

功能梯度圆板和环板受周边力作用的 弹性力学解*

张莹, 梅靖, 陈鼎, 杨博

(浙江理工大学 土木工程系, 杭州 310018)

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摘要: 在推广后的 England-Spencer 板理论基础上,研究了功能梯度圆板和环板受周边力作用的三维弹性场.材料参数沿板厚方向可以任意连续变化,利用复变函数理论给出了求解该问题所需要的 4 个复势表达式,其中含有的待定常数利用板的柱面边界条件确定.当功能梯度环板的内径趋向于零时便退化到圆板问题的解答.通过算例分析,讨论了材料梯度、荷载类型及板厚跨比等因素对功能梯度圆环板静力响应的影响.

关键词: 功能梯度材料; 圆环板; 边界力; 弹性力学解

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引言

圆板和环板是工程中常见的结构构件形式,对圆板和环板在不同荷载和边界条件下的弹性场研究一直是个重要的课题,其中圆板和环板受周边拉压和力偶作用的变形问题就是其中的一个重要问题^[1].另外,实际的工程构件或电子器件在有些工况下也可以简化为板受周边力作用的模型^[2].因此,对圆板和环板在周边力作用下的弹性场研究便具有了重要的理论与应用价值.

功能梯度材料是一种新型非均匀复合材料(以下简称为 FGM),可以控制其组分材料的体积含量在空间上呈光滑和连续变化,这样就可以避免传统层合材料会遇到的应力集中和界面脱离等问题.目前,功能梯度材料已在诸多工程领域得到了应用.在功能梯度板的研究方面,有许多学者做出了一些有意义的研究工作.例如夏巍等^[3]基于 Mindlin 板理论和 Von Karman 应变-位移关系导出功能梯度板的非线性静力平衡方程,采用有限元法分析了典型功能梯度板的热屈曲稳定性和热后屈曲变形问题;杨权等^[4]基于复变函数理论,结合最小二乘边界配点法研究了具有功能梯度加强环的有限尺寸开孔板在任意均布载荷作用下的应力集中问题;

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作者简介: 张莹(1993—),女,硕士生(E-mail: 1013516504@qq.com);

梅靖(1994—),男,硕士生(E-mail: 958801857@qq.com);

陈鼎(1993—),男,硕士生(E-mail: 267402159@qq.com);

杨博(1979—),男,副教授,硕士生导师(通讯作者. Tel: 0571-86843376;

E-mail: youngbo@zstu.edu.cn).

Lu 等^[5]利用推广后的 Plevako 解答获得了横观各向同性 FGM 圆板受轴对称荷载作用的弹性力学解; Alibeigloo^[6]利用状态空间微分求积方法研究了含有功能梯度核的三明治圆板的轴对称热弹性问题.在 FGM 圆板受周边力作用方面, Li 等^[7-8]利用应力法和位移法分别得到了 FGM 圆板受纯弯曲和功能梯度压电圆板受边界拉伸和弯曲时的弹性力学解.据笔者了解,目前还没有关于 FGM 环板受边界力作用研究的报道.

笔者及其合作者将 England-Spencer 理论^[9]推广到了横观各向同性材料,获得了 FGM 矩形板弯曲问题、无限大板含有圆孔和椭圆孔及含有夹杂问题的三维弹性力学解^[10-13].本文基于推广后的 England-Spencer 板理论,通过构造轴对称圆板和环板时的 4 个复势表达式,研究了圆板和环板在周边力作用下的弹性场,获得了该问题的三维弹性力学解.通过算例分析,讨论了梯度因子、荷载形式及板厚跨比对功能梯度圆环板静力响应的影响.

1 理论方程

忽略体积力,平衡方程可以表示为

$$\sigma_{ij,j} = 0, \quad (1)$$

式中下标“,”表示对其后的变量求导数.

横观各向同性材料的应力-位移关系可以表示为^[10]

$$\begin{cases} \sigma_x = c_{11}u_{,x} + c_{12}v_{,y} + c_{13}w_{,z}, & \sigma_y = c_{12}u_{,x} + c_{11}v_{,y} + c_{13}w_{,z}, \\ \sigma_{xx} = c_{44}(w_{,x} + u_{,z}), & \sigma_z = c_{13}u_{,x} + c_{13}v_{,y} + c_{33}w_{,z}, \\ \sigma_{yz} = c_{44}(v_{,z} + w_{,y}), & \sigma_{xy} = c_{66}(u_{,y} + v_{,x}), \end{cases} \quad (2)$$

式中 c_{ij} 和 $2c_{66} = c_{11} - c_{12}$ 为材料的弹性常数,对于功能梯度材料,均为坐标 z 的函数,即 $c_{ij} = c_{ij}(z)$.如果令 $c_{11} = c_{33}, c_{12} = c_{13}$ 及 $c_{44} = c_{66}$,材料便会退化为各向同性材料.

