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双功能梯度纳米梁系统振动分析的辛方法*

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摘要: 在辛力学与非局部 Timoshenko(铁木辛柯)梁理论的基础上,针对黏弹性介质中的双功能梯度纳米梁系统的自由振动问题,提出了一种全新的解析求解方法.在 Hamilton(哈密顿)体系下,位移与广义剪力、转角与广义弯矩互为对偶变量.以对偶变量为基本未知量,Lagrange(拉格朗日)体系下的高阶偏微分控制方程简化为一系列常微分方程.该纳米梁系统的振动问题归结为辛空间下的本征问题,解析频率方程和振动模态可以通过辛本征解和边界条件直接获得.数值结果验证了该方法的正确性与有效性,并针对纳米梁系统的小尺度效应、纳米梁间的相互作用以及黏弹性地基的影响进行了系统的参数分析.

关 键 词: Hamilton 体系; 辛方法; 双功能梯度纳米梁系统; 自由振动; 解析解

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引 言

自从日本学者 Lijima 发现碳纳米管以来,纳米材料引起了学术界的广泛重视.由双纳米梁构成的复杂纳米梁系统作为一类重要的基础结构具有巨大的发展潜力,并已经成功应用于纳米光机系统(NOMS)中[1-2].近年来,为了获得更佳的结构性能,功能梯度型纳米材料被逐渐引入到纳米装置的研发中,如微纳机电系统(MEMS/NAMS)[3]和原子力显微镜(AFM)[4].因此,研究双功能梯度纳米梁系统的力学性能具有重要的实际意义.

在纳米尺度下,经典连续介质力学理论已经不再适用,分子模拟与实验方法又面临着成本过高及无法处理大尺寸复杂结构的困难.为解决上述问题,Eringen 提出了一种可以考虑尺度效应的非局部场理论模型,该模型已经在纳米结构相关研究中获得广泛应用[5-6].目前,国内外学者在非局部理论框架下对纳米梁的力学行为进行了系统研究[7-9].功能梯度纳米梁自由振动方面也已经积累了大量的研究成果[10-20].然而,从现有文献可以发现,解析研究工作还比较少[11-12, 18],解析方法受到求解体系限制均为逆法或半逆法,双功能梯度纳米梁系统的振动问题尚未提及.因此,发展一种能够有效分析双功能梯度纳米梁自由振动问题的解析方法具有重要的理论意义.

应用力学中的 Hamilton 体系理论由钟万勰院士等首次提出[21],并已经成功应用于力学中

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的各个领域[22].本文将 Hamilton 力学与非局部 Timoshenko 梁理论相结合,提出一种适用于双功能梯度梁自由振动分析的解析求解方法,求解其自由振动频率及解析振型函数,并研究尺度效应、材料功能梯度分布以及周围黏弹性介质参数对该类纳米梁系统的影响和作用规律.

1 非局部场理论

根据 Eringen 的非局部场理论,应力-应变关系可以表示为

$$\tilde{\sigma}_{ij}=\int_{V'}\alpha_0(|x-x'|,e_0a_0)\sigma_{ij}(x')\mathrm{d}V'(x'),\qquad \forall x\in V,\tag{1}$$

其中 σ 为应力张量, $\alpha_0(|x-x'|,e_0a_0)$ 为非局部函数, e_0a_0 为表征尺度效应的长度量纲系数,上标“~”表示非局部变量.式(1)中的积分型本构给数学求解带来巨大困难.为解决该问题,Eringen 进一步推导出等效的微分型本构,即

$$(1-\xi^2\nabla^2)\tilde{\sigma}_{ij}=C_{ijkl}\varepsilon_{kl},\tag{2}$$

其中 ε 为应变张量, $\xi=e_0a_0$ 为非局部参数, ∇^2 为 Laplace(拉普拉斯)算子.

2 功能梯度材料

考虑一个放置于黏弹性介质中的双功能梯度纳米梁系统,其坐标如图 1 所示.黏弹性介质的刚度和黏弹性系数分别为 K_w 和 c.纳米梁之间的 Van der Waals(范德华)力、电场力或弹性介质引起的力,均由垂直于纳米梁的一组 Winkler 弹性介质表示,其刚度系数为 K.两个纳米梁的几何和物理参数完全相同.记梁长为 a,截面宽为 b,高度为 h,截面积为 A,密度为 ρ,弹性模量为 E,剪切模量为 $G=E/[2(1+\nu)]$,ν 为 Poisson(泊松)比.对于功能梯度纳米梁,其材料参数沿厚度方向的变化规律为

$$P(z)=(P_1-P_2)\left(\frac{z}{h}+\frac{1}{2}\right)^k+P_2,\tag{3}$$

其中 P_1 和 P_2 分别为梁上下表面的材料参数,P 可以表示等效弹性模量 E、密度 ρ 和剪切模量 G,k 为功能梯度指数.

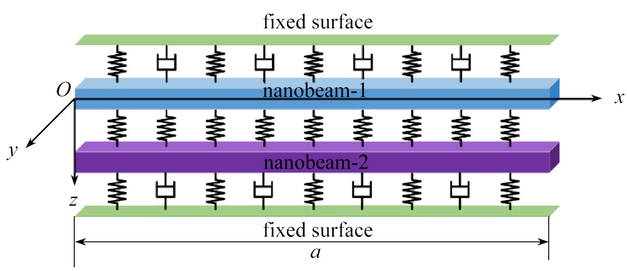

图 1 放置于黏弹性介质中的双功能梯度纳米梁系统

Fig. 1 A functionally graded double-nanobeam system embedded in viscoelastic medium

3 基本方程

对于非局部 Timoshenko 梁,其沿 x,y,z 轴的位移分量分别表示为 $u_1=z\theta,u_2=0,u_3=W(x,t)$,$W$ 为横向位移,θ 为转角.对于自由振动问题,令 $W(x,t)=w(x)\mathrm{e}^{\mathrm{i}\omega t}$,$\omega$ 为自振频率,则功能梯度纳米梁的自由振动方程可以表示为

$$\frac{\partial \tilde{M}_i}{\partial x}=\tilde{Q}_i-I_2\omega^2\theta_i,\ \frac{\partial \tilde{Q}_i}{\partial x}=m\frac{\partial^2 w_i}{\partial t^2}-p_i, \tag{4}$$

其中,i 为虚数单位;$\tilde{M}$ 和 $\tilde{Q}$ 分别为纳米梁的非局部弯矩和剪力;$I_2=\int_A\rho(z)z^2\mathrm{d}A$,$m=\int_A\rho(z)\mathrm{d}A$ 为质量;$p_i=-K(w_i-w_{i'})-K_\mathrm{w}w_i-c\partial w_i/\partial t$,$i=1,2$ 代表纳米梁 1 和 2,$i'=2,1$.由式(2),功能梯度纳米梁的本构方程可以表示为

$$\mathcal{L}\tilde{\sigma}_{xx}=E\varepsilon_{xx}=zE\frac{\partial\theta}{\partial x},\ \mathcal{L}\tilde{\tau}_{xz}=G\gamma_{xz}=G\left(\frac{\partial w}{\partial x}+\theta\right), \tag{5}$$

其中,$\mathcal{L}=1-\xi^2\partial^2/\partial x^2$ 为线性算子,$G=E/[2(1+\nu)]$.

