文章编号:1000-0887(2018)03-0355-09

一类非线性奇异摄动自治微分系统的渐近解*

冯依虎[1], 陈怀军[2], 莫嘉琪[2]

(1. 亳州学院 电子与信息工程系, 安徽 亳州 236800;
2. 安徽师范大学 数学计算机科学学院, 安徽 芜湖 241003)

摘要: 研究了一类广义 Lienard 奇异摄动系统.首先, 求出了系统的退化解;其次, 利用奇异摄动方法得到了系统的外部解,并用伸长变量方法, 求得了系统的初始层校正项;最后, 得到了系统解的任意次渐近解析展开式,并证明了解的一致有效性.该文所用的方法和理论, 具有广泛的实际应用价值.

关 键 词: 伸长变量; 奇异摄动; 自治系统

中图分类号: O175.19　　**文献标志码:** A　　DOI: 10.21656/1000-0887.380110

引　　言

非线性问题自治微分系统广泛存在于物理学、力学、生态学、工程数学和其他应用科学的很多领域中.诸多学者在理论物理、应用力学、电路、生化系统等方面都作了一些研究和应用[1-8].例如, 对于鸭解轨线的非线性系统的探讨,在自治微分系统定性理论相应的实际问题中具有深远意义[9-14].由于非线性自治微分系统通常不能用有限项的初等函数来表示它的精确解,所以需用近似的方法来得到其近似解析解.莫嘉琪、冯依虎等也曾经讨论了某些非线性反应扩散、孤波、激光脉冲、鸭解、生态和大气物理等相关问题[15-27].本文利用微分系统的比较定理和摄动方法来求得一类广义 Lienard 自治微分系统的奇异摄动渐近解析解.

研究如下一类广义 Lienard 奇异摄动自治微分系统的初值问题:

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}t} - (ax + by) = f(x,y), \qquad t \in [0,T], \tag{1}$$

$$\varepsilon \frac{\mathrm{d}y}{\mathrm{d}t} - (cx + dy) = g(x,y), \qquad t \in [0,T], \tag{2}$$

$$x(0) = \bar{x},\ y(0) = \bar{y}, \tag{3}$$

其中 ε 为小的正参数, T 为足够大的正常数, $a,b,c,d,\bar{x},\bar{y}$ 为常数, t 为自变量, $x,y \in D \subset R^2$ 为状态变量, D 为二维单通有限光滑凸域.

对广义 Lienard 奇异摄动自治系统初值问题(1)~(3),假设:

(H1) f,g 是关于其变量为充分光滑的函数;

* **收稿日期:** 2017-04-25; **修订日期:** 2017-06-07
基金项目: 国家自然科学基金(11202106);安徽省教育厅自然科学重点基金(KJ2015A347;KJ2017A702);安徽省高校优秀青年人才支持计划重点项目(gxyqZD2016520)
作者简介: 冯依虎(1982—),男,副教授,硕士(E-mail: fengyihubzsz@163.com);
莫嘉琪(1937—),男,教授(通讯作者. E-mail: mojiaqi@mail.ahnu.edu.cn).

(H2) 存在一个正常数 δ, 成立

$$\max[f_x(X,Y)+f_y(X,Y),g_x(X,Y)+g_y(X,Y)]\geqslant\delta;$$

(H3) 存在正常数 $C_{ij}(i,j=1,2)$, 满足不等式

$$|f_x|\leqslant C_{11},\ |f_y|\leqslant C_{12},\ |g_x|\leqslant C_{21},\ |g_y|\leqslant C_{22},$$

且矩阵 $\begin{pmatrix} a-C_{11} & b-C_{12} \\ c-C_{21} & d-C_{22}\end{pmatrix}$ 的特征根的实部小于零.

由上述假设知, 广义 Lienard 奇异摄动自治微分系统(1)、(2)的退化情形

$$ax+by=f(x,y), \tag{4}$$

$$cx+dy=g(x,y) \tag{5}$$

有如下一组解:

$$x=X_0(t),\ y=Y_0(t). \tag{6}$$

1 广义 Lienard 系统外部解

设广义 Lienard 奇异摄动自治微分系统初值问题(1)~(3)的外部解 (X,Y) 为

$$X(t)=\sum_{i=0}^{\infty}X_i(t)\varepsilon^i,\ Y(t)=\sum_{i=0}^{\infty}Y_i(t)\varepsilon^i. \tag{7}$$

将式(7)代入奇异摄动自治微分系统(1)、(2), 并按 ε 的幂级数展开且合并 ε 的同次幂的系数, 令 ε^0 的系数为 0,即为退化系统(4)、(5).于是它的解可由式(6)表示.

