dfrac{\partial}{\partial t}\right)\varphi_1 - \nabla^2\left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right]\varphi_2 + \\ \qquad \dfrac{n}{k}\left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right]\varphi_3. \end{cases} \tag{22}$$

将式(20)代入式(19)可得

$$\nabla^2\left[\frac{2(1-\nu)}{1-2\nu}\nabla^2 + \frac{\rho_{\mathrm{f}} - \rho}{\mu}\frac{\partial^2}{\partial t^2} - \frac{n}{\mu k}\frac{\partial}{\partial t}\right]\varphi_i = 0, \qquad i = 1,2,3. \tag{23}$$

令

$$\xi_1 = \nabla^2\varphi_1 + \frac{\nabla^2\varphi_2}{\mu},\ \xi_2 = \varphi_3, \tag{24}$$

由式(23)可知

$$\left[\nabla^2\frac{2(1-\nu)}{1-2\nu} + \frac{\rho_f - \rho}{\mu}\frac{\partial^2}{\partial t^2} - \frac{n}{\mu k}\frac{\partial}{\partial t}\right]\xi_1 = 0, \tag{25}$$

$$\nabla^2\left[\nabla^2\frac{2(1-\nu)}{1-2\nu} + \frac{\rho_f - \rho}{\mu}\frac{\partial^2}{\partial t^2} - \frac{n}{\mu k}\frac{\partial}{\partial t}\right]\xi_2 = 0. \tag{26}$$

根据式(23),通解(22)可改写为

$$\begin{cases} v_x = -\xi_1 + \dfrac{n}{k\mu}\xi_2, \\ w_x = \nabla^2\dfrac{\partial\xi_1}{\partial t} - \left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2} + \dfrac{\rho_f}{\mu}\dfrac{\partial^2}{\partial t^2}\right]\xi_2, \\ p = \left(\rho_f\dfrac{\partial^2}{\partial t^2} - \dfrac{n}{k}\dfrac{\partial}{\partial t}\right)\xi_1 + \dfrac{n}{k}\left[\nabla^2\dfrac{2(1-\nu)}{1-2\nu} - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right]\xi_2. \end{cases} \tag{27}$$

v_y 和 w_y 满足方程(13)第二式和方程(14)第二式,这些方程属于扭转波方程,令

$$\xi_3 = v_y. \tag{28}$$

根据方程(13)第二式和方程(14)第二式,可知

$$w_y = -\frac{k\rho_f}{n}\frac{\partial^2\xi_3}{\partial t^2}, \tag{29}$$

$$\left(\nabla^2 - \frac{\rho}{\mu}\frac{\partial^2}{\partial t^2}\right)\xi_3 = 0. \tag{30}$$

从式(25)、(26)、(30)可以看出,退化时间项后,$\xi_i(i = 1,2,3)$ 具有调和函数或双调和函数,为了有所区分,称之为类调和函数.

将式(27)~(29)代入式(5),可以整理出动力学通解:

$$\begin{cases} u_x = -\dfrac{\partial\xi_1}{\partial x} + \dfrac{n}{k\mu}\dfrac{\partial\xi_2}{\partial x} + \dfrac{\partial\xi_3}{\partial y}, \\ u_y = -\dfrac{\partial\xi_1}{\partial y} + \dfrac{n}{k\mu}\dfrac{\partial\xi_2}{\partial y} - \dfrac{\partial\xi_3}{\partial x}, \\ V_x = \dfrac{\partial}{\partial t}\dfrac{\partial\xi_1}{\partial x} - \left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 + \dfrac{\rho_f - \rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right]\dfrac{\partial\xi_2}{\partial x} - \dfrac{k\rho_f}{n}\dfrac{\partial^2}{\partial t^2}\dfrac{\partial\xi_3}{\partial y}, \\ V_y = \dfrac{\partial}{\partial t}\dfrac{\partial\xi_1}{\partial y} - \left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 + \dfrac{\rho_f - \rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right]\dfrac{\partial\xi_2}{\partial y} + \dfrac{k\rho_f}{n}\dfrac{\partial^2}{\partial t^2}\dfrac{\partial\xi_3}{\partial x}, \\ p = \left(\rho_f\dfrac{\partial^2}{\partial t^2} - \dfrac{n}{k}\dfrac{\partial}{\partial t}\right)\xi_1 + \dfrac{n}{k}\left[\dfrac{2(1-\nu)}{1-2\nu}\nabla^2 - \dfrac{\rho}{\mu}\dfrac{\partial^2}{\partial t^2}\right]\xi_2, \end{cases} \tag{31}$$