非局部内力与经典内力之间的关系为

$$\tilde{M}_i=\int_A z(\tilde{\sigma}_{xx})_i\mathrm{d}A=J_1\frac{\partial\theta_i}{\partial x}=\mathcal{L}^{-1}M_i, \tag{6a}$$

$$\tilde{Q}_i=\int_A K_\mathrm{s}(\tilde{\tau}_{xz})_i\mathrm{d}A=K_\mathrm{s}J_2\left(\frac{\partial w_i}{\partial x}+\theta_i\right)=\mathcal{L}^{-1}Q_i, \tag{6b}$$

其中,$J_1=\int_{-h/2}^{h/2}-E(z)z^2\mathrm{d}A$,$J_2=\int_{-h/2}^{h/2}G(z)\mathrm{d}A$,$K_\mathrm{s}$ 为剪切修正系数.将式(6)代入式(4),可得

$$\frac{\partial M_i}{\partial x}=Q_i-\mathcal{L}I_2\omega^2\theta_i,\ \frac{\partial Q_i}{\partial x}=-\mathcal{L}m\omega^2w_i-\mathcal{L}p_i. \tag{7}$$

根据双纳米梁系统的振动特点,方程(7)可以分为两类子问题,即同向振动和反向振动.令 $w_\mathrm{IP}=w_1+w_2$,$\theta_\mathrm{IP}=\theta_1+\theta_2$ 和 $w_\mathrm{OP}=w_1-w_2$,$\theta_\mathrm{OP}=\theta_1+\theta_2$[23],则方程(7)可以改写为统一形式:

$$\frac{\partial M}{\partial x}=Q-\mathcal{L}I_2\omega^2\theta, \tag{8a}$$

$$\frac{\partial Q}{\partial x}=\mathcal{L}(-m\omega^2+\phi K+K_\mathrm{w}+\mathrm{i}c\omega)w, \tag{8b}$$

其中,当 $\phi=0$ 时,$w=w_\mathrm{IP}$,$\theta=\theta_\mathrm{IP}$,为同向振动问题;当 $\phi=2$ 时,$w=w_\mathrm{OP}$,$\theta=\theta_\mathrm{OP}$,为反向振动问题.方程(8)即为以经典变量表示的双功能梯度纳米梁系统自由振动的控制方程.

令 S、C 和 F 分别表示简支、固支和自由边界,$x=0,a$ 处的边界条件可以表示为

① S:$w=0,\ M=0,\quad \text{when } x=0,a;$ (9a)

② C:$w=0,\ \theta=0,\quad \text{when } x=0,a;$ (9b)

③ F:$M=0,\ Q=0,\quad \text{when } x=0,a.$ (9c)

4 Hamilton 体系

为引入双功能梯度纳米梁系统自由振动问题的 Hamilton 求解体系,定义变量上一点表示对 x 方向的微分,即 $\dot{(\ )}=\partial(\ )/\partial x$,并记 $\boldsymbol{q}=\{w,\theta\}^\mathrm{T}$,则 Lagrange 密度函数可以表示为[21]

$$\begin{aligned}L(\boldsymbol{q},\dot{\boldsymbol{q}})&=\frac{1}{2}\dot{\boldsymbol{q}}^\mathrm{T}\boldsymbol{K}_{22}\dot{\boldsymbol{q}}+\dot{\boldsymbol{q}}^\mathrm{T}\boldsymbol{K}_{21}\boldsymbol{q}+\frac{1}{2}\boldsymbol{q}^\mathrm{T}\boldsymbol{K}_{11}\boldsymbol{q}=\\&\frac{1}{2}[K_\mathrm{s}J_2-(m\omega^2-\phi K-K_\mathrm{w}-\mathrm{i}c\omega)\xi^2]\dot{w}^2+\frac{1}{2}(J_1-I_2\omega^2\xi^2)\dot{\theta}^2+\\&K_\mathrm{s}J_2\dot{w}\theta+\frac{1}{2}(-m\omega^2+\phi K+K_\mathrm{w}+\mathrm{i}c\omega)w^2+\frac{1}{2}(K_\mathrm{s}J_2-I_2\omega^2)\theta^2,\end{aligned} \tag{10}$$

其中

$$\boldsymbol{K}_{11}=\begin{bmatrix}-m\omega^2+\phi K+K_{\mathrm{w}}+\mathrm{i}c\omega & 0\\ 0 & K_{\mathrm{s}}J_2-I_2\omega^2\end{bmatrix},\ \boldsymbol{K}_{21}=\begin{bmatrix}0 & K_{\mathrm{s}}J_2\\ 0 & 0\end{bmatrix},$$

$$\boldsymbol{K}_{12}=\boldsymbol{K}_{21}^{\mathrm{T}},\ \boldsymbol{K}_{22}=\begin{bmatrix}K_{\mathrm{s}}J_2+(-m\omega^2+\phi K+K_{\mathrm{w}}+\mathrm{i}c\omega)\xi^2 & 0\\ 0 & J_1-I_2\omega^2\xi^2\end{bmatrix}.$$

根据 Legendre(勒让德)变换, $\boldsymbol{q}$ 的对偶变量为

$$\boldsymbol{p}=\frac{\partial L}{\partial\dot{\boldsymbol{q}}}=\boldsymbol{K}_{22}\dot{\boldsymbol{q}}+\boldsymbol{K}_{21}\boldsymbol{q}=\{\bar{Q},\bar{M}\}^{\mathrm{T}},\tag{11}$$

其中 $\bar{Q}=Q-(m\omega^2-\phi K-K_{\mathrm{w}}-\mathrm{i}c\omega)\xi^2\dot{w}$ 为广义剪力, $\bar{M}=M-I_2\omega^2\xi^2\dot{\theta}$ 为广义弯矩.