将式(7)代入奇异摄动自治微分系统(1)、(2), 由 ε^1 的系数为 0,可得

$$(a+f_x(X_0,Y_0))X_1+(b+f_y(X_0,Y_0))Y_1=\frac{dX_0}{dt}, \tag{8}$$

$$(c+g_x(X_0,Y_0))X_1+(d+g_y(X_0,Y_0))Y_1=\frac{dY_0}{dt}. \tag{9}$$

显然,上述微分系统(8)、(9)的解为

$$X_1(t)=\frac{\Delta_{11}}{\Delta},\ Y_1(t)=\frac{\Delta_{12}}{\Delta}, \tag{10}$$

其中 $\Delta,\Delta_{11},\Delta_{12}$ 分别为

$$\Delta=\begin{vmatrix} a+f_x(X_0,Y_0) & b+f_y(X_0,Y_0) \\ c+g_x(X_0,Y_0) & d+g_y(X_0,Y_0)\end{vmatrix},$$

$$\Delta_{11}=\begin{vmatrix} \frac{dX_0}{dt} & b+f_y(X_0,Y_0) \\ \frac{dY_0}{dt} & d+g_y(X_0,Y_0)\end{vmatrix},\ \Delta_{12}=\begin{vmatrix} a+f_x(X_0,Y_0) & \frac{dX_0}{dt} \\ c+g_x(X_0,Y_0) & \frac{dY_0}{dt}\end{vmatrix}.$$

不妨再补充假设:

(H4) $\Delta\neq 0$.

同样, 将式(7)代入奇异摄动自治微分系统(1)、(2),有

$$(a+f_x(X_0,Y_0))X_i+(b+f_y(X_0,Y_0))Y_i=\frac{dX_{i-1}}{dt}+F_i,\qquad i=2,3,\cdots, \tag{11}$$

$$(c+g_x(X_0,Y_0))X_i+(d+g_y(X_0,Y_0))Y_i=\frac{dX_{i-1}}{dt}+G_i,\qquad i=2,3,\cdots, \tag{12}$$

其中 $F_i, G_i (i=2,3,\cdots)$ 为关于 i 逐次已知的函数，它们分别为

$$F_i = \frac{1}{i!}\left[\frac{\partial^i}{\partial \varepsilon^i} f\left(\sum_{i=0}^{\infty} X_i(t)\varepsilon^i, \sum_{i=0}^{\infty} Y_i(t)\varepsilon^i\right)\right]_{\varepsilon=0} - [f_x(X_0,Y_0)X_i + f_y(X_0,Y_0)Y_i],$$

$$G_i = \frac{1}{i!}\left[\frac{\partial^i}{\partial \varepsilon^i} g\left(\sum_{i=0}^{\infty} X_i(t)\varepsilon^i, \sum_{i=0}^{\infty} Y_i(t)\varepsilon^i\right)\right]_{\varepsilon=0} - [g_x(X_0,Y_0)X_i + g_y(X_0,Y_0)Y_i].$$

于是微分系统(11)、(12)的解为

$$X_i(t) = \frac{\Delta_{1i}}{\Delta},\ Y_i(t) = \frac{\Delta_{2i}}{\Delta}, \qquad i = 2,3,\cdots, \tag{13}$$

其中 $\Delta_{1i}, \Delta_{2i} (i=2,3,\cdots)$ 分别为

$$\Delta_{1i} = \begin{vmatrix} \dfrac{\mathrm{d}X_{i-1}}{\mathrm{d}t} + F_i & b + f_y(X_0,Y_0) \\ \dfrac{\mathrm{d}Y_{i-1}}{\mathrm{d}t} + G_i & d + g_y(X_0,Y_0) \end{vmatrix},\ \Delta_{2i} = \begin{vmatrix} a + f_x(X_0,Y_0) & \dfrac{\mathrm{d}X_{i-1}}{\mathrm{d}t} + F_i \\ c + g_x(X_0,Y_0) & \dfrac{\mathrm{d}Y_{i-1}}{\mathrm{d}t} + G_i \end{vmatrix}.$$

2 自治系统初始层校正项

设广义 Lienard 奇异摄动自治系统初值问题(1)~(3)的解 (x,y) 为

$$x = X + \tilde{X},\ y = Y + \tilde{Y}, \tag{14}$$

其中 (X,Y) 为微分系统(1)~(3)的外部解，而 $(\tilde{X},\tilde{Y})$ 为相应的初始层校正项，并设它们为

$$\tilde{X}(\tau) = \sum_{i=0}^{\infty} \tilde{X}_i(\tau)\varepsilon^i,\ \tilde{Y}(\tau) = \sum_{i=0}^{\infty} \tilde{Y}_i(\tau)\varepsilon^i, \tag{15}$$

这里 τ 为伸长变量：$\tau = t/\varepsilon$.

