

谱元法求解 Helmholtz 方程 透射特征值问题*

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摘要: 研究了 Helmholtz 方程透射特征值问题,提出一种 Chebyshev 谱元法求解,该方法兼具了有限元法处理边界及区域的灵活性和谱方法的快速收敛特性.运用加权余量原理,得到了 Chebyshev 谱元法用于透射特征值问题的基本理论以及数学公式,将原问题转化为二次特征值问题.最后通过数值实验算例验证了 Chebyshev 谱元法的有效性.

关键词: 透射特征值问题; 二次特征值问题; 谱元法; Chebyshev 基函数

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引言

透射特征问题来源于非均匀的逆散射理论^[1],在近几年的发展中已经成为逆散射理论的一个重要组成部分^[2].特征值能够通过远场数据分析得到并且可以用来分析散射对象的材料属性^[3],在很多领域有广泛的应用.目前,对于透射特征值问题的数值模拟有比较多的方法,如有限元法、边界元法^[4],Colton 和 Sun 等对透射特征值问题提出了不同的离散方法^[5-8]:混合有限元法、二次特征值法和连续有限元法等.有限元法能够很好地处理边界和区域复杂结构,有很好的适应性,但是随着对象波数的增大,需要更多的网络节点数目,这样会使得计算量增大,耗费大量计算资源.对于这类问题,谱方法是一种比较好的处理方法^[9].它将插值基函数取为无限可微全局函数,具有较好的收敛性.理论上谱方法可以在最短波长上取最少的网格点得到所需的精度^[10],但是谱方法对于复杂区域上的问题求解效果不好,区域适应性不高.谱元法是由 Patera^[11]在结合谱方法的精度和有限元的思想的基础上提出的一种新方法,它兼具了有限元处理边界和区域的灵活性和谱方法的快速收敛特性.由于在单元上采用无穷光滑的插值基函数,谱元法的精度可以根据需求和计算条件来调节^[12],并且具有和有限元同样优势的几何区域适应性,所以该方法非常适应求解透射特征值问题.

本文结构如下:第1节通过变分原理得到原问题的弱形式,并对二维有界单联通区域 D 进行了单元划分,构造 Chebyshev 谱元逼近函数^[13],得到总刚度矩阵和总荷载矩阵;第2节给出多组数值实验,通过对比其他方法,验证了谱元法的有效性;第3节对文章进行了小结和展望.

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1 二维透射特征值问题数值方法

1.1 数值模型

考虑声波在一个有界单连通的各向同性介质 $D \in R^2$ 上的 Helmholtz 方程透射特征值问题,即求解 $k \in \mathbf{C}, v \in L^2(D), w \in L^2(D), v - w \in H_0^2(D)$, 满足方程:

$$\Delta w + k^2 n(x)w = 0, \quad \text{in } D, \quad (1a)$$

$$\Delta v + k^2 v = 0, \quad \text{in } D, \quad (1b)$$

$$w - v = 0, \quad \text{on } \partial D, \quad (1c)$$

$$\frac{\partial w}{\partial \boldsymbol{\gamma}} - \frac{\partial v}{\partial \boldsymbol{\gamma}} = \mathbf{0}, \quad \text{on } \partial D, \quad (1d)$$

其中 $\boldsymbol{\gamma}$ 为散射体边界上的单位法向量, $n(x)$ 表示折射率, 非零常数 k 为传输特征值, 相应的 w, v 称为特征函数, 称 (k, w, v) 为特征对.

$$H_0^2(D) = \left\{ u \in H^2(D) : u = 0, \frac{\partial u}{\partial \boldsymbol{\gamma}} = \mathbf{0} \text{ on } \partial D \right\}. \quad (2)$$

记 $u = w - v \in H_0^2(D)$, 用式(1a)减去式(1b)可得

$$(\Delta + k^2)u = -k^2(n(x) - 1)w. \quad (3)$$

式(3)左右两边同时乘以 $\Delta + k^2 n(x)$ 算子可以得到

$$(\Delta + k^2 n(x)) \frac{1}{n(x) - 1} (\Delta + k^2)u = 0. \quad (4)$$

这样,该透射特征值问题可以表述为^[14], 求 $(k^2 \neq 0, u) \in \mathbf{C} \times H_0^2(D)$ 满足对于任意 $v \in H_0^2$, 有

$$\left(\frac{1}{n(x) - 1} (\Delta + k^2)u, (\Delta + k^2 n(x))v \right) = 0. \quad (5)$$