由上式可进一步求得

$$\dot{\boldsymbol{q}}=\{\dot{w},\dot{\theta}\}^{\mathrm{T}}=-\boldsymbol{K}_{22}^{-1}\boldsymbol{K}_{21}\boldsymbol{q}+\boldsymbol{K}_{22}^{-1}\boldsymbol{p}.\tag{12}$$

由式(10)~(12),Hamilton 密度函数为

$$H(\boldsymbol{q},\boldsymbol{p})=\boldsymbol{p}^{\mathrm{T}}\dot{\boldsymbol{q}}-L(\boldsymbol{q},\dot{\boldsymbol{q}})=\boldsymbol{p}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{q}-\frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{q}+\frac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{p},\tag{13}$$

其中

$$\boldsymbol{A}=-\boldsymbol{K}_{22}^{-1}\boldsymbol{K}_{21}=\begin{bmatrix}0 & A_{12}\\ 0 & 0\end{bmatrix},\ \boldsymbol{B}=\boldsymbol{K}_{11}-\boldsymbol{K}_{12}\boldsymbol{K}_{22}^{-1}\boldsymbol{K}_{21}=\begin{bmatrix}B_{11} & 0\\ 0 & B_{22}\end{bmatrix},$$

$$\boldsymbol{D}=\boldsymbol{K}_{22}^{-1}=\begin{bmatrix}D_{11} & 0\\ 0 & D_{22}\end{bmatrix},$$

$$A_{12}=-K_{\mathrm{s}}J_2/[K_{\mathrm{s}}J_2-(m\omega^2-\phi K-K_{\mathrm{w}}-\mathrm{i}c\omega)\xi^2],\ B_{11}=-m\omega^2+\phi K+K_{\mathrm{w}}+\mathrm{i}c\omega,$$

$$B_{22}=-K_{\mathrm{s}}J_2(m\omega^2-\phi K-K_{\mathrm{w}}-\mathrm{i}c\omega)\xi^2/[K_{\mathrm{s}}J_2-(m\omega^2-\phi K-K_{\mathrm{w}}-\mathrm{i}c\omega)\xi^2]-I_2\omega^2,$$

$$D_{11}=1/[K_{\mathrm{s}}J_2-(m\omega^2-\phi K-K_{\mathrm{w}}-\mathrm{i}c\omega)\xi^2],\ D_{22}=1/(J_1-I_2\omega^2\xi^2).$$

由式(13),Hamilton 体系下的控制方程可以表示为

$$\dot{\boldsymbol{q}}=\frac{\partial H}{\partial\boldsymbol{p}}=\boldsymbol{A}\boldsymbol{q}+\boldsymbol{D}\boldsymbol{p},\ \dot{\boldsymbol{p}}=-\frac{\partial H}{\partial\boldsymbol{q}}=\boldsymbol{B}\boldsymbol{q}-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{p}.\tag{14}$$

定义全状态向量 $\boldsymbol{\Psi}=\{\boldsymbol{q}^{\mathrm{T}},\boldsymbol{p}^{\mathrm{T}}\}^{\mathrm{T}}$, 则上式可以写为矩阵形式,即

$$\dot{\boldsymbol{\Psi}}=\boldsymbol{H}\boldsymbol{\Psi},\tag{15}$$

式中 $\boldsymbol{H}=\begin{bmatrix}\boldsymbol{A} & \boldsymbol{D}\\ \boldsymbol{B} & -\boldsymbol{A}^{\mathrm{T}}\end{bmatrix}$ 为 Hamilton 矩阵.对应的边界条件可以由式(9)获得

$$\boldsymbol{h}_{\mathrm{g}}\boldsymbol{\Psi}=\boldsymbol{0},\quad \text{when } x=0,a,\tag{16}$$

其中 $\boldsymbol{h}$ 为边界指示矩阵,g=S,C 和 F,

$$\boldsymbol{h}_{\mathrm{S}}=\begin{bmatrix}1&0&0&0\\0&0&0&1\end{bmatrix},\ \boldsymbol{h}_{\mathrm{C}}=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix},$$

$$\boldsymbol{h}_{\mathrm{F}}=\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & -(m\omega^2-\phi K-K_{\mathrm{w}}-\mathrm{i}c\omega)\xi^2 & 1 & 0\end{bmatrix}.$$

5　辛本征值和本征解

在辛空间下,Hamilton 方程(15)可用分离变量方法求解.令 $\boldsymbol{\Psi}(x)=\boldsymbol{\psi}_j\mathrm{e}^{\mu_j x}$, 本征方程为

$$\boldsymbol{H}\boldsymbol{\psi}_j=\mu_j\boldsymbol{\psi}_j,\tag{17}$$

其中，μ_j 和 $\boldsymbol{\psi}_j$ 分别为 Hamilton 矩阵的本征值和本征向量.由方程(17)可得

$$\mu^4 - \gamma\mu^2 + \zeta = 0, \tag{18}$$

其中

$$\gamma = D_{11}B_{11} + D_{22}B_{22},\ \zeta = A_{12}^2 D_{22}B_{11} + D_{11}D_{22}B_{11}B_{22}.$$

方程(18)的 4 个根为

$$\mu_{1,2} = \mp\sqrt{\frac{\gamma + \sqrt{\gamma^2 - 4\zeta}}{2}},\ \mu_{3,4} = \mp\sqrt{\frac{\gamma - \sqrt{\gamma^2 - 4\zeta}}{2}}. \tag{19}$$

可以证明，当且仅当 $\mu_{1,2} \neq 0,\ \mu_{3,4} \neq 0,\ \mu_1 \neq \mu_3$ 时，本征值具有物理意义[23].此时，辛本征解向量为

$$\boldsymbol{\psi}_j = \left\{1 \quad \frac{\mu_j^2 - D_{11}B_{11}}{A_{12}\mu_j} \quad \frac{B_{11}}{\mu_j} \quad \frac{\mu_j^2 - D_{11}B_{11}}{A_{12}D_{22}}\right\}^{\mathrm{T}} \qquad (j = 1,2,3,4). \tag{20}$$

根据式(20)，方程(15)的解可以表示为

$$\boldsymbol{\Psi} = \sum_{j=1}^{4} d_j \boldsymbol{\psi}_j \mathrm{e}^{\mu_j x}, \tag{21}$$

其中 d_j 为待定系数.