基于推广后的 England-Spencer 板理论,取如下形式的位移场展开式:

$$\begin{cases} u(x, y, z) = \bar{u} + R_1\Delta_{,x} + R_0\bar{w}_{,x} + R_2\nabla^2\bar{w}_{,x} + R_3\nabla^4\bar{w}_{,x} + R_4\nabla^6\bar{w}_{,x}, \\ v(x, y, z) = \bar{v} + R_1\Delta_{,y} + R_0\bar{w}_{,y} + R_2\nabla^2\bar{w}_{,y} + R_3\nabla^4\bar{w}_{,y} + R_4\nabla^6\bar{w}_{,y}, \\ w(x, y, z) = \bar{w} + T_1\Delta + T_2\nabla^2\bar{w} + T_3\nabla^4\bar{w} + T_4\nabla^6\bar{w}, \end{cases} \quad (3)$$

式中 $R_j(j=0,1,\dots,4)$ 和 $T_k(k=1,2,3,4)$ 为坐标 z 的函数; $\bar{u} = \bar{u}(x, y)$, $\bar{v} = \bar{v}(x, y)$ 和 $\bar{w} = \bar{w}(x, y)$ 为板的中面位移;且

$$\Delta = \bar{u}_{,x} + \bar{v}_{,y}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla^4 = \nabla^2\nabla^2, \quad \nabla^6 = \nabla^2\nabla^4. \quad (4)$$

函数 $R_j(j=0,1,2)$ 和 $T_k(k=1,2)$ 的表达式可以利用板上下表面的应力边界条件确定(参见文献[10]).板中面位移分量的控制方程为

$$S_1(h/2)\nabla^4\bar{w} = -p(x, y) + S_{21}\nabla^2p(x, y), \quad (5)$$

$$\frac{\partial E(x, y)}{\partial x} - \frac{\partial \Omega}{\partial y} = 0, \quad \frac{\partial E(x, y)}{\partial y} + \frac{\partial \Omega}{\partial x} = 0, \quad (6)$$

式中 $S_{21} = S_2(h/2)/S_1(h/2)$, $p(x, y)$ 为作用在板表面 $z = h/2$ 处的任意双调和荷载,

$$E(x, y) = \kappa_1\Delta + \kappa_2\nabla^2\bar{w} + \kappa_3\nabla^4\bar{w} + \kappa_4\nabla^6\bar{w}, \quad \Omega = \bar{v}_{,x} - \bar{u}_{,y}, \quad (7)$$

$$S_1(z) = c_{33}T'_3 + c_{13}\left(R_2 - \frac{\kappa_2}{\kappa_1}R_1\right), \quad S_2(z) = c_{33}T'_4 + c_{13}\left(R_3 - \frac{\kappa_3}{\kappa_1}R_1\right), \quad (8)$$

式中 $\kappa_1, \kappa_2, \kappa_3$ 和 κ_4 为常数^[10].

根据文献[10],板的中面位移可以利用4个解析函数 $\alpha(\zeta), \beta(\zeta), \phi(\zeta)$ 和 $\psi(\zeta)$ 表示为

$$\bar{w} = \alpha(\zeta) + \overline{\alpha(\zeta)} + \bar{\zeta}\beta(\zeta) + \zeta\overline{\beta(\zeta)} + W(\zeta, \bar{\zeta}), \quad (9)$$

$$D = \bar{u} + i\bar{v} = \frac{\kappa_1 + 1}{\kappa_1 - 1} \phi(\zeta) - \zeta \overline{\phi'(\zeta)} - \overline{\psi(\zeta)} - 2 \frac{\kappa_2}{\kappa_1} [\beta(\zeta) + \zeta \overline{\beta'(\zeta)}] - \frac{2}{\kappa_1} (\kappa_2 + \kappa_3 \nabla^2 + \kappa_4 \nabla^4) \frac{\partial W}{\partial \bar{\zeta}}, \quad (10)$$

式中 $W(\zeta, \bar{\zeta})$ 为式(5)中与荷载有关的一个特解,

$$\zeta = x + iy, \bar{\zeta} = x - iy, 2 \frac{\partial}{\partial \zeta} = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}, 2 \frac{\partial}{\partial \bar{\zeta}} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}. \quad (11)$$