将式(15)代入微分系统初值问题(1)~(3)，得

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}\tau} - (a\tilde{X} + b\tilde{Y}) = f(X+\tilde{X}, Y+\tilde{Y}) - f(X,Y), \tag{16}$$

$$\frac{\mathrm{d}\tilde{Y}}{\mathrm{d}\tau} - (c\tilde{X} + d\tilde{Y}) = g(X+\tilde{X}, Y+\tilde{Y}) - g(X,Y), \tag{17}$$

$$\tilde{X}(0) = \bar{x} - X(0),\ \tilde{Y}(0) = \bar{y} - Y(0). \tag{18}$$

再将式(15)代入式(16)~(18)，按 ε 的幂级数展开，合并 ε 各同次幂的系数，由 ε^0 的系数为0，得

$$\frac{\mathrm{d}\tilde{X}_0}{\mathrm{d}\tau} - (a\tilde{X}_0 + b\tilde{Y}_0) = f(X_0+\tilde{X}_0, Y_0+\tilde{Y}_0) - f(X_0,Y_0), \tag{19}$$

$$\frac{\mathrm{d}\tilde{Y}_0}{\mathrm{d}\tau} - (c\tilde{X}_0 + d\tilde{Y}_0) = g(X_0+\tilde{X}_0, Y_0+\tilde{Y}_0) - g(X_0,Y_0), \tag{20}$$

$$\tilde{X}_0(0) = \bar{x} - X_0(0),\ \tilde{Y}_0(0) = \bar{y} - Y_0(0). \tag{21}$$

由假设知，问题(19)~(21)有解 $(\tilde{X}_0(\tau), \tilde{Y}_0(\tau))$.

将式(15)代入微分系统(16)~(18)，由 ε^1 的系数为0，得

$$\frac{\mathrm{d}\tilde{X}_1}{\mathrm{d}\tau} - (a\tilde{X}_1 + b\tilde{Y}_1) = f_x(\bar{x},\bar{y})\tilde{X}_1 + f_y(\bar{x},\bar{y})\tilde{Y}_1, \tag{22}$$

$$\frac{\mathrm{d}\tilde{Y}_1}{\mathrm{d}\tau} - (c\tilde{X}_1 + d\tilde{Y}_1) = g_x(\bar{x},\bar{y})\tilde{X}_1 + g_y(\bar{x},\bar{y})\tilde{Y}_1, \tag{23}$$

$$\tilde{X}_1(0) = -X_1(0),\ \tilde{Y}_1(0) = -Y_1(0). \tag{24}$$

不难得到线性微分系统初值问题(22)~(24)的解为

$$\tilde{X}_1(\tau) = -X_1(0)\exp(\lambda_1\tau), \tag{25}$$

$$\tilde{Y}_1(\tau) = -Y_1(0)\exp(\lambda_2\tau), \tag{26}$$

其中 $\lambda_i(i=1,2)$ 为矩阵

$$\begin{pmatrix} a-f_x(\bar{x},\bar{y}) & b-f_y(\bar{x},\bar{y}) \\ c-g_x(\bar{x},\bar{y}) & d-g_y(\bar{x},\bar{y}) \end{pmatrix}$$

的特征根.

将式(15)代入微分系统(16)~(18),由各 $\varepsilon^i(i=2,3,\cdots)$ 的系数为0,得

$$\frac{\mathrm{d}\tilde{X}_i}{\mathrm{d}\tau} - (a\tilde{X}_i + b\tilde{Y}_i) = f_x(\bar{x},\bar{y})\tilde{X}_i + f_y(\bar{x},\bar{y})\tilde{Y}_i + \tilde{F}_i,\qquad i=2,3,\cdots, \tag{27}$$

$$\frac{\mathrm{d}\tilde{Y}_i}{\mathrm{d}\tau} - (c\tilde{X}_i + d\tilde{Y}_i) = g_x(\bar{x},\bar{y})\tilde{X}_i + g_y(\bar{x},\bar{y})\tilde{Y}_i + \tilde{G}_i,\qquad i=2,3,\cdots, \tag{28}$$

$$\tilde{X}_i(0) = -X_i(0),\ \tilde{Y}_i(0) = -Y_i(0), \tag{29}$$

其中 $\tilde{F}_i,\tilde{G}_i(i=2,3,\cdots)$ 为关于 i 逐次已知的函数, 它们分别为

$$\tilde{F}_i = \left[\frac{\partial^i}{\partial\varepsilon^i}\left(f\left(\sum_{i=0}^{\infty}(X_i(t)+\tilde{X}_i(\tau))\varepsilon^i, \sum_{i=0}^{\infty}(Y_i(t)+\tilde{Y}_i(\tau))\varepsilon^i\right) - \right.\right.$$
$$\left.\left. f\left(\sum_{i=0}^{\infty}X_i(t)\varepsilon^i, \sum_{i=0}^{\infty}Y_i(t)\varepsilon^i\right)\right)\right]_{\varepsilon=0} - f_x(\bar{x},\bar{y})\tilde{X}_i + f_y(\bar{x},\bar{y})\tilde{Y}_i,$$

$$\tilde{G}_i = \left[\frac{\partial^i}{\partial\varepsilon^i}\left(g\left[\sum_{i=0}^{\infty}(X_i(t)+\tilde{X}_i(\tau))\varepsilon^i, \sum_{i=0}^{\infty}(Y_i(t)+\tilde{Y}_i(\tau))\varepsilon^i\right) - \right.\right.$$
$$\left.\left. g\left(\sum_{i=0}^{\infty}X_i(t)\varepsilon^i, \sum_{i=0}^{\infty}Y_i(t)\varepsilon^i\right)\right)\right]_{\varepsilon=0} - g_x(\bar{x},\bar{y})\tilde{X}_i + g_y(\bar{x},\bar{y})\tilde{Y}_i.$$