6 自 由 振 动

为确定纳米梁的自由振动频率，通解(21)可以进一步表示为等价形式，即

$$\boldsymbol{\Psi}_j(x) = \begin{bmatrix} A_{w1} & A_{w2} & A_{w3} & A_{w4} \\ A_{\theta 1} & A_{\theta 2} & A_{\theta 3} & A_{\theta 4} \\ A_{Q1} & A_{Q2} & A_{Q3} & A_{Q4} \\ A_{M1} & A_{M2} & A_{M3} & A_{M4} \end{bmatrix} \begin{Bmatrix} \sinh(\mu x) \\ \cosh(\mu x) \\ \sinh(\bar{\mu} x) \\ \cosh(\bar{\mu} x) \end{Bmatrix}, \tag{22}$$

这里，$\mu = \mu_1,\ \bar{\mu} = \mu_3$. $A_{wj}, A_{\theta j}, A_{Mj}, A_{Qj}(j = 1,\ 2,\ 3,\ 4)$ 为待定系数，它们之间满足

$$A_{\theta j} = \kappa_{11}A_{wn},\ A_{Mj} = \kappa_{12}A_{wj},\ A_{Qj} = \kappa_{13}A_{wn} \qquad (j = 1,2;\ n = 2,1), \tag{23a}$$

$$A_{\theta j} = \kappa_{21}A_{wn},\ A_{Mj} = \kappa_{22}A_{wj},\ A_{Qj} = \kappa_{23}A_{wn} \qquad (j = 3,4;\ n = 4,3), \tag{23b}$$

其中

$$\kappa_{11} = A_{12}D_{22}\mu/[A_{12}^2 D_{22} + D_{11}(D_{22}B_{22} - \mu^2)],$$

$$\kappa_{12} = A_{12}\mu^2/[A_{12}^2 D_{22} + D_{11}(D_{22}B_{22} - \mu^2)],$$

$$\kappa_{13} = (D_{22}B_{22}\mu - \mu^3)/[A_{12}^2 D_{22} + D_{11}(D_{22}B_{22} - \mu^2)],$$

$$\kappa_{21} = A_{12}D_{22}\bar{\mu}/[A_{12}^2 D_{22} + D_{11}(D_{22}B_{22} - \bar{\mu}^2)],$$

$$\kappa_{22} = A_{12}\bar{\mu}/[A_{12}^2 D_{22} + D_{11}(D_{22}B_{22} - \bar{\mu}^2)],$$

$$\kappa_{23} = (D_{22}B_{22}\bar{\mu} - \bar{\mu}^3)/[A_{12}^2 D_{22} + D_{11}(D_{22}B_{22} - \bar{\mu}^2)].$$

下面以两端简支边界条件为例，推导该双纳米梁系统自由振动的频率方程.

(i) SS

将通解(22)代入边界条件(16)(g=S)，可得

$$\boldsymbol{Z}\boldsymbol{A}_w = \boldsymbol{0}, \tag{24}$$

这里，$\boldsymbol{A}_w = \{A_{w1}, A_{w2}, A_{w3}, A_{w4}\}^{\mathrm{T}}$，$\boldsymbol{Z}$ 矩阵为自振频率 ω 的函数.根据方程(24)的非零解条件 $\det(\boldsymbol{Z}) = 0$，可得两端简支非局部双功能梯度纳米梁系统的频率方程：

$$f_1(\omega) = 0, \tag{25}$$

其中

$$f_1(\omega)=\sinh(\mu a)\sinh(\bar{\mu}a).$$

同理,其他边界条件下双功能梯度纳米梁系统的频率方程可推导如下:

(ii) CC

$$2\kappa_{11}\kappa_{21}[f_2(\omega)-1]-(\kappa_{11}^2+\kappa_{21}^2)f_1(\omega)=0; \tag{26}$$

(iii) CS

$$\kappa_{11}f_3(\omega)-\kappa_{21}f_4(\omega)=0; \tag{27}$$

(iv) CF

$$\kappa_{21}\kappa_{12}\kappa_{14}+\kappa_{22}\kappa_{11}\kappa_{24}+(\kappa_{11}\kappa_{22}\kappa_{14}+\kappa_{21}\kappa_{12}\kappa_{24})f_1(\omega)-(\kappa_{11}\kappa_{12}\kappa_{24}+\kappa_{21}\kappa_{22}\kappa_{14})f_2(\omega)=0; \tag{28}$$

(V) SF

$$\kappa_{12}\kappa_{24}f_4(\omega)-\kappa_{22}\kappa_{14}f_3(\omega)=0; \tag{29}$$

(vi) FF

$$(\kappa_{12}^2\kappa_{22}^2+\kappa_{14}^2\kappa_{24}^2)f_1(\omega)+2\kappa_{12}\kappa_{14}\kappa_{22}\kappa_{24}(1-f_2(\omega))=0, \tag{30}$$

其中

$$\kappa_{14}=\kappa_{13}+\mu(m\omega^2-\phi K-K_{\mathrm{w}}-\mathrm{i}c\omega)\xi^2,$$

$$\kappa_{24}=\kappa_{23}+\bar{\mu}(m\omega^2-\phi K-K_{\mathrm{w}}-\mathrm{i}c\omega)\xi^2,$$

$$f_2(\omega)=\cosh(\mu a)\cosh(\bar{\mu}a),$$

$$f_3(\omega)=\cosh(\mu a)\sinh(\bar{\mu}a),$$

$$f_4(\omega)=\sinh(\mu a)\cosh(\bar{\mu}a).$$

7 数值算例

为方便分析和计算,数值算例中均采用无量纲参数.$\bar{\omega}=\omega a^2\sqrt{\rho_1 A/(E_1 I)}$,$\bar{K}=J_1 K/a^4$,$\bar{K}_{\mathrm{w}}=J_1 K_{\mathrm{w}}/a^4$,$\bar{c}=c\sqrt{J_1 m}/a^2$,$I$ 为转动惯量.