当 $p(x, y) = 0$ 时,可得 $W(\zeta, \bar{\zeta}) = 0$,此时板的三维位移和应力分量表达式为

$$w(x, y, z) = \bar{\zeta}\beta(\zeta) + \zeta\overline{\beta(\zeta)} + \alpha(\zeta) + \overline{\alpha(\zeta)} + \frac{2T_1}{\kappa_1 - 1} [\phi'(\zeta) + \overline{\phi'(\zeta)}] + 4 \left(T_2 - \frac{\kappa_2}{\kappa_1} T_1 \right) [\beta'(\zeta) + \overline{\beta'(\zeta)}], \quad (12)$$

$$u(x, y, z) + iv(x, y, z) = \frac{\kappa_1 + 1}{\kappa_1 - 1} \phi(\zeta) - \zeta \overline{\phi'(\zeta)} - \overline{\psi(\zeta)} + \frac{4}{\kappa_1 - 1} R_1 \overline{\phi''(\zeta)} + 2 \left(R_0 - \frac{\kappa_2}{\kappa_1} \right) [\beta(\zeta) + \zeta \overline{\beta'(\zeta)}] + 2R_0 \overline{\alpha'(\zeta)} + 8 \left(R_2 - \frac{\kappa_2}{\kappa_1} R_1 \right) \overline{\beta''(\zeta)}, \quad (13)$$

$$\sigma_x + \sigma_y = \left(c_{11} + c_{12} - 2 \frac{c_{13}^2}{c_{33}} \right) \left\{ \frac{2}{\kappa_1 - 1} [\phi'(\zeta) + \overline{\phi'(\zeta)}] - 4 \left(z + \frac{\kappa_2}{\kappa_1} \right) [\beta'(\zeta) + \overline{\beta'(\zeta)}] \right\}, \quad (14)$$

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 4c_{66} [\bar{\zeta}\phi''(\zeta) + \psi'(\zeta)] - \frac{16c_{66}}{\kappa_1 - 1} R_1 \phi'''(\zeta) + 8c_{66} \left(z + \frac{\kappa_2}{\kappa_1} \right) \bar{\zeta}\beta''(\zeta) + 8c_{66} z \alpha''(\zeta) - 32c_{66} \left(R_2 - \frac{\kappa_2}{\kappa_1} R_1 \right) \beta'''(\zeta), \quad (15)$$

$$\sigma_{xz} - i\sigma_{yz} = \frac{4c_{44}}{\kappa_1 - 1} (R'_1 + T_1) \phi''(\zeta) + 8c_{44} \left[T_2 + R'_2 - \frac{\kappa_2}{\kappa_1} (T_1 + R'_1) \right] \beta''(\zeta). \quad (16)$$

积分式(14)~(16)可得板的轴力、弯矩、剪力和扭矩表达式,可见文献[10]中附录B.

2 FGM圆板和环板受周边力作用

考虑一个等厚度FGM环板,在柱坐标系 (r, θ, z) 下,坐标 $r-\theta$ 平面与板中面重合,如图1所示,板的内外半径和厚度分别为 b, a 和 h .板的上下表面处于无应力状态,在板的柱面边界上分别受轴力 N 和力偶矩 M 作用.当 $b \rightarrow 0$ 时,FGM环板即变为FGM圆板.接下来分别求解圆板和环板在此轴对称变形状态下的弹性力学解.

2.1 FGM圆板受周边力作用

考虑一个简支FGM圆板受周边轴力 N 和力偶距 M 作用,在 $r = a$ 处的边界条件为

$$\bar{w}(a) = 0, N_r(a) = N, M_r(a) = M. \quad (17)$$

此时圆板 ($0 \leq r \leq a$) 处于轴对称变形状态, 取

$$\alpha(\zeta) = \alpha_0, \beta(\zeta) = \beta_1 \zeta, \phi(\zeta) = \phi_1 \zeta, \psi(\zeta) = 0, \quad (18)$$

式中 α_0, β_1 和 ϕ_1 均为待定实常数.

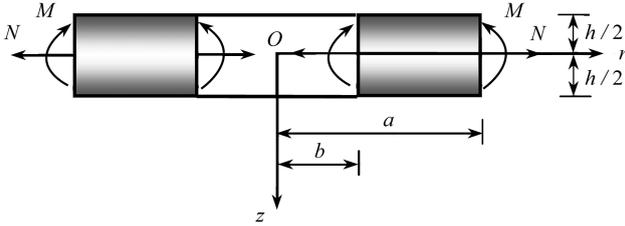


图1 FGM圆板和环板受周边面力和力偶矩作用

Fig. 1 An FGM circular/annular plate subjected to in-plane forces and moments

由式(18)可得如下圆板的中面位移和内力表达式:

$$\bar{w}(r) = 2(\alpha_0 + \beta_1 r^2), \quad (19)$$

$$N_r(r) = 4a_2\beta_1 + a_1\phi_1, \quad (20)$$

$$M_r(r) = 4b_2\beta_1 - b_1\phi_1. \quad (21)$$

将式(19)~(21)代入式(17), 得

$$\alpha_0 + \beta_1 a^2 = 0, 4a_2\beta_1 + a_1\phi_1 = N, 4b_2\beta_1 - b_1\phi_1 = M. \quad (22)$$

联立求解式(22)中的3个方程, 得

$$\begin{Bmatrix} \alpha_0 \\ \beta_1 \\ \phi_1 \end{Bmatrix} = -\frac{1}{4(a_2b_1 + b_2a_1)} \begin{bmatrix} -4a_2b_1 - 4a_1b_2 & a^2b_1 & a^2a_1 \\ 0 & -b_1 & -a_1 \\ 0 & -4b_2 & 4a_2 \end{bmatrix} \begin{Bmatrix} 0 \\ N \\ M \end{Bmatrix}. \quad (23)$$