因此线性系统初值问题(27)~(29)的解为

$$\tilde{X}_i(\tau) = -X_i(0)\int_0^{\tau}\tilde{F}_i(\tilde{X}_j(\tau_1),\tilde{Y}_j(\tau_1))\exp(\lambda_1(\tau-\tau_1))\mathrm{d}\tau_1,\qquad i=2,3,\cdots, \tag{30}$$

$$\tilde{Y}_i(\tau) = -Y_i(0)\int_0^{\tau}\tilde{G}_i(\tilde{X}_j(\tau_1),\tilde{Y}_j(\tau_1))\exp(\lambda_2(\tau-\tau_1))\mathrm{d}\tau_1,\qquad i=2,3,\cdots. \tag{31}$$

由假设还不难知道, 得到的初始层校正项具有如下负指数幂的性态:

$$\tilde{X}_i(\tau) = O(\exp(-k_i\tau)) = O\left(\exp\left(-k_i\frac{t}{\varepsilon}\right)\right),$$
$$\tau = \frac{t}{\varepsilon},\ t\in[0,T],\ 0<\varepsilon\ll 1, \tag{32}$$

$$\tilde{Y}_i(\tau) = O(\exp(-k_i\tau)) = O\left(\exp\left(-k_i\frac{t}{\varepsilon}\right)\right),$$
$$\tau = \frac{t}{\varepsilon},\ t\in[0,T],\ 0<\varepsilon\ll 1, \tag{33}$$

其中 $k_i(i=1,2,\cdots)$ 为适当大的正常数.

由此, 便得到了广义 Lienard 奇异摄动自治微分系统初值问题(1)~(3)的解 (x,y) 的渐近展开式:

$$x(t)=\sum_{i=0}^{\infty}\left[X_i(t)+\tilde{X}_i\left(\frac{t}{\varepsilon}\right)\right]\varepsilon^i,\qquad t\in[0,T],\ 0<\varepsilon\ll 1,\tag{34}$$

$$y(t)=\sum_{i=0}^{\infty}\left[Y_i(t)+\tilde{Y}_i\left(\frac{t}{\varepsilon}\right)\right]\varepsilon^i,\qquad t\in[0,T],\ 0<\varepsilon\ll 1.\tag{35}$$

3 比较定理

定义 1 设有函数 $\bar{x}(t)$,$\underline{x}(t)$,$\bar{y}(t)$,$\underline{y}(t)$ 在 $t\in[0,T]$ 上为无限光滑的函数,使得 $\underline{x}(t)\leqslant x(t)\leqslant\bar{x}(t)$,$\underline{y}(t)\leqslant y(t)\leqslant\bar{y}(t)$,并成立

$$\varepsilon\frac{\mathrm{d}\underline{x}}{\mathrm{d}t}-(a\underline{x}+b\underline{y})-f(\underline{x},\underline{y})\leqslant 0,\ \varepsilon\frac{\mathrm{d}\underline{y}}{\mathrm{d}t}-(c\underline{x}+d\underline{y})=g(\underline{x},\underline{y})\leqslant 0,\tag{36}$$

$$\varepsilon\frac{\mathrm{d}\bar{x}}{\mathrm{d}t}-(a\bar{x}+b\bar{y})-f(\bar{x},\bar{y})\geqslant 0,\ \varepsilon\frac{\mathrm{d}\bar{y}}{\mathrm{d}t}-(c\bar{x}+d\bar{y})=g(\bar{x},\bar{y})\geqslant 0,\tag{37}$$

$$\underline{x}(0)\leqslant\bar{x}\leqslant\bar{x}(0),\ \underline{y}(0)\leqslant\bar{y}\leqslant\bar{y}(0),\tag{38}$$

则 $(\underline{x}(t),\underline{y}(t))$ 和 $(\bar{x}(t),\bar{y}(t))$ 分别称广义 Lienard 奇异摄动自治微分系统初值问题(1)~(3)的下解和上解.

现证如下比较定理.

定理 1 在假设(H1)~(H4)下,当 $\forall\varepsilon\in(0,\varepsilon_0]$ 时(ε_0 为正常数),若广义 Lienard 奇异摄动自治微分系统初值问题(1)~(3)有一组下解 $(\underline{x}(t),\underline{y}(t))$ 和一组上解 $(\bar{x}(t),\bar{y}(t))$,则广义 Lienard 奇异摄动自治系统初值问题(1)~(3)存在一组解 $(x(t),y(t))$,并且成立

$$\underline{x}(t)\leqslant x(t)\leqslant\bar{x}(t),\ \underline{y}(t)\leqslant y(t)\leqslant\bar{y}(t),\qquad t\in[0,T].$$