7.1 算例 1

为验证本文提出方法的正确性和精确性,本小节将计算结果与现有文献数据进行对比.首先,考虑一个两端简支的非局部功能梯度 Timoshenko 梁,其材料参数为

$$E_1=390\ \mathrm{GPa},\ E_2=210\ \mathrm{GPa},\ \rho_1=3\ 960\ \mathrm{kg/m^3},\ \rho_2=7\ 800\ \mathrm{kg/m^3},$$

$$\nu_1=0.24,\ \nu_2=0.30,\ K_{\mathrm{s}}=5/6.$$

选用矩形截面,其宽与高均为 $b=h=1\ 000$ nm,长为 $a=10\ 000$ nm.表 1 给出了该纳米梁在不同非局部参数和长宽比下对应的无量纲基频.从表中数据可以看出,本文的计算结果与文献[12]的结果吻合得非常好,最大误差仅为 1.12%.其次,考虑一个由两端简支的各向同性非局部 Timoshenko 梁组成的双纳米梁系统,其计算参数为

$$b=1\ \mathrm{nm},\ h=1\ \mathrm{nm},\ \rho=1\ \mathrm{kg/m^3},\ E=30\ \mathrm{MPa},\ K_{\mathrm{s}}=5/6,\ \bar{K}=10.$$

表 2 给出了不同非局部参数下双纳米梁系统的前六阶同向振动频率.从对比结果可以发现,本文的计算结果不仅与现有的非局理论模型结果一致,还与经典连续介质力学的结果(ANSYS, $\xi^2=0$)吻合较好.上述表 1 和表 2 的数据对比表明,本文提出的辛方法适用于双功能梯度纳米系统的自由振动分析,并可以得到精度较高的结果.

表 1 两端简支功能梯度纳米梁的基频

Table 1 Fundamental frequencies of an FG nanobeam with 2 ends simply supported

k	$(\xi^2/10^{-12})/\mathrm{m}^2$	a/h					
		20		50		100	
		present	ref. [12]	present	ref. [12]	present	ref. [12]
0	0	9.828 1	9.829 6	9.862 9	9.863 1	9.867 9	9.868 0
	1	9.376 4	9.377 7	9.409 5	9.409 7	9.414 3	9.414 3
	2	8.981 7	8.982 9	9.013 4	9.013 6	9.018 0	9.018 0
	3	8.633 1	8.634 1	8.663 4	8.663 6	8.667 8	8.667 8
0.2	0	8.679 1	8.660 0	8.708 7	8.689 5	8.712 9	8.693 8
	1	8.280 1	8.262 0	8.308 3	8.290 1	8.312 4	8.294 1
	2	7.931 6	7.914 0	7.958 6	7.941 1	7.962 4	7.944 9
	3	7.623 8	7.606 8	7.649 6	7.632 7	7.653 3	7.636 5
1	0	7.074 6	6.967 6	7.099 0	6.991 7	7.102 6	6.995 2
	1	6.749 4	6.647 3	6.772 7	6.670 3	6.776 0	6.673 6
	2	6.465 3	6.367 4	6.487 6	6.389 5	6.490 8	6.392 7
	3	6.214 3	6.120 2	6.235 7	6.141 4	6.238 8	6.144 4
5	0	5.983 2	5.917 2	6.005 2	5.938 9	6.008 4	5.942 1
	1	5.708 1	5.645 2	5.729 2	5.665 9	5.732 2	5.668 9
	2	5.467 9	5.407 5	5.488 0	5.427 4	5.490 9	5.430 2
	3	5.255 6	5.197 5	5.274 9	5.216 6	5.277 7	5.219 4

表 2 两端简支双纳米梁系统的前六阶同向振动频率

Table 2 First 6 in-phase natural frequencies of a double-nanobeam system with 2 ends simply supported

length-width ratio	ξ^2/nm^2	mode(data in ref. [24] given in parentheses; data in ANSYS given in brackets)					
		1	2	3	4	5	6
$a/h=10$	0	9.707 5	37.096 2	78.154 7	128.666 0	185.318 3	245.832 3
		(9.744 3)	(36.840 6)				
		[9.712 2]	[37.162 5]	[78.447 5]	[129.451 7]	[186.957 0]	[248.750 2]
	1	9.261 2	31.410 5	56.875 3	80.117 4	127.792 8	169.947 8
		(9.293 1)	(31.236 6)				
	2	8.871 3	27.730 3	46.903 4	63.096 8	94.502 0	130.078 6
		(8.899 4)	(27.587 0)				
	3	8.526 9	25.099 6	40.825 4	53.716 3	83.519 0	115.521 8
		(8.551 7)	(24.972 7)				
$a/h=20$	0	9.828 1	38.829 9	85.661 9	148.384 6	224.779 4	312.618 9
		(9.838 1)	(38.964 5)	(85.748 3)			
		[9.832 7]	[38.900 7]	[86.006 5]	[149.416 2]	[227.144 4]	[317.191 5]
	1	9.709 1	37.044 8	77.489 1	125.642 2	176.775 3	227.501 3
		(9.718 7)	(37.161 4)	(77.529 1)			
	2	9.594 2	35.485 3	71.282 7	110.921 3	150.398 7	187.613 6
		(9.603 6)	(35.587 5)	(71.292 2)			
	3	9.483 4	34.107 4	66.362 8	100.398 6	133.134 9	163.301 6
		(9.492 4)	(34.197 9)	(66.351 5)			

续表 2

length-width ratio	ξ^2 /nm^2	mode(data in ref. [24] given in parentheses; data in ANSYS given in brackets)					
		1	2	3	4	5	6
a/h = 50	0	9.862 9 (9.864 5) [9.867 5]	39.371 9 (39.397 6) [39.444 7]	88.290 7 (88.414 7) [88.656 3]	156.235 2 [157.383 7]	242.686 9 [245.453 9]	347.011 1 [352.687 8]
	1	9.843 5 (9.845 1)	39.064 7 (39.089 7)	86.762 8 (86.880 4)	151.522 9	231.530 2	324.703 5
	2	9.824 2 (9.825 8)	38.764 5 (38.789 0)	85.311 6 (85.423 3)	147.212 9	221.783 0	306.210 1
	3	9.805 0 (9.806 6)	38.471 2 (38.495 1)	83.930 8 (84.037 2)	143.250 9	213.171 5	290.554 6

7.2 算例 2

本小节将针对尺度效应、功能梯度材料指数、介质黏弹性参数进行详细讨论.考虑一个放置于黏弹性介质中的双功能梯度纳米梁系统,其计算参数为 E_1 = 390 GPa, E_2 = 210 GPa, ρ_1 = 3 960 kg/m^3, ρ_2 = 7 800 kg/m^3, ν_1 = 0.24, ν_2 = 0.30, $h = b$ = 1 000 nm, L = 10 000 nm, K_s = 5/6, $\bar{K}$ = 10, $\bar{K}_w$ = 20.