由式(23)可得板的位移和正应力表达式为

$$u(r, z) = \frac{2r(b_2N - a_2M)}{(\kappa_1 - 1)(a_1b_2 + a_2b_1)} + \frac{r\left(R_0 - \frac{\kappa_2}{\kappa_1}\right)(a_1M + b_1N)}{a_1b_2 + a_2b_1}, \quad (24)$$

$$w(r, z) = \frac{(a_1M + b_1N)\left(r^2 - a^2 + 4T_2 - 4T_1\frac{\kappa_2}{\kappa_1}\right)}{2(a_1b_2 + a_2b_1)} + \frac{4T_1(b_2N - a_2M)}{(\kappa_1 - 1)(a_1b_2 + a_2b_1)}, \quad (25)$$

$$\sigma_r = \sigma_\theta = \left(c_{11} + c_{12} - 2\frac{c_{13}^2}{c_{33}}\right) \left\{ \frac{2(b_2N - a_2M)}{(\kappa_1 - 1)(a_1b_2 + a_2b_1)} - \frac{(z + \kappa_2/\kappa_1)(b_1N + a_1M)}{a_1b_2 + a_2b_1} \right\}. \quad (26)$$

当材料退化为各向同性均匀材料时, 有

$$\begin{cases} a_1 = \frac{2Eh}{1 + \nu}, a_2 = b_1 = 0, b_2 = -\frac{Eh^3}{12(1 - \nu)}, \kappa_1 = \frac{2}{1 - \nu}, \kappa_2 = 0, \\ T_1(z) = -\frac{\nu}{1 - \nu}z, T_2(z) = \frac{1}{2}\frac{\nu}{1 - \nu}z^2, R_0 = -z, \end{cases} \quad (27)$$

式中 E 和 ν 为材料的弹性模量和 Poisson(泊松)比. 将式(27)代入式(24)~(26)可得

$$u(r, \theta) = \frac{12(1 - \nu)rzM}{Eh^3} + \frac{r(1 - \nu)N}{Eh}, \quad (28)$$

$$w(r, \theta) = \frac{6(1 - \nu)(a^2 - r^2)M - 12\nu z^2 M}{Eh^3} - \frac{2\nu z N}{Eh}, \quad (29)$$

$$\sigma_r = \sigma_\theta = \frac{12Mz}{h^3} + \frac{N}{h}. \quad (30)$$

Li 等^[7]给出了各向同性均匀材料圆板受纯弯曲时的解答:

$$u = 12(1 - \nu)zrM/(Eh^3), \quad (31)$$

$$w = 6(1 - \nu)(a^2 - r^2)M/(Eh^3) - 12\nu z^2 M/(Eh^3), \quad (32)$$

$$\sigma_r = \sigma_\theta = 12Mz/h^3. \quad (33)$$

可以发现, 当不考虑轴力 N 作用时, 本文的解答(28)~(30)与上述解答完全一致.

2.2 FGM 环板受周边力作用

现考虑一个 FGM 环板, 在 $r = b$ 处受周边力 N_1 和力偶距 M_1 作用; 在 $r = a$ 处受周边力 N_2 和力偶距 M_2 作用. 该板具有如下边界条件:

$$N_r(b) = N_1, M_r(b) = M_1, Q_{rz}(b) = 0, \quad (34)$$

$$\bar{w}(a) = 0, N_r(a) = N_2, M_r(a) = M_2. \quad (35)$$

此时 FGM 环板 ($b \leq r \leq a$) 处于轴对称变形状态, 取

$$\alpha(\zeta) = \alpha_0 + \gamma_0 \ln \frac{\zeta}{a}, \beta(\zeta) = \beta_1 \zeta + \gamma_1 \zeta \ln \frac{\zeta}{a}, \phi(\zeta) = \phi_1 \zeta, \psi(\zeta) = \psi_{-1} \zeta^{-1}, \quad (36)$$

式中 $\alpha_0, \gamma_0, \beta_1, \gamma_1, \phi_1$ 和 ψ_{-1} 为待定的实常数.

由式(36)可得环板的位移和内力表达式:

$$\bar{w}(r) = 2\alpha_0 + 2\beta_1 r^2 + 2(\gamma_0 + \gamma_1 r^2) \ln \frac{r}{a}, \quad (37)$$

$$N_r(r) = a_1 \phi_1 + 4a_2 \beta_1 + \frac{1}{2} a_1 \psi_{-1} r^{-2} + a_6 \gamma_0 r^{-2} + \frac{1}{2} \left[4a_2 \left(1 + 2 \ln \frac{r}{a} \right) - a_7 r^{-2} \right] \gamma_1, \quad (38)$$

$$M_r(r) = -b_1 \phi_1 + 4b_2 \beta_1 + \frac{1}{2} a_6 \psi_{-1} r^{-2} + \frac{1}{2} b_7 \gamma_0 r^{-2} + \frac{1}{2} \left[8b_2 \left(1 + \ln \frac{r}{a} \right) - b_6 - b_8 r^{-2} \right] \gamma_1, \quad (39)$$

$$Q_{rz}(r) = 8Q_{z2} \gamma_1 r^{-1}, \quad (40)$$

式中 Q_{z2} 为常数^[10].