也即

$$\int_D \frac{1}{n(x) - 1} (\Delta u + k^2 u) (\Delta v + k^2 n(x)v) dx = 0. \quad (6)$$

1.2 标准单元内函数 Chebyshev 谱逼近

将有界单连通区域 D 分解为若干个互不重叠的矩形区域, 记单元总数为 $N_G, D = \bigcup_{i=1}^{N_G} D_i$, 考察第 i 个单元 D_i , 并设 $L_x^i = x_{i+1} - x_i, L_y^i = y_{i+1} - y_i$, 将此单元 D_i 化成标准正方形求解单元 e_i , 则

$$\xi = \frac{2}{L_x^i} (x - x_i) - 1 \quad \text{or} \quad x = \frac{1}{2} L_x^i (\xi + 1) + x_i, \quad (7)$$

$$\eta = \frac{2}{L_y^i} (y - y_i) - 1 \quad \text{or} \quad y = \frac{1}{2} L_y^i (\eta + 1) + y_i. \quad (8)$$

用谱方法进行方程离散时, 通常使用 Legendre 正交多项式或者 Chebyshev 正交多项式构造全局基函数^[15]. 本文选择 Chebyshev 正交多项式配置插值点, 在单元 e_i 内, 在 x 方向选取 $N_x^i + 1$ 个点, 在 y 方向选取 $N_y^i + 1$ 个点作为 $u(x, y)$ 的插值点, 此处用 Chebyshev 多项式的极值点作为插值点:

$$\xi_j^i = \cos(j\pi/N_x^i), \quad j = 0, 1, \dots, N_x^i,$$

$$\eta_k^i = \cos(k\pi/N_y^i), \quad k = 0, 1, \dots, N_y^i.$$

在单元 e_i 内, $u^i(\xi, \eta)$ 的插值式可表达为

$$u^i(\xi, \eta) = \sum_{i=0}^{N_x^i} \sum_{k=0}^{N_y^i} u_{jk}^i h_j^i(\xi) h_k^i(\eta) = \sum_{j=0}^{N_x^i} \sum_{k=0}^{N_y^i} u_{jk}^i \phi_{jk}^i(\xi, \eta). \quad (9)$$

在单元 e_i 外, $h_j^i(\xi), h_k^i(\eta)$ 为零; 在 e_i 内满足 $h_j^i(\xi_p) = \delta_{jp}, h_k^i(\eta_q) = \delta_{kq}, (\xi_j, \eta_k)$ 和 (ξ_p, η_q) 为插值点, 且

$$h_j^i(\xi) = \frac{2}{N_x^i} \sum_{m=0}^{N_x^i} \frac{1}{c_j c_m} T_m(\xi_j^i) T_m(\xi) h_k^i(\eta) = \frac{2}{N_x^i} \sum_{n=0}^{N_x^i} \frac{1}{c_k c_n} T_n(\eta_k^i) T_m(\eta). \quad (10)$$

同理, 试函数有以下形式:

$$v^i(\xi, \eta) = \sum_{p=0}^{N_x^i} \sum_{q=0}^{N_y^i} v_{pq}^i h_p^i(\xi) h_q^i(\eta) = \sum_{p=0}^{N_x^i} \sum_{q=0}^{N_y^i} v_{pq}^i \phi_{pq}^i(\xi, \eta), \quad (11)$$

$$h_p^i(\xi) = \frac{2}{N_x^i} \sum_{l=0}^{N_x^i} \frac{1}{c_p c_l} T_l(\xi_p^i) T_l(\xi), h_q^i(\eta) = \frac{2}{N_y^i} \sum_{r=0}^{N_y^i} \frac{1}{c_q c_r} T_r(\eta_q^i) T_r(\eta), \quad (12)$$

其中 $T_k(\xi) = \cos(k \arccos x)$ 为 k 阶 Chebyshev 多项式, c_m 满足

$$c_m = \begin{cases} 2, & m = 0, N_x^i, \\ 1, & m \neq 0, N_x^i. \end{cases} \quad (13)$$

1.3 单元刚度矩阵的生成

将插值函数 $u^i(\xi, \eta), v^i(\xi, \mu)$ 代入式(6)中, 并运用 v_{pq}^i 的任意性, 可得到

$$(\mathbf{A}^i + k^2 \mathbf{B}^i + k^4 \mathbf{C}^i) \bar{\mathbf{u}} = 0, \quad (14)$$

其中 $\bar{\mathbf{u}}$ 是由全局节点值矩阵 \mathbf{u} 的列向量组成的.