表 3 不同端部条件下双纳米梁系统的前四阶同向振动频率

Table 3 First 4 in-phase natural frequencies of a functionally graded double-nanobeam system with various end conditions

end condition	(ε^2 /10^{-12})/m^2	mode			
		1	2	3	4
SS	0	6.810 7+3.570 8i	26.695 4+3.503 0i	56.382 8+3.425 9i	92.962 0+3.360 8i
	1	6.480 6+3.570 8i	22.601 6+3.502 8i	41.121 7+3.424 4i	58.219 2+3.356 0i
	2	6.191 6+3.570 8i	19.956 5+3.502 5i	33.985 0+3.422 8i	46.034 8+3.352 1i
CS	0	10.573 4+3.568 3i	32.646 4+3.505 0i	63.353 6+3.436 7i	99.997 4+3.378 7i
	1	10.005 2+3.565 8i	27.388 0+3.494 9i	45.856 4+3.418 1i	62.339 6+3.354 1i
	2	9.516 4+3.563 7i	24.065 5+3.489 1i	37.814 8+3.409 6i	49.278 3+3.343 5i
CC	0	15.039 2+3.569 8i	38.782 7+3.509 6i	70.233 0+3.449 1i	106.790 6+3.396 3i
	1	14.196 3+3.562 2i	32.247 9+3.486 9i	50.483 2+3.413 0i	66.263 1+3.353 4i
	2	13.476 4+3.556 0i	28.176 9+3.474 7i	41.542 8+3.397 6i	52.336 8+3.336 8i
CF	0	1.942 8+3.585 2i	14.991 3+3.513 2i	39.648 5+3.430 3i	72.048 0+3.349 6i
	1	1.956 6+3.584 9i	14.052 1+3.512 6i	32.824 8+3.432 3i	51.269 3+3.357 7i
	2	1.970 6+3.584 5i	13.247 0+3.512 0i	28.682 3+3.431 5i	42.075 3+3.354 9i
FF	0	15.482 8+3.541 9i	40.486 2+3.463 5i	73.979 0+3.391 2i	113.065 5+3.335 3i
	1	14.035 2+3.541 6i	31.844 9+3.462 8i	49.976 5+3.388 4i	65.801 4+3.328 3i
	2	12.940 6+3.541 3i	27.096 3+3.462 1i	40.319 0+3.385 7i	51.206 1+3.323 0i
SF	0	10.718 3+3.522 7i	33.319 8+3.428 4i	64.989 5+3.337 4i	102.877 9 +3.265 8i
	1	10.129 1+3.525 1i	27.805 2+3.440 5i	46.590 2+3.362 3i	63.331 9+3.301 8i
	2	9.625 3+3.526 9i	24.379 2+3.445 6i	38.341 3+3.366 9i	50.003 6+3.302 9i

首先,考虑尺度效应对该纳米梁系统自由振动的影响.表 3 和表 4 分别给出了不同端部条件和非局部参数对应的同向振动和反向振动对应的自由振动频率.计算过程中,黏弹性介质参数为 $\bar{c}$ = 10,功能梯度指数为 k = 1.从表中数据可以发现,除了 CF 边界以外,所有边界对应的

各阶频率的实部均随着非局部参数的增加而减小，而非局部参数对所有边界对应的频率的虚部影响都很小.该现象与单纳米梁自由振动的特点一致[25].此外，在选取相同非局部参数时，反向振动的自振频率明显大于同向振动的结果，该现象应为双纳米梁间相互作用力引起的，相当于改变了纳米梁系统的等效结构刚度.为了直观展示该类双纳米梁系统的自由振动形式，图 2 和图 3 分别给出了两端简支功能梯度双纳米梁系统的前四阶同向振动和反向振动的模态.

表 4　不同端部条件下双纳米梁系统的前四阶反向振动频率

Table 4　First 4 out-of-phase natural frequencies of a functionally graded double-nanobeam system with various end conditions

end condition	$(\varepsilon^2/10^{-12})/\mathrm{m}^2$	mode			
		1	2	3	4
SS	0	7.527 7+3.570 8i	26.883 7+3.503 0i	56.470 2+3.425 9i	93.014 0+3.360 7i
	1	7.230 4+3.570 8i	22.823 6+3.502 8i	41.241 4+3.424 4i	58.302 1+3.356 0i
	2	6.972 5+3.570 8i	20.207 5+3.502 5i	34.129 6+3.422 8i	46.139 5+3.352 0i
CS	0	11.048 5+3.568 3i	32.800 6+3.505 0i	63.431 6+3.436 7i	100.046 0+3.378 7i
	1	10.505 7+3.565 8i	27.571 1+3.494 9i	45.963 6+3.418 1i	62.417 0+3.354 1i
	2	10.041 0+3.563 7i	24.273 3+3.489 1i	37.944 4+3.409 5i	49.375 9+3.343 5i
CC	0	15.377 1+3.569 8i	38.912 7+3.509 6i	70.303 6+3.449 1i	106.836 4+3.396 3i
	1	14.553 0+3.562 2i	32.403 2+3.486 9i	50.580 5+3.413 0i	66.335 9+3.353 4i
	2	13.851 0+3.556 0i	28.353 9+3.474 7i	41.660 4+3.397 5i	52.428 5+3.336 8i
CF	0	3.754 5+3.585 2i	15.324 9+3.513 2i	39.772 9+3.430 3i	72.114 9+3.349 6i
	1	3.761 5+3.584 8i	14.407 5+3.512 5i	32.975 0+3.432 3i	51.363 5+3.357 6i
	2	3.768 7+3.584 4i	13.623 3+3.512 0i	28.854 1+3.431 5i	42.189 9+3.354 9i
FF	0	15.808 7+3.541 9i	40.609 1+3.463 5i	74.045 0+3.391 2i	113.108 0+3.335 3i
	1	14.393 8+3.541 6i	32.001 0+3.462 8i	50.074 0+3.388 4i	65.874 2+3.328 3i
	2	13.328 7+3.541 3i	27.279 6+3.462 0i	40.439 7+3.385 7i	51.299 4+3.322 9i
SF	0	11.181 4+3.522 7i	33.467 6+3.428 4i	65.063 3+3.337 4i	102.923 6+3.265 8i
	1	10.618 3+3.525 1i	27.982 7+3.440 5i	46.694 0+3.362 3i	63.406 9+3.301 8i
	2	10.139 0+3.526 9i	24.581 8+3.445 5i	38.467 5+3.366 8i	50.098 6+3.302 8i