将式(37)~(40)代入式(34)和(35), 得

$$\alpha_0 + \beta_1 a^2 = 0, \quad (41)$$

$$2a_1 \phi_1 + 8a_2 \beta_1 + a_1 \psi_{-1} a^{-2} + 2a_6 \gamma_0 a^{-2} + (4a_2 - a_7 a^{-2}) \gamma_1 = 2N_2, \quad (42)$$

$$2a_1 \phi_1 + 8a_2 \beta_1 + a_1 \psi_{-1} b^{-2} + 2a_6 \gamma_0 b^{-2} + \left[4a_2 \left(1 + 2 \ln \frac{b}{a} \right) - a_7 b^{-2} \right] \gamma_1 = 2N_1, \quad (43)$$

$$-2b_1 \phi_1 + 8b_2 \beta_1 + a_6 \psi_{-1} a^{-2} + b_7 \gamma_0 a^{-2} + (8b_2 - b_6 - b_8 a^{-2}) \gamma_1 = 2M_2, \quad (44)$$

$$-2b_1 \phi_1 + 8b_2 \beta_1 + a_6 \psi_{-1} b^{-2} + b_7 \gamma_0 b^{-2} + \left[8b_2 \left(1 + \ln \frac{b}{a} \right) - b_6 - b_8 b^{-2} \right] \gamma_1 = 2M_1, \quad (45)$$

$$\gamma_1 = 0. \quad (46)$$

联立求解式(41)~(45),并利用式(46),得

$$\alpha_0 = -\frac{a^2 [b_1(a^2 N_2 - b^2 N_1) + a_1(a^2 M_2 - b^2 M_1)]}{4(a^2 - b^2)(a_1 b_2 + a_2 b_1)}, \quad (47)$$

$$\begin{Bmatrix} \psi_{-1} \\ \gamma_0 \end{Bmatrix} = \frac{2a^2 b^2}{(a^2 - b^2)(a_1 b_7 - 2a_6^2)} \begin{pmatrix} b_7 & -2a_6 \\ -a_6 & a_1 \end{pmatrix} \begin{Bmatrix} N_1 - N_2 \\ M_1 - M_2 \end{Bmatrix}, \quad (48)$$

$$\begin{Bmatrix} \phi_1 \\ \beta_1 \end{Bmatrix} = \frac{1}{4(a^2 - b^2)(a_1 b_2 + a_2 b_1)} \begin{pmatrix} 4b_2 & -4a_2 \\ b_1 & a_1 \end{pmatrix} \begin{Bmatrix} a^2 N_2 - b^2 N_1 \\ a^2 M_2 - b^2 M_1 \end{Bmatrix}. \quad (49)$$

这样,将式(46)~(49)代入式(36)就可确定该4个解析函数,进而得到FGM环板受周边力 N 和力偶矩 M 作用时的三维弹性力学解.当 $N_1 = N_2 = N$, $M_1 = M_2 = M$ 时,可以发现FGM环板的位移表达式与FGM圆板的完全一致.

当 $b \rightarrow 0$ 时,由式(46)~(49)可得

$$\begin{cases} \psi_{-1} = 0, \gamma_0 = 0, \\ \begin{Bmatrix} \phi_1 \\ \beta_1 \end{Bmatrix} = \frac{1}{4(a_1 b_2 + a_2 b_1)} \begin{pmatrix} 4b_2 & -4a_2 \\ b_1 & a_1 \end{pmatrix} \begin{Bmatrix} N_2 \\ M_2 \end{Bmatrix}, \\ \alpha_0 = -\frac{a^2(b_1 N_2 + a_1 M_2)}{4(a_1 b_2 + a_2 b_1)}. \end{cases} \quad (50)$$

通过比较可以发现,式(50)的4个解析函数表达式与式(18)完全一致,进而证明了FGM环板解的正确性.利用确定后的4个解析函数表达式(18)和(36),可以获得周边轴力和力偶矩作用下FGM圆板和环板中任意位置处的三维位移和应力解答.