下面定义以上 3 个矩阵, 见表 1.

表 1 标准单元区域上几种矩阵定义

Table 1 Several matrix definitions in the standard cell area

matrix	dimension	definition
\mathbf{A}^i	$(N_x^i + 1)^2 \times (N_y^i + 1)^2$	$A_{j_k p q}^i = \int_{D_i} \frac{1}{n-1} \Delta \phi_{jk}^i \cdot \Delta \phi_{pq}^i dx$
\mathbf{B}^i	$(N_x^i + 1)^2 \times (N_y^i + 1)^2$	$B_{j_k p q}^i = \int_{D_i} \left(\frac{1}{n-1} \Delta \phi_{jk}^i \phi_{pq}^i + \frac{n}{n-1} \Delta \phi_{jk}^i \phi_{pq}^i \right) dx$
\mathbf{C}^i	$(N_x^i + 1)^2 \times (N_y^i + 1)^2$	$C_{j_k p q}^i = \int_{D_i} \frac{n}{n-1} \phi_{jk}^i \phi_{pq}^i dx$

将 $\mathbf{A}^i, \mathbf{B}^i, \mathbf{C}^i$ 代入式(14), 可得

$$\sum_{j=0}^{N_x^i} \sum_{k=0}^{N_y^i} [\mathbf{A}^i [h_j^i(\xi) h_k^i(\eta), h_p^i(\xi) h_q^i(\eta)] + k^2 \mathbf{B}^i [h_j^i(\xi) h_k^i(\eta), h_p^i(\xi) h_q^i(\eta)] + k^4 \mathbf{C}^i [h_j^i(\xi) h_k^i(\eta), h_p^i(\xi) h_q^i(\eta)]] u_{jk}^i = 0, \quad (15)$$

其中 $j = 0, 1, \dots, N_x, k = 0, 1, \dots, N_y$.

$$\begin{aligned} \mathbf{A}^i [h_j^i(\xi) h_k^i(\eta), h_p^i(\xi) h_q^i(\eta)] = & \int_{-1}^1 \int_{-1}^1 \frac{1}{n-1} \cdot \frac{4}{L_x^i L_y^i} \cdot \frac{16}{(N_x^i N_y^i)^2} \left[h_k^i \frac{\partial^2 h_j^i(\xi)}{\partial \xi^2} + h_j^i(\xi) \frac{\partial^2 h_k^i(\eta)}{\partial \eta^2} \right] \times \\ & \left[h_q^i(\eta) \frac{\partial^2 h_p^i(\xi)}{\partial \xi^2} + h_p^i(\xi) \frac{\partial^2 h_q^i(\eta)}{\partial \eta^2} \right] d\xi d\eta, \end{aligned} \quad (16)$$

$$\mathbf{B}^i [h_j^i(\xi) h_k^i(\eta), h_p^i(\xi) h_q^i(\eta)] =$$

$$\begin{aligned}
 & - \frac{2}{L_x^i} \frac{L_y^i}{2} \frac{n+1}{n-1} \cdot \frac{16}{(N_x^i N_y^i)^2} \sum_{m=0}^{N_x^i} \sum_{n=0}^{N_y^i} \sum_{l=0}^{N_x^i} \sum_{r=0}^{N_y^i} \frac{1}{c_j c_m c_k c_n c_p c_l c_q c_r} \times \\
 & T_m(\xi_j) T_n(\eta_k) T_l(\xi_p) T_r(\eta_q) a_{ml} b_{nr} - \\
 & \frac{2}{L_x^i} \frac{L_y^i}{2} \frac{n+1}{n-1} \cdot \frac{16}{(N_x^i N_y^i)^2} \sum_{m=0}^{N_x^i} \sum_{n=0}^{N_y^i} \sum_{l=0}^{N_x^i} \sum_{r=0}^{N_y^i} \frac{1}{c_j c_m c_k c_n c_p c_l c_q c_r} \times \\
 & T_m(\xi_j) T_n(\eta_k) T_l(\xi_p) T_r(\eta_q) b_{ml} a_{nr}, \tag{17}
 \end{aligned}$$