证明 设 $(\bar{x}^0(t),\bar{y}^0(t))$ 为一组初始迭代.首先由下列线性微分系统构造一组序列 $\{\bar{x}^j(t),\bar{y}^j(t)\}$:

$$\varepsilon\frac{\mathrm{d}\bar{x}^j}{\mathrm{d}t}-(a\bar{x}^j+b\bar{y}^j)+N\bar{x}^j=N\bar{x}^{j-1}+f(\bar{x}^{j-1},\bar{y}^{j-1}),\qquad t\in[0,T],$$

$$\varepsilon\frac{\mathrm{d}\bar{y}^j}{\mathrm{d}t}-(c\bar{x}^j+d\bar{y}^j)+N\bar{y}^j=N\bar{y}^{j-1}+g(\bar{x}^{j-1},\bar{y}^{j-1}),\qquad t\in[0,T],$$

$$\bar{x}(0)=\bar{x},\ \bar{y}(0)=\bar{y}.$$

这里 N 为足够大的正常数,取 $\bar{u}_0=\bar{x}^0-\bar{x}^1$,$\bar{v}_0=\bar{y}^0-\bar{y}^1$,则由微分系统(36)~(38)知

$$\varepsilon\frac{\mathrm{d}\bar{u}_0}{\mathrm{d}t}-(a\bar{u}_0+b\bar{v}_0)+N\bar{u}_0=N(\bar{u}_0-\bar{u}_0)+f(\bar{u}_0,\bar{v}_0)-f(\bar{u}_0,\bar{v}_0)=0,$$

$$\varepsilon\frac{\mathrm{d}\bar{v}_0}{\mathrm{d}t}-(c\bar{u}_0+d\bar{v}_0)+N\bar{v}_0=N(\bar{v}_0-\bar{v}_0)+g(\bar{u}_0,\bar{v}_0)-g(\bar{u}_0,\bar{v}_0)=0,$$

$$\bar{u}_0(0)=0,\ \bar{v}_0(0)=0.$$

因此 $\bar{u}_0\geqslant 0$,$\bar{v}_0\geqslant 0$,即 $\bar{x}^0(t)\geqslant\bar{x}^1(t)$,$\bar{y}^0(t)\geqslant\bar{y}^1(t)$.

取 $\bar{u}_{j-1}=\bar{x}^{j-1}-\bar{x}^j$,$\bar{v}_{j-1}=\bar{y}^{j-1}-\bar{y}^j(j=2,3,\cdots)$,则由微分系统(36)~(38)知

$$\varepsilon\frac{\mathrm{d}\bar{u}_{j-1}}{\mathrm{d}t}-(a\bar{u}_{j-1}+b\bar{v}_{j-1})+N\bar{u}_{j-1}=$$
$$N(\bar{u}_{j-2}-\bar{u}_{j-1})+f(\bar{u}_{j-1},\bar{v}_{j-1})-f(\bar{u}_{j-2},\bar{v}_{j-2})\geqslant 0,$$

$$\varepsilon\frac{\mathrm{d}\bar{v}_{j-1}}{\mathrm{d}t}-(c\bar{u}_{j-1}+d\bar{v}_{j-1})+N\bar{v}_{j-1}=$$
$$N(\bar{v}_{j-2}-\bar{v}_{j-1})+g(\bar{u}_{j-1},\bar{v}_{j-1})-g(\bar{u}_{j-2},\bar{v}_{j-2})\geqslant 0,$$
$$\bar{u}_{j-1}(0)=0,\ \bar{v}_{j-1}(0)=0.$$

因此 $\bar{u}_{j-1}\geqslant 0$, $\bar{v}_{j-1}\geqslant 0$, 即

$$\bar{x}^{j-1}(t)\geqslant\bar{x}^{j}(t),\ \bar{y}^{j-1}(t)\geqslant\bar{y}^{j}(t),\qquad j=2,3,\cdots.$$

类似地,可构造序列 $(\underline{x}^{j}(t),\underline{y}^{j}(t))$, 得

$$\underline{x}^{j}(t)\geqslant\underline{x}^{j-1}(t),\ \underline{y}^{j}(t)\geqslant\underline{y}^{j-1}(t),\qquad j=2,3,\cdots.$$

现证明

$$\bar{x}^{j}(t)\geqslant\underline{x}^{j}(t),\ \bar{y}^{j}(t)\geqslant\underline{y}^{j}(t),\qquad j=1,2,\cdots.$$

令 $\tilde{u}_j=\bar{x}^{j}(t)-\underline{x}^{j}(t)$, $\tilde{v}_j=\bar{y}^{j}(t)-\underline{y}^{j}(t)$, 于是有

$$\varepsilon\frac{\mathrm{d}\tilde{u}_{j}}{\mathrm{d}t}-(a\tilde{u}_{j}+b\tilde{v}_{j})+N\tilde{u}_{j}=$$
$$N(\tilde{u}_{j-1}-\tilde{u}_{j})+f(\tilde{u}_{j},\tilde{v}_{j})-f(\tilde{u}_{j-1},\tilde{v}_{j-1})\geqslant 0,$$
$$\varepsilon\frac{\mathrm{d}\tilde{v}_{j}}{\mathrm{d}t}-(c\tilde{u}_{j}+d\tilde{v}_{j})+N\tilde{v}_{j}=$$
$$N(\tilde{v}_{j-1}-\tilde{v}_{j})+g(\tilde{u}_{j},\tilde{v}_{j})-g(\tilde{u}_{j-1},\tilde{v}_{j-1})\geqslant 0,$$
$$\tilde{u}_{j}(0)=0,\ \tilde{v}_{j}(0)=0.$$