3 数值算例

考虑一个受周边力作用的功能梯度圆板或环板.取 $N = 100 \text{ N/m}$, $M = 600 \text{ N}\cdot\text{m/m}^{[8]}$; $a = 0.1 \text{ m}$, $b = 0.025 \text{ m}$.如无特别说明,板的厚度取 $h = 0.015 \text{ m}$,计算时圆板取 $r = 0$ 处,环板取 $r = (a + b)/2$ 处.假设材料常数沿板的厚度方向按如下幂律模式变化^[14]:

$$c_{ij} = c_{ij}^{0(\text{A})} (0.5 - z/h)^\lambda + c_{ij}^{0(\text{S})} [1 - (0.5 - z/h)^\lambda] \quad (i, j = 1, 2, \dots, 6), \quad (51)$$

式中 $c_{ij}^{0(\text{A})}$ 表示在 $z = -h/2$ 处铝(aluminum)的弹性常数, $c_{ij}^{0(\text{S})}$ 表示在 $z = h/2$ 处碳化硅(silicon carbide)的弹性常数,见表1所示. λ 为材料的梯度因子,反映材料的非均匀程度.显然,当 $\lambda = 0$ 时为均匀材料.

表1 两种各向同性材料的弹性常数(单位:GPa)
Table 1 Elastic constants of 2 isotropic materials(unit: GPa)

material	c_{11}	c_{13}	c_{44}
aluminum	94.23	40.38	26.92
silicon carbide	565.38	242.31	161.54

表2给出了 λ 取不同值时,FGM环板受力偶矩 M 作用和轴力 N 与力偶矩 M 同时作用时的无量纲挠度 $w(\bar{r}, h/2)/h$, $\bar{r} = (a + b)/2$.可以发现,随着 λ 的增大,挠度减小,这是由于板的抗弯刚度增加的缘故.轴力 N 对板挠度的影响不大,但对于FGM环板而言,随着 λ 的增大,这种影响逐渐增加,即两种工况下挠度值的差异在减小.

表 2 环板无量纲挠度 $w(\bar{r}, h/2)/h$
Table 2 Dimensionless deflection $w(\bar{r}, h/2)/h$

λ	load condition	
	M	N, M
0	$4.298\ 95 \times 10^{-3}$	$4.298\ 92 \times 10^{-3}$
1	$1.485\ 62 \times 10^{-3}$	$1.485\ 17 \times 10^{-3}$
2	$1.225\ 34 \times 10^{-3}$	$1.225\ 04 \times 10^{-3}$
3	$1.106\ 60 \times 10^{-3}$	$1.106\ 37 \times 10^{-3}$
4	$1.035\ 16 \times 10^{-3}$	$1.034\ 98 \times 10^{-3}$
5	$9.865\ 5 \times 10^{-4}$	$9.864\ 0 \times 10^{-4}$
10	$8.706\ 2 \times 10^{-4}$	$8.705\ 4 \times 10^{-4}$
20	$7.995\ 9 \times 10^{-4}$	$7.995\ 4 \times 10^{-4}$

图 2 和图 3 分别给出了 FGM 圆板受 M 作用时沿厚度方向和径向的无量纲挠度分布.可以发现,沿厚度方向挠度分布基本上为直线,说明沿厚度方向的无量纲挠度值基本上是相等的;正如预期的那样,在板中心处的挠度值最大;在分布模式(51)下,材料梯度分布可以显著提高板的弯曲刚度,从而大大降低板的最大挠度.

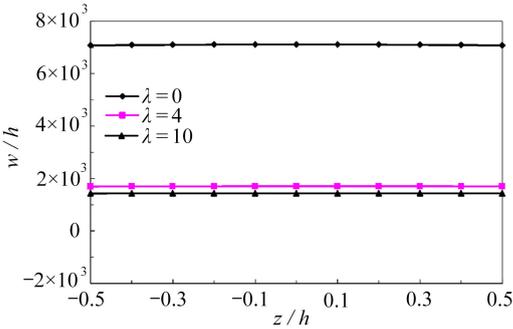


图 2 FGM 圆板受 M 作用时沿厚度方向的挠度分布
Fig. 2 Through-thickness deflection distributions of the FGM circular plate subjected to M

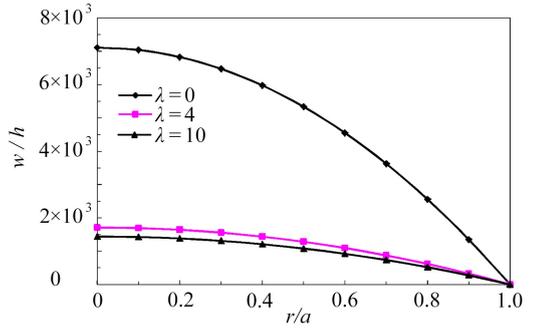


图 3 FGM 圆板受 M 作用时沿径向的挠度分布
Fig. 3 Through-radius deflection distributions of the FGM circular plate subjected to M