其中

$$a_{ml} = \int_{-1}^1 \frac{\partial T_m(\xi)}{\partial \xi} \frac{\partial T_l(\xi)}{\partial \xi} d\xi, \quad b_{nr} = \int_{-1}^1 T_n(\eta) T_r(\eta) d\eta, \tag{18}$$

$$a_{nr} = \int_{-1}^1 \frac{\partial T_n(\eta)}{\partial \eta} \frac{\partial T_r(\eta)}{\partial \eta} d\eta, \quad b_{ml} = \int_{-1}^1 T_m(\xi) T_l(\xi) d\xi, \tag{19}$$

$$\begin{aligned}
 & C^i [h_j^i(\xi) h_k^i(\eta), h_p^i(\xi) h_q^i(\eta)] = \\
 & \int_{-1}^1 \int_{-1}^1 \frac{4}{L_x^i L_y^i} \cdot \frac{n}{n-1} \frac{16}{(N_x^i N_y^i)^2} \sum_{m=0}^{N_x^i} \sum_{n=0}^{N_y^i} \sum_{l=0}^{N_x^i} \sum_{r=0}^{N_y^i} \frac{1}{c_j c_k c_m c_n c_p c_q c_l c_r} \times \\
 & T_m(\xi_j) T_m(\xi) T_n(\eta_k) T_n(\eta) T_l(\xi_p) T_l(\xi) T_r(\eta_q) T_r(\eta) d\xi d\eta. \tag{20}
 \end{aligned}$$

1.4 总体矩阵的生成

将各单元上矩阵合成为总体矩阵，最终可以得到以下一般二次特征值问题：

$$(\mathbf{A} + \lambda \mathbf{B} + \lambda^2 \mathbf{C}) \bar{\mathbf{u}} = 0, \tag{21}$$

其中 $\lambda = k^2$ 。

2 数值分析

为了验证 Chebyshev 谱元法求解透射特征值问题的有效性，本文中考虑 3 种数学模型：① 以原点为中心的边长为 2 的正方形；② 以原点为中心的环型，外边界为边长为 2 的正方形，内边界为边长为 1 的正方形，环宽度为 1；③ L 型，长为 3，宽为 3，L 型区域宽度为 1。区域划分时将整个区域分解为等规模的若干个矩形，取网格大小 $h = 1/12$ ，在单元矩形上利用 Chebyshev 多项式的极值点作为插值点，本文均选择 3 阶 Chebyshev 多项式进行离散，如图 1 所示。

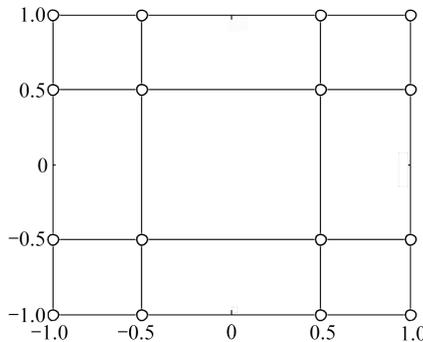


图 1 3 次 Chebyshev 多项式配置插值点示意图

Fig. 1 Collocation points of the 3rd-order Chebyshev interpolation

本文考虑 $n(x)$ 取常数和非常数两种情况，在 3 个模型区域上得到 4 个最小的正实透射特征值，结果如表 2、3 所示。

表 2 $n(x) = 16$ 时不同区域上所对应的 4 个最小的正实透射特征值Table 2 The first real transmission eigenvalues of some different regions. The index of refraction $n(x) = 16$

	the square area	the ring area	the L area
k_1	1.819 4	1.949 2	1.744 9
k_2	2.342 0	2.474 4	2.125 5
k_3	2.343 7	2.475 3	2.175 3
k_4	2.828 4	2.850 9	2.688 3