（a）第一阶 （a）The 1st order　（b）第二阶 （b）The 2nd order　（c）第三阶 （c）The 3rd order　（d）第四阶 （d）The 4th order

图 2　前四阶简支同向振动模态

Fig. 2　First 4 in-phase vibration mode shapes

（a）第一阶 （a）The 1st order　（b）第二阶 （b）The 2nd order　（c）第三阶 （c）The 3rd order　（d）第四阶 （d）The 4th order

图 3　前四阶简支反向振动模态

Fig. 3　First 4 out-of-phase vibration mode shapes

其次,表 5 分析了功能梯度指数对双功能梯度纳米梁系统自由振动的影响.这里,计算参数为 $\xi^2 = 10^{-12}\ \mathrm{m}^2$, $\bar{c} = 10$, 端部条件选取为两端简支.表 5 中数据变化趋势与表 3 和表 4 类似,各阶振动频率的实部和虚部均随着 k 的增加而减小.该现象说明,功能梯度指数对该类功能梯度双纳米梁系统具有显著影响,是该类纳米结构设计中不可忽略的关键因素.

最后,表 6 分析了地基黏弹性参数对双功能梯度纳米梁系统的作用规律.计算参数为 $\xi^2 = 10^{-12}\ \mathrm{m}^2$, $k = 1$.从计算结果可以看出,地基黏弹性参数对各阶频率的虚部影响较大,虚部数值均随着 $\bar{c}$ 的增大表现出明显的上升趋势,这与其表征阻尼的物理意义相符.地基黏弹性参数对基频的实部影响较大,对高阶频率的实部影响较小.这说明黏弹性地基系数仅对基频造成影响.

表 5 不同功能梯度指数对应两端简支功能梯度双纳米梁系统的前四阶振动频率

Table 5 First 4 natural frequencies of an SS functionally graded double-nanobeam system with various power-law indexes

	k	mode			
		1	2	3	4
in-phase vibration	0	9.005 7+4.961 1i	31.423 3+4.866 3i	57.208 0+4.756 4i	81.046 1+4.659 9i
	0.5	7.126 7+3.924 9i	24.868 1+3.847 6i	45.279 0+3.757 9i	64.157 0+3.678 6i
	1	6.480 6+3.570 8i	22.601 6+3.502 8i	41.121 7+3.424 4i	58.219 2+3.356 0i
	5	5.475 7+3.022 4i	19.047 0+2.969 7i	34.532 8+2.911 0i	48.708 4+2.862 2i
out-of-phase vibration	0	10.047 2+4.961 1i	31.731 5+4.866 2i	57.374 0+4.756 4i	81.161 0+4.659 9i
	0.5	7.950 7+3.924 9i	25.111 8+3.847 6i	45.410 2+3.757 8i	64.247 7+3.678 6i
	1	7.230 4+3.570 8i	22.823 6+3.502 8i	41.241 4+3.424 4i	58.302 1+3.356 0i
	5	6.111 0+3.022 4i	19.235 9+2.969 7i	34.635 3+2.910 9i	48.779 9+2.862 1i

表 6 不同黏性弹性系数对应两端简支功能梯度双纳米梁系统的前四阶振动频率

Table 6 First 4 natural frequencies of an SS functionally graded double-nanobeam system with various viscoelastic coefficients

	$\bar{c}$	mode			
		1	2	3	4
in-phase vibration	0	7.399 2	22.871 3	41.263 8	58.315 1
	1	7.390 6+0.357 1i	22.868 7+0.350 3i	41.262 4+0.342 4i	58.314 2+0.335 6i
	5	7.180 6+1.785 4i	22.804 2+1.751 4i	41.228 3+1.712 2i	58.291 2+1.678 0i
	10	6.480 6+3.570 8i	22.601 6+3.502 8i	41.121 7+3.424 4i	58.219 2+3.356 0i
out-of-phase vibration	0	8.064 1	23.090 8	41.383 1	58.397 9
	1	8.056 2+0.357 1i	23.088 1+0.350 3i	41.381 6+0.342 4i	58.397 0+0.335 6i
	5	7.863 9+1.785 4i	23.024 3+1.751 4i	41.347 7+1.712 2i	58.374 0+1.678 0i
	10	7.230 4+3.570 8i	22.823 6+3.502 8i	41.241 4+3.424 4i	58.302 1+3.356 0i

8 结　　论

本文针对一类双功能梯度纳米梁系统的自由振动问题提出一种全新的解析求解方法.利用 Eringen 提出的非局部 Timoshenko 梁理论,以经典变量表示该类双纳米梁系统振动控制微分方程.通过变量代换,将原问题分解为同向振动和反向振动,从而进一步获得统一形式的控制方程.在 Hamilton 体系下,基本未知量为原变量及其对偶变量组成的全状态向量.传统 Lagrange 体系下的高阶偏微分振动控制方程转化为一组常微分方程组.因此,分离变量法得以应用,原自由振动问题归结为 Hamilton 矩阵的本征值和本征解求解问题,并可以直接获得 6 种端

部条件下的解析频率方程和振动模态.对比算例验证了本文提出方法的正确性与精确性,并针对关键影响参数进行了系统分析.研究结果表明,功能梯度指数对双功能梯度纳米梁系统自振频率的实部和虚部影响效果显著,而尺度效应仅对自振频率实部影响较大,介质黏弹性参数仅对自振频率虚部作用效果明显.上述结论可以为该类纳米梁系统的设计提供科学指导,并为其安全评估提供可靠依据.此外,本文提出的辛方法可以推广至其他纳米结构的动力分析中,为相关领域提供解析求解方法.

参考文献(References):

[1] EICHENFIELD M, CAMACHO R, CHAN J, et al. A picogram and nanometre-scale photonic-crystal optomechanical cavity[J]. *Nature*, 2009, **459**(7246): 550-555.

[2] LIN Q, ROSENBERG J, CHANG D, et al. Coherent mixing of mechanical excitations in nano-optomechanical structures[J]. *Nature Photonics*, 2010, **4**(4): 236-242

[3] FU Y, DU H, HUANG W, et al. TiNi-based thin films in MEMS applications: a review[J]. *Sensors and Actuators A: Physical*, 2004, **112**(2): 395-408.