因此 $\tilde{u}_j\geqslant 0$, $\tilde{v}_j\geqslant 0$, 即

$$\bar{x}^{j}(t)\geqslant\underline{x}^{j}(t),\ \bar{y}^{j}(t)\geqslant\underline{y}^{j}(t),\qquad j=1,2,\cdots.$$

由上述的结果, 便有

$$\underline{x}^{0}(t)\leqslant\underline{x}^{1}(t)\leqslant\cdots\leqslant\underline{x}^{j}(t)\leqslant\cdots\leqslant\bar{x}^{j}(t)\leqslant\cdots\leqslant\bar{x}^{1}(t)\leqslant\bar{x}^{0}(t),\qquad t\in[0,T],$$
$$\underline{y}^{0}(t)\leqslant\underline{y}^{1}(t)\leqslant\cdots\leqslant\underline{y}^{j}(t)\leqslant\cdots\leqslant\bar{y}^{j}(t)\leqslant\cdots\leqslant\bar{y}^{1}(t)\leqslant\bar{y}^{0}(t),\qquad t\in[0,T].$$

再由假设,有

$$\lim_{j\to\infty}\underline{x}^{j}(t)=\lim_{j\to\infty}\bar{x}^{j}(t)=x(t),\ \lim_{j\to\infty}\underline{y}^{j}(t)=\lim_{j\to\infty}\bar{y}^{j}(t)=y(t),\qquad t\in[0,T].$$

即广义 Lienard 奇异摄动自治微分系统初值问题(1)~(3)存在一组解 $(x(t),y(t))$, 并成立 $\underline{x}(t)\leqslant x(t)\leqslant\bar{x}(t)$, $\underline{y}(t)\leqslant y(t)\leqslant\bar{y}(t)$, $t\in[0,T]$. 定理 1 证毕.

4 解的一致有效性

定理 2 广义 Lienard 奇异摄动自治微分系统初值问题(1)~(3), 在假设(H1)~(H4)下,存在一组解 $(x(t),y(t))$, 并在 $t\in[0,T]$ 上成立一致有效的渐近展开式(34)、(35).

证明 首先,构造两组辅助函数 (α_i,β_i) $(i=1,2)$:

$$\alpha_i=Z_{im}-r\varepsilon^{m+1},\ \beta_i=Z_{im}+r\varepsilon^{m+1},\qquad i=1,2,\tag{39}$$

这里 r 为一个足够大的正常数,它将在下面确定, m 为任意大的正整数,而

$$Z_{1m}=\sum_{i=0}^{m}\left[X_i(t)+\tilde{X}_i\left(\frac{t}{\varepsilon}\right)\right]\varepsilon^{i},\ Z_{2m}=\sum_{i=0}^{m}\left[Y_i(t)+\tilde{Y}_i\left(\frac{t}{\varepsilon}\right)\right]\varepsilon^{i}.$$

显然, 由辅助函数式(39)不难看出

$$\alpha_i(t)\leqslant\beta_i(t),\qquad i=1,2,\ t\in[0,T],\tag{40}$$

$$\alpha_1(0) \leqslant \bar{x} \leqslant \beta_1(0),\ \alpha_2(0) \leqslant \bar{y} \leqslant \beta_1(0). \tag{41}$$

现证明

$$\varepsilon \frac{\mathrm{d}\alpha_1}{\mathrm{d}t} - (a\alpha_1 + b\alpha_2) - f(\alpha_1, \alpha_2) \leqslant 0, \tag{42}$$

$$\varepsilon \frac{\mathrm{d}\alpha_2}{\mathrm{d}t} - (c\alpha_1 + d\alpha_2) - g(\alpha_1, \alpha_2) \leqslant 0, \tag{43}$$

$$\varepsilon \frac{\mathrm{d}\beta_1}{\mathrm{d}t} - (a\beta_1 + b\beta_2) - f(\beta_1, \beta_2) \geqslant 0, \tag{44}$$

$$\varepsilon \frac{\mathrm{d}\beta_2}{\mathrm{d}t} - (c\beta_1 + d\beta_2) - g(\beta_1, \beta_2) \geqslant 0. \tag{45}$$