表 3 $n(x) = x_1 - x_2$ 时不同区域上所对应的 4 个最小的正实透射特征值Table 3 The first real transmission eigenvalues of some different regions. The index of refraction $n(x) = x_1 - x_2$

	the square area	the ring area	the L area
k_1	3.583 1	2.074 5	3.066 5
k_2	4.096 5	3.972 1	3.984 6
k_3	5.213 9	5.268 4	4.874 2
k_4	5.727 2	5.931 9	5.370 6

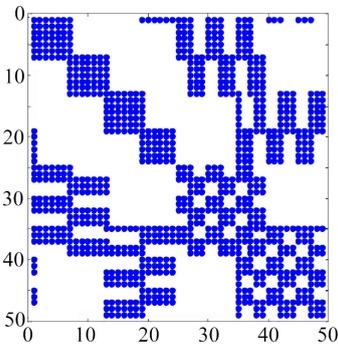
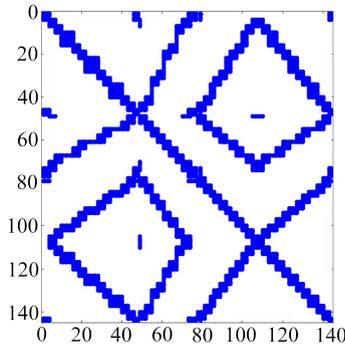
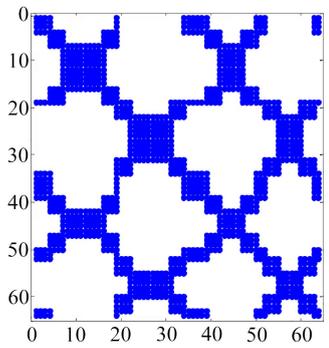
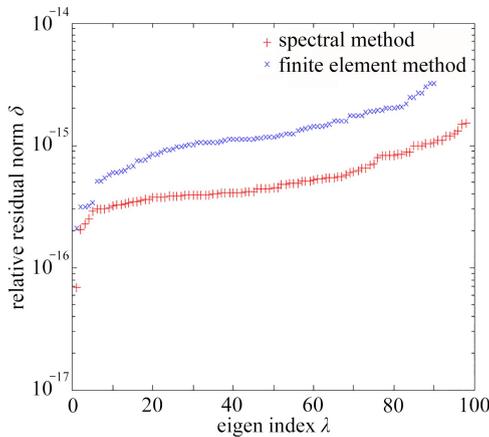
图 2 正方形区域上总体矩阵
($n_z = 967$)Fig. 2 The global mass matrix in the square area ($n_z = 967$)图 3 环形区域上总体矩阵
($n_z = 2\ 862$)Fig. 3 The global mass matrix in the ring area ($n_z = 2\ 862$)图 4 L 形区域上总体矩阵
($n_z = 1\ 262$)Fig. 4 The global mass matrix in the L area ($n_z = 1\ 262$)

图 5 正方形模型上谱元法与有限元法结果对比

Fig. 5 Comparison between the spectral element method and the finite element method in the square area

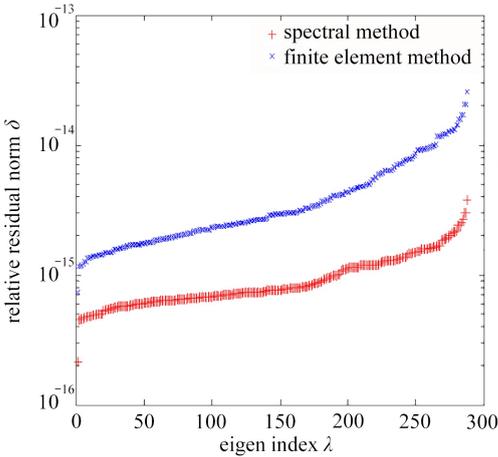


图6 环型模型上谱元法和有限元法结果对比

Fig. 6 Comparison between the spectral element method and the finite element method in the ring area