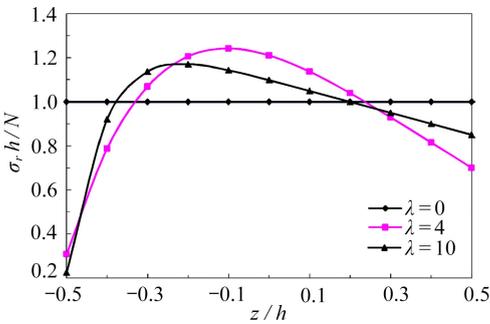


图 4 FGM 圆板受 N 作用时沿厚度方向的正应力分布
Fig. 4 Through-thickness normal stress distributions of the FGM circular plate subjected to N

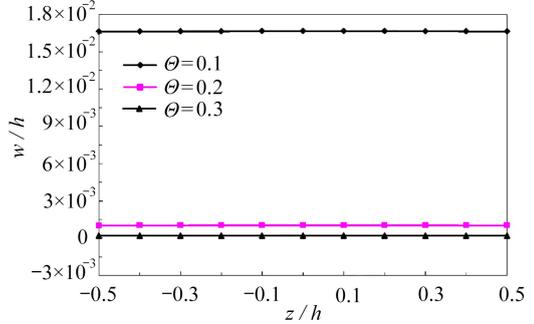


图 5 FGM 环板受 N 和 M 作用时沿厚度方向的挠度分布
Fig. 5 Through-thickness deflection distributions of the FGM annular plate subjected to N and M

图 4 给出了 FGM 圆板受 N 作用时沿厚度方向的无量纲正应力.可以发现,对于均匀材料板,厚度方向上每点的正应力都相等;对于 FGM 板,正应力沿厚度方向呈非线性变化,应力

的最大值在板中面附近,随着 λ 的增大,应力的最大值向板的上表面($z = -h/2$)靠近。

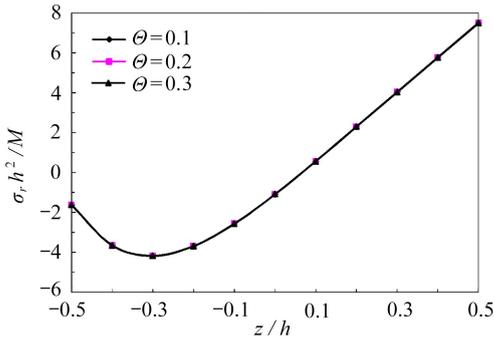


图6 FGM环板受 N 和 M 作用时沿厚度方向的正应力分布(Θ 变化)

Fig. 6 Through-thickness normal stress distributions of the FGM annular plate subjected to N and M vs. Θ

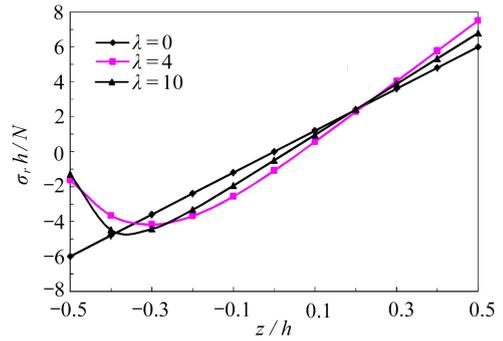


图7 FGM环板受 N 和 M 作用时沿厚度方向的正应力分布(λ 变化)

Fig. 7 Through-thickness normal stress distributions of the FGM annular plate subjected to N and M vs. λ

图5和图6给出了FGM环板受 N 和 M 作用时沿厚度方向的无量纲挠度和正应力。这里 $\Theta = h/(a-b)$ 表示环板的厚径比, $\lambda = 4$ 。可以发现,随着 Θ 增大,挠度在减小,这是由于板的厚度增加使得板的抗弯刚度也相应地增大。对于无量纲正应力分布而言, Θ 的影响可以忽略不计。由式(30)可知,厚径比对圆板的无量纲正应力影响也是如此。

图7给出了FGM环板受 N 和 M 作用时沿厚度方向的无量纲正应力。可以发现,对于均匀材料板,上下表面的最大拉压应力是相等的;对于FGM板,正应力沿厚度方向呈非线性变化特征,最大拉应力还是出现在板的下表面($z = h/2$)处,但最大压应力则不再出现在板的上表面($z = -h/2$),而是随着 λ 的增大,最大压应力向板的上表面($z = -h/2$)靠近。

4 结 论

本文利用England-Spencer板理论研究了FGM圆板和环板在边界力作用下的弹性场问题。材料参数沿厚度方向可以任意变化。利用复变函数解法分别获得了在边界轴力和力偶矩作用下FGM圆板和环板的三维弹性力学解。数值结果表明,材料梯度因子对FGM圆环板的弹性场具有显著的影响,板的厚径比对其位移有影响,而对无量纲正应力的影响可以忽略不计。因此,可以在工程应用中通过调整上述影响因素进而对FGM圆环板在边界力作用下的问题予以优化设计。