其中,ξ_1,ξ_2 和 ξ_3 分别满足方程(25)、(26)和(30).把方程(31)代入固体骨架的运动方程(2)、流体的运动方程(3)、流体的连续性方程(4),方程皆成立,故方程(31)为弹性动力学通解.

3 稳态通解及完备性

将动力学通解(31)中的时间项退化,可得其稳态通解:

$$\begin{cases} u_x = -\dfrac{\partial \xi_1}{\partial x} + \dfrac{n}{k\mu}\dfrac{\partial \xi_2}{\partial x} + \dfrac{\partial \xi_3}{\partial y}, \\ u_y = -\dfrac{\partial \xi_1}{\partial y} + \dfrac{n}{k\mu}\dfrac{\partial \xi_2}{\partial y} - \dfrac{\partial \xi_3}{\partial x}, \\ V_x = -\dfrac{2(1-\nu)}{1-2\nu}\nabla^2\dfrac{\partial \xi_2}{\partial x}, \\ V_y = -\dfrac{2(1-\nu)}{1-2\nu}\nabla^2\dfrac{\partial \xi_2}{\partial y}, \\ p = \dfrac{2(1-\nu)n}{(1-2\nu)k}\nabla^2 \xi_2, \end{cases} \tag{32}$$

其中

$$\nabla^2 \xi_1 = 0, \ \nabla^2\nabla^2 \xi_2 = 0, \ \nabla^2 \xi_3 = 0. \tag{33}$$

由式(3)和(4)可知

$$\nabla^2 p = 0. \tag{34}$$

对任意 p 总可以找到 ξ_2 使

$$\bar{p} = \frac{2(1-\nu)n}{(1-2\nu)k}\nabla^2 \xi_2,$$

可获一组特解:

$$\begin{cases} \bar{u}_x = \dfrac{n}{k\mu}\dfrac{\partial \xi_2}{\partial x}, \\ \bar{u}_y = \dfrac{n}{k\mu}\dfrac{\partial \xi_2}{\partial y}, \\ \bar{p} = \dfrac{2(1-\nu)n}{(1-2\nu)k}\nabla^2 \xi_2, \\ \bar{V}_x = -\dfrac{2(1-\nu)}{1-2\nu}\nabla^2\dfrac{\partial \xi_2}{\partial x}, \\ \bar{V}_y = -\dfrac{2(1-\nu)}{1-2\nu}\nabla^2\dfrac{\partial \xi_2}{\partial y}. \end{cases} \tag{35}$$

令

$$\begin{cases} u_x^* = u_x - \bar{u}_x, \ u_y^* = u_y - \bar{u}_y, \\ V_x^* = V_x - \bar{V}_x, \ V_y^* = V_y - \bar{V}_y, \ p^* = p - \bar{p}, \end{cases} \tag{36}$$

由式(32)、(33)可知

$$V_x^* = 0, \ V_y^* = 0, \ p^* = 0.$$

把式(36)代入式(2)中得

$$\begin{cases} \nabla^2 u_x^* + \dfrac{1}{1-2\nu}\dfrac{\partial}{\partial x}\left(\dfrac{\partial u_x^*}{\partial x} + \dfrac{\partial u_y^*}{\partial y}\right) = 0, \\ \nabla^2 u_y^* + \dfrac{1}{1-2\nu}\dfrac{\partial}{\partial y}\left(\dfrac{\partial u_x^*}{\partial x} + \dfrac{\partial u_y^*}{\partial y}\right) = 0. \end{cases} \tag{37}$$