事实上,由辅助函数(39), 存在一组正常数 $M_i(i = 1,2)$, 有

$$\begin{aligned}
&\varepsilon \frac{\mathrm{d}\alpha_1}{\mathrm{d}t} - (a\alpha_1 + b\alpha_2) - f(\alpha_1, \alpha_2) = \\
&\quad \varepsilon \frac{\mathrm{d}Z_{1m}}{\mathrm{d}t_1} - (aZ_{1m} + bZ_{2m}) - f(Z_{1m}, Z_{2m}) - \\
&\quad [f(Z_{1m} - r\varepsilon^{m+1}, Z_{2m} - r\varepsilon^{m+1}) - f(Z_{1m}, Z_{2m})] \leqslant \\
&\quad [aX_0 + bY_0 - f(X_0, Y_0)] + \sum_{j=1}^{m} \left[\frac{\Delta_{1i}}{\Delta}\right]\varepsilon^i + \\
&\quad \left[\frac{\mathrm{d}\tilde{X}_0}{\mathrm{d}\tau} - (a\tilde{X}_0 + b\tilde{Y}_0) - f(X_0 + \tilde{X}_0, Y_0 + \tilde{Y}_0) + f(X_0, Y_0)\right] + \\
&\quad \sum_{j=1}^{m}\left[-(a + f_x(X_0, Y_0))X_j - (b + f_y(X_0, Y_0))Y_j + \frac{\mathrm{d}X_{j-1}}{\mathrm{d}t} - F_j\right]\varepsilon^j - \\
&\quad (f_x(Z_{1m}, Z_{2m}) + f_y(Z_{1m}, Z_{2m})r + M)\varepsilon^{m+1} \leqslant (M - r\delta)\varepsilon^{m+1}.
\end{aligned}$$

选取 $r \geqslant M/\delta$, 这时不等式(42)成立.

用同样的方法可以证明不等式(43)~(45).

由式(40)~(45)和定理 1,广义 Lienard 奇异摄动自治微分系统初值问题(1)~(3)存在一组解 $(x(t), y(t))$, 使得

$$\alpha_1(t) \leqslant x(t) \leqslant \beta_1(t),\ \alpha_2(t) \leqslant y(t) \leqslant \beta_2(t), \qquad t \in [0, T].$$

再由式(39)知, 解 $(x(t), y(t))$ 在 $t \in [0, T]$ 上成立一致有效的渐近展开式(34)、(35). 定理 2 证毕.

5 结　　论

广义 Lienard 奇异摄动自治微分系统在连续介质力学、凝聚态物理、应用电路、反应扩散等方面都有重要的应用,是典型的非线性微分系统.对它的研究和求解具有很大的实用价值.利用比较定理的理论并结合奇异摄动方法来求得问题的渐近解析解,可以继续利用解析的方法和运算来进一步深入地探讨相应实际问题的解,能够更加深入地了解相关物理量的解析性态.因此对它的研究具有很重要的意义,弥补了单纯用数值模拟方法得到的解的不足.

致谢　本文作者衷心感谢亳州学院重点教学研究项目(2017zdjy02)和亳州学院重点科学研究项目(BYZ2017B02)对本文的资助.

参考文献(References):

[1] CALLOT J L, DIENER F, DIENER M. Le probleme deia "chase au canard"[J]. *C R Acad Sci Paris*, *Ser 1*, 1978, **286**(22): 1059-1061.

[2] ZVOKIN A K, SHUBI M A. Nonstandard analysis and singular perturbation of ordinary differential equations[J]. *Usp Mat Nauk*, 1984, **39**(2): 77-127.

[3] ECKHAUS W. *Relaxation Oscillations Including a Standard Chase on French Ducks*[M]. *Lect Notes in Math*, Vol **985**. Berlin: Springer-Verlag, 1983: 449-494.

[4] BENOIT E, LOBRY C. Les canards de R3[J]. *C R Acad Sci Paris*, *Ser 1*, 1982, **294**(14): 483-488.

[5] XU Yun, ZHANG Jianxia, ZU Xia, et al. Nonchaotic random behaviour in the second order autonomous system[J]. *Chin Phys*, 2007, **16**(8): 2285-2290.

[6] MISCHENKO E F, ROZOV N. *Differential Equations With Small Parameters and Relaxation Oscillations*[M]. Moscow: Academic Press, 1975.

[7] LUTZ R, GOZE M. *Nonstandard Analysis*, *a Practical Guide With Applications*[M]. *Lect Notes in Math*, Vol **881**. Berlin : Springer-Verlag, 1981.

[8] EI S I, MIMURA M, NAGAYAMA M. Interacting spots in reaction diffusion systems[J]. *Discrete Contin Dyn Syst*, 2006, **14**(1): 31-62.

[9] 李翠萍. 奇异摄动与鸭解[J]. 北京航空航天大学学报, 1993, **4**(1): 84-89.(LI Cuiping. Singular perturbation and duck solutions[J]. *Journal of Beijing University of Aeronautics and Astronautics*, 1993, **4**(1): 84-89.(in Chinese))

[10] LI Cuiping. Duck solutions: a new kind of bifurcation phenomenon in relaxation oscillations [J]. *Acta Math Sinica* (*New Ser*), 1996, **12**(1): 89-104.

[11] 李翠萍. 奇异摄动中的鸭解问题[J]. 中国科学(A 辑), 1999, **29**(12): 1084-1093.(LI Cuiping. The duck solution problems in singular perturbation[J]. *Science in China*(*Series A*), 1999, **29**(12): 1084-1093.(in Chinese))

[12] XIE Feng, HAN Maoan, ZHANG Weijiang. The persistence of canards in 3-D singularly perturbed systems with two fast variables[J]. *Asymptotic Anal*, 2006, **47**(1/2): 95-106.