[4] KAHROBAIYAN M H, ASGHARI M, RAHAEIFARD M, et al. Investigation of the size-dependent dynamic characteristics of atomic force microscope microcantilevers based on the modified couple stress theory[J]. *International Journal of Engineering Science*, 2010, **48**(12): 1985-1994.

[5] ERINGEN A C. On differential-equations of nonlocal elasticity and solutions of screw dislocation and surface-waves[J]. *Journal of Applied Physics*, 1983, **54**(9): 4703-4710.

[6] ERINGEN A C. *Nonlocal Continuum Field Theories*[M]. New York: Springer, 2002.

[7] RAFII-TABAR H, GHAVANLOO E, FAZELZADEH S A. Nonlocal continuum-based modeling of mechanical characteristics of nanoscopic structures[J]. *Physics Reports*, 2016, **638**: 1-97.

[8] 林 C W. 基于非局部弹性应力场理论的纳米尺度效应研究:纳米梁的平衡条件、控制方程以及静态挠度[J]. 应用数学和力学, 2010, **31**(1): 35-50.(LIM C W. On the truth of nanoscale for nanobeams based on nonlocal elastic stress field theory: equilibrium, governing equation and static deflection[J]. *Applied Mathematics and Mechanics*, 2010, **31**(1): 35-50. (in Chinese))

[9] 尹春松, 杨洋. 考虑非局部剪切效应的碳纳米管弯曲特性研究[J]. 应用数学和力学, 2015, **36**(6): 600-606.(YIN Chunsong, YANG Yang. Shear deformable bending of carbon nanotubes based on a new analytical nonlocal Timoshenko beam model[J]. *Applied Mathematics and Mechanics*, 2015, **36**(6): 600-606.(in Chinese))

[10] ELTAHER M A, EMAM S A, MAHMOUD F F. Free vibration analysis of functionally graded size-dependent nanobeams[J]. *Applied Mathematics and Computation*, 2012, **218**(14): 7406-7420.

[11] LI L, LI X, HU Y. Free vibration analysis of nonlocal strain gradient beams made of functionally graded material[J]. *International Journal of Engineering Science*, 2016, **102**: 77-92.

[12] RAHMANI O, PEDRAM O. Analysis and modeling the size effect on vibration of functionally graded nanobeams based on nonlocal Timoshenko beam theory[J]. *International Journal of Engineering Science*, 2014, **77**(7): 55-70.

[13] ELTAHER M A, ALSHORBAGY A E, MAHMOUD F F. Determination of neutral axis position and its effect on natural frequencies of functionally graded macro/nanobeams[J]. *Composite*

Structures, 2013, **99**(5): 193-201.

[14] NIKNAM H, AGHDAM M M. A semi analytical approach for large amplitude free vibration and buckling of nonlocal FG beams resting on elastic foundation[J]. *Composite Structures*, 2015, **119**: 452-462.

[15] EL-BORGI S, FERNANDES R, REDDY J N. Non-local free and forced vibrations of graded nanobeams resting on a non-linear elastic foundation[J]. *International Journal of Non-Linear Mechanics*, 2015, **77**: 348-363.

[16] NAZEMNEZHAD R, HOSSEINI-HASHEMI S. Nonlocal nonlinear free vibration of functionally graded nanobeams[J]. *Composite Structures*, 2014, **110**: 192-199.

[17] ELTAHER M A, KHAIRY A, SADOUN A M, et al. Static and buckling analysis of functionally graded Timoshenko nanobeams[J]. *Applied Mathematics and Computation*, 2014, **229**: 283-295.

[18] LI L, HU Y. Nonlinear bending and free vibration analyses of nonlocal strain gradient beams made of functionally graded material[J]. *International Journal of Engineering Science*, 2016, **107**: 77-97.

[19] KE L L, WANG Y S, YANG J, et al. Nonlinear free vibration of size-dependent functionally graded microbeams[J]. *International Journal of Engineering Science*, 2012, **50**(1): 256-267.

[20] 杨晓东, 林志华. 利用多尺度方法分析基于非局部效应纳米梁的非线性振动[J]. 中国科学: 技术科学, 2010, **40**(2): 152-156.(YANG Xiaodong, LIN Zhihua. The nonlinear vibration of nanoscale beam based on nonlocal effect by multiscale method[J]. *Chinese Science: Technical Science*, 2010, **40**(2): 152-156. (in Chinese))

[21] YAO W A, ZHONG W X, LIM C W. *Symplectic Elasticity*[M]. Hackensack: World Scientific Pubinshing Company, 2009.

[22] LIM C W, XU X S. Symplectic elasticity: theory and applications[J]. *Applied Mechanics Reviews*, 2011, **63**(5): 050802.

[23] ZHOU Z, RONG D, YANG C, et al. Rigorous vibration analysis of double-layered orthotropic nanoplate system[J]. *International Journal of Mechanical Sciences*, 2017, **123**: 84-93.

[24] AYDOGDU M. A general nonlocal beam theory: its application to nanobeam bending, buckling and vibration[J]. *Physica E: Low-Dimensional Systems and Nanostructures*, 2009, **41**(9): 1651-1655.

[25] ARASH B, WANG Q. A review on the application of nonlocal elastic models in modeling of carbon nanotubes and graphenes[J]. *Computational Materials Science*, 2012, **51**(1): 303-313.

A Symplectic Approach for Free Vibration of Functionally Graded Double-Nanobeam Systems Embedded in Viscoelastic Medium

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Abstract: A new analytical approach was proposed for free vibration of functionally graded (FG) double-nanobeam systems (DNBSs) embedded in viscoelastic medium under the framework of symplectic mechanics and the nonlocal Timoshenko beam theory. In the Hamiltonian system, the dual variables of the displacement and the rotation angle are the generalized shear force and bending moment, respectively. The high-order governing partial differential equations in the classical Lagrangian system were simplified into a set of ordinary differential equations through introduction of an unknown vector composed of the fundamental variables and their dual variables. The free vibration of DNBSs was finally reduced to an eigenproblem in the symplectic space. Analytical frequency equations and vibration mode functions were directly obtained with the symplectic eigensolutions and boundary conditions. Numerical results verify the accuracy and efficiency of the presented method. A systematic parametric study on the small size effect, the interaction between the double nanobeams and the viscoelastic foundation influence, was also provided.

Key words: Hamiltonian system; symplectic method; functionally graded double-nanobeam system; free vibration; analytical solution

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