式(37)为二维弹性位移方程,与第三方向无关,如将这两个方程中的第二项系数 1/(1 -

2ν）替换为由横观各向同性材料常数表示的（$C_{12}+C_{66}$）$/C_{66}$，在数学形式上与横观各向同性材料在各向同性面内的平面应变问题的方程一致，故方程（37）的解具有二维 Elliott-Lodge 通解的形式：

$$\begin{cases} u_x^* = -\dfrac{\partial \xi_1}{\partial x} + \dfrac{\partial \xi_3}{\partial y}, \\ u_y^* = -\dfrac{\partial \xi_1}{\partial y} - \dfrac{\partial \xi_3}{\partial x}. \end{cases} \tag{38}$$

Wang 等[14]证明了 Elliott-Lodge 通解的完备性，即对任意位移场 u_x^* 和 u_y^* 总可找到满足式（32）和（33）的 ξ_1 和 ξ_3，故二维各向同性稳态通解（32）是完备的.

4 结果与讨论

4.1 讨论

本文给出了二维多孔介质的动力学通解（31），利用本构关系

$$\begin{cases} \sigma_{xx} = \dfrac{2\mu\nu}{1-2\nu}(\varepsilon_{xx}+\varepsilon_{yy}) + 2\mu\varepsilon_{xx} - p, \\ \sigma_{yy} = \dfrac{2\mu\nu}{1-2\nu}(\varepsilon_{xx}+\varepsilon_{yy}) + 2\mu\varepsilon_{yy} - p, \\ \sigma_{xy} = 2\mu\varepsilon_{xy}, \end{cases} \tag{39}$$

可以得到利用类调和函数 $\xi_i(i=1,2,3)$ 表示的应力场：

$$\begin{cases} \sigma_{xx} = 2\mu\dfrac{\partial^2 \xi_1}{\partial y^2} - \rho\dfrac{\partial^2 \xi_1}{\partial t^2} - \dfrac{2n}{k}\dfrac{\partial^2 \xi_2}{\partial y^2} + \dfrac{n\rho}{k\mu}\dfrac{\partial^2 \xi_2}{\partial t^2} + 2\mu\dfrac{\partial^2 \xi_3}{\partial x\partial y}, \\ \sigma_{yy} = 2\mu\dfrac{\partial^2 \xi_1}{\partial x^2} - \rho\dfrac{\partial^2 \xi_1}{\partial t^2} - \dfrac{2n}{k}\dfrac{\partial^2 \xi_2}{\partial x^2} + \dfrac{n\rho}{k\mu}\dfrac{\partial^2 \xi_2}{\partial t^2} - 2\mu\dfrac{\partial^2 \xi_3}{\partial x\partial y}, \\ \sigma_{xy} = -2\mu\dfrac{\partial^2 \xi_1}{\partial x\partial y} + \dfrac{2n}{k}\dfrac{\partial^2 \xi_2}{\partial x\partial y} + \mu\dfrac{\partial^2 \xi_3}{\partial y^2} - \mu\dfrac{\partial^2 \xi_3}{\partial x^2}. \end{cases} \tag{40}$$

将时间项退化后，可以得到

$$\begin{cases} \sigma_{xx} = 2\mu\dfrac{\partial^2 \xi_1}{\partial y^2} - \dfrac{2n}{k}\dfrac{\partial^2 \xi_2}{\partial y^2} + 2\mu\dfrac{\partial^2 \xi_3}{\partial x\partial y}, \\ \sigma_{yy} = 2\mu\dfrac{\partial^2 \xi_1}{\partial x^2} - \dfrac{2n}{k}\dfrac{\partial^2 \xi_2}{\partial x^2} - 2\mu\dfrac{\partial^2 \xi_3}{\partial x\partial y}, \\ \sigma_{xy} = -2\mu\dfrac{\partial^2 \xi_1}{\partial x\partial y} + \dfrac{2n}{k}\dfrac{\partial^2 \xi_2}{\partial x\partial y} + \mu\dfrac{\partial^2 \xi_3}{\partial y^2} - \mu\dfrac{\partial^2 \xi_3}{\partial x^2}. \end{cases} \tag{41}$$

此时的各类调和函数退化为（双）调和函数，满足式（33）.