[13] 徐云, 张建峡, 徐霞, 等. Canard 轨迹原理[J]. 物理学报, 2008, **57**(7): 4029-4033.(XU Yun, ZHANG Jianxia, XU Xia, et al. The principle of the phase track of Canard[J]. *Acta Phys Sin*, 2008, **57**(7): 4029-4033.(in Chinese))

[14] 欧阳成, 姚静荪, 温朝晖, 等. 一类广义鸭轨迹系统轨线的构造[J]. 物理学报, 2012, **61**(3): 030202.(OUYANG Cheng, YAO Jingsun, WEN Zhaohui, et al. Constructing path curve for a class of generalized phase tracks of canard system[J]. *Acta Phys Sin*, 2012, **61**(3): 030202.(in Chinese))

[15] MO Jiaqi. A singularly perturbed nonlinear boundary value problem[J]. *J Math Anal Appl*, 1993, **178**(1): 289-293.

[16] MO Jiaqi. Singular perturbation for a class of nonlinear reaction diffusion systems[J]. *Science in China*(*Series A*), 1989, **32**(11): 1306-1315.

[17] MO Jiaqi, LIN Wantao. A nonlinear singular perturbed problem for reaction diffusion equations with boundary perturbation[J]. *Acta Math Appl Sin*, 2005, **21**(1): 101-104.

[18] MO Jiaqi, LIN Wantao, ZHU Jiang. A variational iteration method for studying the ENSO mechanism[J]. *Progress in Natural Science*, 2004, **14**(12): 1126-1128.

[19] MO Jiaqi, LIN Wantao. Asymptotic solution for a class of sea-air oscillator model for El-Nino-southern oscillation[J]. *Chinese Physics B*, 2008, **17**(2): 370-372.

[20] MO Jiaqi, LIN Wantao. Asymptotic solution of activator inhibitor systems for nonlinear reac-

tion diffusion equations[J]. *J Sys Sci & Complexity*, 2008, **20**(1): 119-128.

[21] MO Jiaqi. Variational iteration solving method for a class of generalized Boussinesq equation [J]. *Chin Phys Lett*, 2009, **26**(6): 060202.

[22] MO Jiaqi. Homotopiv mapping solving method for gain fluency of a laser pulse amplifier[J]. *Science in China*, *Ser G*, 2009, **52**(7): 1007-1010.

[23] FENG Yihu, MO Jiaqi. The shock asymptotic solution for nonlinear elliptic equatiom with two parameters[J]. *Math Appl*, 2015, **27**(3): 579-585.

[24] 冯依虎, 石兰芳, 汪维刚, 等. 一类广义非线性强阻尼扰动发展方程的行波解[J]. 应用数学和力学, 2015, **36**(3): 315-324.(FENG Yihu, SHI Lanfang, WANG Weigang, et al. The traveling wave solution for a class of generalized nonlinear strong damping disturbed evolution equations[J]. *Applied Mathematics and Mechanics*, 2015, **36**(3): 315-324.(in Chinese))

[25] 冯依虎, 莫嘉琪. 一类非线性非局部扰动 LGH 方程的孤立子行波解[J]. 应用数学和力学, 2016, **37**(4): 426-433.(FENG Yihu, MO Jiaqi. A class of soliton travelling wave solution for the nonlinear nonlocal disturbed LGH equation[J]. *Applied Mathematics and Mechanics*, 2016, **37**(4): 426-433.(in Chinese))

[26] FENG Yihu, MO Jiaqi. Asymptotic solution for singularly perturbed fractional order differential equation[J]. *J Math*, 2016, **36**(2): 239-245.

[27] FENG Yihu, CHEN Xianfeng, MO Jiaqi. The generalized interior shock layer solution of a class of nonlinear singularly perturbed reaction diffusion problem[J]. *Math Appl*, 2016, **29**(1): 161-165.

Asymptotic Solution to a Class of Nonlinear Singular Perturbation Autonomous Differential Systems

FENG Yihu[1], CHEN Huaijun[2], MO Jiaqi[2]

(1. *Department of Electronics and Information Engineering*, *Bozhou College*, *Bozhou*, *Anhui* 236800, *P.R.China*;
2. *School of Mathematics & Computer Science*, *Anhui Normal University*, *Wuhu*, *Anhui* 241003, *P.R.China*)

Abstract: A class of generalized Lienard singular perturbation systems were considered. Firstly, the reduced solution to the system was obtained. Next, the outer solution was constructed by means of the singular perturbation method. Then, a stretch variable was introduced and the initial layer corrective term was found. Finally, the arbitrary-order asymptotic analytic expansion of the system solution was given and the uniform validity of the solution was proved. The proposed method with the basic theory has wide application values.

Key words: stretch variable; singular perturbation; autonomous system

Foundation item: The National Natural Science Foundation of China(11202106)

引用本文/Cite this paper:

冯依虎, 陈怀军, 莫嘉琪. 一类非线性奇异摄动自治微分系统的渐近解[J]. 应用数学和力学, 2018, **39**(3): 355-363.

FENG Yihu, CHEN Huaijun, MO Jiaqi. Asymptotic solution to a class of nonlinear singular perturbation autonomous differential systems[J]. *Applied Mathematics and Mechanics*, 2018, **39**(3): 355-363.