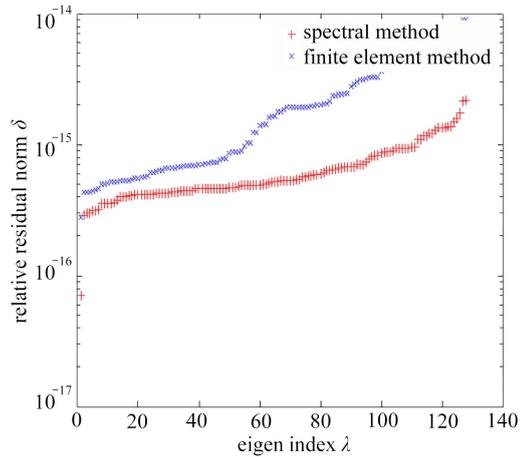


图7 L型模型上谱元法与有限元法结果对比

Fig. 7 Comparison between the spectral element method and the finite element method in the L area

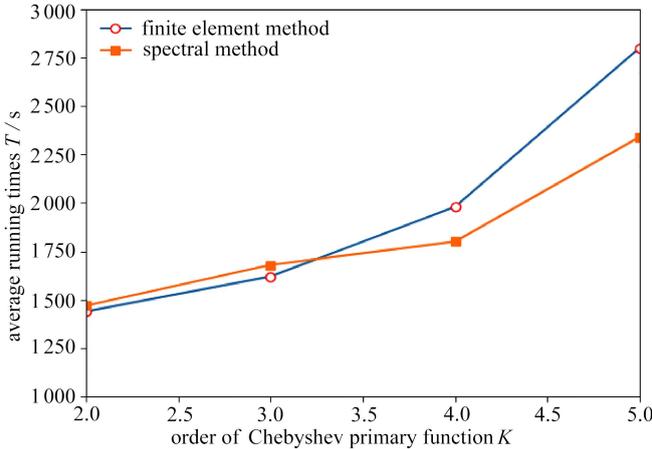


图8 在相同基函数阶数下谱元法与有限元运行时间对比

Fig. 8 The time consumptions of the spectral method and the finite element method in the same order of Chebyshev primary function

图2~图4分别给出了3种区域上总体矩阵结构,需要说明的是,由于谱元法所生成的总体矩阵是对称的、稀疏的、正定的、带状的,因此采用基于MPI的并行运算^[16],对于总体矩阵采用分布式存储,该方法可以较大地节约运行内存,从而提高执行效率,本文所有数值实验都是在MATLAB R2014a, Windows 10, 系统安装内存2.00 GB下实现,本文将Chebyshev谱元法与传统有限元法进行对比,通过误差分析得到谱元法在计算精度上相对于有限元法较高,如图5~7所示,图8给出了谱元法和有限元法执行时间对比。

通过以上数据以及对比图像反映,Chebyshev谱元法求解二维区域透射特征值问题是合理有效的,对区域适应性也比较强,在相同精度要求下,谱元法通过较稀疏的单元划分可减少计算时间和运行内存。

3 总结和发展

本文利用 Chebyshev 谱元法对二维区域上透射特征值问题进行处理,首先通过变分法将原问题变为二次特征值问题.将总体区域划分为若干规则矩形,单元上采用三阶精度的 Chebyshev 谱函数进行逼近,谱元法通过在每一个单元上应用谱展开从而结合了谱方法和有限元的优点,将每个单元上得到的矩阵集成为总体矩阵.最后再选取 3 个特定模型进行数值求解,通过实验结果验证了 Chebyshev 谱元法求解透射特征值问题的有效性.

今后的工作可以从以下几个方面展开:加密网格,并采用更高阶 Chebyshev 谱逼近;针对形状更为复杂的散射体进行计算分析;三维区域上 Chebyshev 谱元法推广.

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A Spectral Element Method for Transmission Eigenvalue Problems of the Helmholtz Equation

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Abstract: A Chebyshev spectral element method for the transmission eigenvalue problems of the Helmholtz equation was proposed, which combined the flexibility of the finite element method to deal with the boundary and region and the fast convergence of the spectral method. By means of the principle of weighted residuals, the basic theory and mathematical formulae of the Chebyshev spectral element method for transmission eigenvalue problems were obtained. The original problem was transformed into quadratic eigenvalue problems. Furthermore, several numerical examples were given to illustrate the effectiveness of the proposed method.

Key words: transmission eigenvalue problem; quadratic eigenvalue problem; spectral element method; Chebyshev basis function

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