令

$$H_1 = \int_{y_0(x)}^{y} \xi_3(x,\zeta)\,\mathrm{d}\zeta,$$

其中，$y_0(x)$ 是研究区域边界上一点，它是关于 x 的函数[15].

可以知道

$$\frac{\partial H_1}{\partial y}=\xi_3. \tag{42}$$

根据式(36)和式(33)的第三个方程,可以推导出

$$\begin{aligned}\nabla^2 H_1 &= \int_{y_0}^{y}\frac{\partial^2\xi_3(x,\zeta)}{\partial x^2}\mathrm{d}\zeta+\frac{\partial\xi_3}{\partial y}+A_1(x)=\\ &-\int_{y_0}^{y}\frac{\partial^2\xi_3(x,\zeta)}{\partial y^2}\mathrm{d}\zeta+\frac{\partial\xi_3}{\partial y}+A_1(x)=\left.\frac{\partial\xi_3}{\partial y}\right|_{y=y_0(x)}+A_1(x)=A_2(x).\end{aligned} \tag{43}$$

该方程与 y 无关.令

$$H_2(x)=\int_{x_0}^{x}\int_{\zeta_0}^{\zeta}A_2(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\zeta. \tag{44}$$

根据式(36)和(37)可知 $\xi_4(x,y)=H_1(x,y)-H_2(x)$ 满足

$$\frac{\partial\xi_4}{\partial y}=\xi_3,\ \nabla^2\xi_4=0. \tag{45}$$

则式(41)可以改写为

$$\begin{cases}\sigma_{xx}=\dfrac{\partial^2}{\partial y^2}\left(2\mu\xi_1-\dfrac{2n}{k}\xi_2+2\mu\dfrac{\partial\xi_4}{\partial x}\right),\\ \sigma_{yy}=\dfrac{\partial^2}{\partial x^2}\left(2\mu\xi_1-\dfrac{2n}{k}\xi_2+2\mu\dfrac{\partial\xi_4}{\partial x}\right),\\ \sigma_{xy}=-\dfrac{\partial^2}{\partial x\partial y}\left(2\mu\xi_1-\dfrac{2n}{k}\xi_2+2\mu\dfrac{\partial\xi_4}{\partial x}\right),\end{cases} \tag{46}$$

其中括号内为双调和函数,与平面应变问题的应力场一致.

4.2 结论

本文给出了二维各向同性多孔介质的弹性动力学通解,该通解退化时间项后可以获得稳态通解,可以证明该稳态通解具有完备性.这些通解的获得将大大降低求解二维多孔介质边值问题的工作量和难度,可以为多孔介质在油气田开采、岩石加固、地下水迁移引起的地基沉降变形、部分软骨组织在多场作用下的膨胀、聚合物胶体的溶胀等重要研究领域的应用提供必要且有效的理论支持.

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General Solutions of Elastodynamics for 2D Isotropic Porous Media

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Abstract: With the general equations of solid displacement and fluid velocity, the general solution of elastodynamics for 2D isotropic porous media were studied. Through introduction of 4 undetermined functions, the equations of motion, fluid velocity and continuity were formulated and divided into 2 parts of expansion wave and torsion wave. Thus, the general solutions expressed with 3 quasi harmonic functions were obtained under the Lur'e operator theory. With the solid displacement and fluid velocity independent of the time, the general steady-state solutions for 2D isotropic porous media were given, and their completeness was proved.

Key words: porous medium; general solution; Lur'e operator matrix; completeness; quasi harmonic function

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