本文的解析解精确满足弹性力学的全部方程及上下板表面的应力边界条件,在板的柱面边界上采用Saint-Venant(圣维南)原理对边界条件进行了放松处理。因此,给出的解答可以作为基准解用以评价对该问题分析所采用的各种近似的弹性板理论以及各种数值解法。

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参考文献(References):

- [1] TIMOSHENKO S P, GOODIER J N. *Theory of Elasticity* [M]. 3rd ed. New York: McGraw-Hill, 1970.
- [2] FANG Daining, WANG Ji, CHEN Weiqu. *Analysis of Piezoelectric Structures and Devices* [M]. Higher Education Press, 2013.
- [3] 夏巍, 赵东伟, 冯宇鹏. 基于Mindlin横剪变形理论的功能梯度板热屈曲分析[J]. 应用力学学

- 报, 2016, **33**(1): 13-18. (XIA Wei, ZHAO Dongwei, FENG Yupeng. Thermal buckling analysis of functionally graded plates based on Mindlin transverse shear deformable theory[J]. *Chinese Journal of Applied Mechanics*, 2016, **33**(1): 13-18. (in Chinese))
- [4] 杨权权, 朱为国, 刘飞. 具有功能梯度加强环的有限尺寸开孔板应力集中问题[J]. *应用数学和力学*, 2016, **37**(11): 1239-1246. (YANG Quanquan, ZHU Weiguo, LIU Fei. Stress concentration in finite plates with functionally graded rings around circular holes[J]. *Applied Mathematics and Mechanics*, 2016, **37**(11): 1239-1246. (in Chinese))
- [5] LU Y Y, SHI J T, NIE G J, et al. An elasticity solution for transversely isotropic, functionally graded circular plates[J]. *Mechanics of Advanced Materials and Structures*, 2016, **23**(4): 451-457.
- [6] ALIBEIGLOO A. Thermo elasticity solution of sandwich circular plate with functionally graded core using generalized differential quadrature method[J]. *Composite Structures*, 2016, **136**: 229-240.
- [7] LI Xiangyu, DING Haojiang, CHEN Weiqiu. Pure bending of simply supported circular plate of transversely isotropic functionally graded material[J]. *Journal of Zhejiang University: Science A*, 2006, **7**(8): 1324-1328.
- [8] LI X Y, WU J, DING H J, et al. 3D analytical solution for a functionally graded transversely isotropic piezoelectric circular plate under tension and bending[J]. *International Journal of Engineering Science*, 2011, **49**(7): 664-676.
- [9] ENGLAND A H. Bending solution for inhomogeneous and laminated elastic plates[J]. *Journal of Elasticity*, 2006, **82**(2): 129-173.
- [10] YANG B, DING H J, CHEN W Q. Elasticity solutions for functionally graded rectangular plates with two opposite edges simply supported[J]. *Applied Mathematical Modelling*, 2012, **36**: 488-503.
- [11] YANG B, CHEN W Q, DING H J. 3D elasticity solutions for equilibrium problems of transversely isotropic FGM plates with holes[J]. *Acta Mechanica*, 2015, **226**(5): 1571-1590.
- [12] YANG B, CHEN W Q, DING H J. Equilibrium of transversely isotropic FGM plates with elliptical holes: 3D elasticity solutions[J]. *Archive of Applied Mechanics*, 2016, **86**(8): 1391-1414.
- [13] YANG B, CHEN W Q, DING H J. Several three-dimensional solutions for transversely isotropic functionally graded material plate welded with circular inclusion[J]. *Applied Mathematics and Mechanics (English Edition)*, 2016, **37**(6): 683-694.
- [14] 杨云芳, 杨博, 陈伟球, 等. 功能梯度板的柱面弯曲弹性力学解[J]. *应用数学和力学*, 2015, **36**(12): 1257-1264. (YANG Yunfang, YANG Bo, CHEN Weiqiu. Elasticity solutions for cylindrical bending of functionally graded plates[J]. *Applied Mathematics and Mechanics*, 2015, **36**(12): 1257-1264. (in Chinese))

Elasticity Solutions for Functionally Graded Circular and Annular Plates Subjected to Boundary Forces and Moments

ZHANG Ying, MEI Jing, CHEN Ding, YANG Bo

(Department of Civil Engineering, Zhejiang Sci-Tech University,
Hangzhou 310018, P.R.China)

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Abstract: On the basis of the generalized England-Spencer plate theory, the 3D elastic fields of functionally graded circular and annular plates subjected to boundary forces were studied. The elastic constants of the material can be arbitrarily continuously variable along the thickness direction. Expressions of 4 complex potentials for solving the problem were given with the complex variable function method in which the undetermined constants were solved from the cylindrical boundary conditions of the plate. When the inner radius of the annular plate approaches zero, solutions of the annular plate reach to those of the corresponding circular plate. Numerical examples conducted show the effects of the material gradient, the load type and the thickness to radius ratio on the static responses of functionally graded plates.

Key words: functionally graded material; annular plate; boundary force; elasticity solution

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