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四边固支正交各向异性矩形薄板弯曲问题的辛叠加方法*

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摘要： 将正交各向异性矩形薄板方程化为 Hamilton 系统,利用分离变量法给出相应的无穷维 Hamilton 算子,进而计算出该无穷维 Hamilton 算子的本征值及对应的本征函数系, 并分别证明了本征函数系的辛正交性及完备性.之后利用辛叠加方法,求出正交各向异性矩形薄板弯曲问题的解析解.最后通过算例验证了所得解析解的正确性.

关 键 词： 正交各向异性薄板； 无穷维 Hamilton 算子； 本征函数系； 解析解

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引　　言

各向异性矩形板广泛应用于国防及民用工业等多个领域,它在各个方向的力学性质是不同的,这种力学上的各向异性为设计人员提供了自由度.但各向异性板的方程一般为高阶多变量的偏微分方程,很难得到其解析解.因此,学者们尝试用各种数值方法解决这类问题,如有限元法[1]、无单元法[2]等.与数值方法相比,解析方法给许多重要的理论问题提供了更便捷的渠道,因此解析方法不可忽略.经过学者们的长期研究,给出了关于正交各向异性板问题的一些解析方法,如有限积分变换法[3]、三角级数法[4]等.但是上述方法均属于半逆解法或基于半逆解法的方法,这类方法需要事先人为选定挠度函数等,这样的选取方法不够理性,无规律可循,因而具有一定的局限性.20 世纪 90 年代,钟万勰院士将无穷维 Hamilton 体系引入弹性力学[5],得到了辛弹性力学方法[6].到目前为止,辛弹性力学方法已经研究了许多力学中的实际问题[7-8],其中薄板相关方面问题也已有很多工作,如矩形薄板、扇形薄板[9]和环形薄板[10]等问题.但应用辛弹性力学方法解决对边固支薄板的弯曲问题时[11],会遇到对应无穷维 Hamilton 算子本征值和本征函数系均为复数的情况,需要用数值的近似计算来解决,因此利用辛弹性力学方法解决该问题不够理想.为了解决这类问题,Liu 和 Li 提出了辛叠加法[12].该方法已在各

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向同性板弯曲[13]、振动[14-15]等一系列问题中得到了广泛应用.然而到目前为止,由于各向异性板问题本身的复杂性使得该方法仍未能应用到此类问题中.基于上述情况,本文研究了四边固支正交各向异性矩形薄板弯曲方程,将其转化为 Hamilton 系统,利用分离变量法得到对应的无穷维 Hamilton 算子.经符号运算,给出对边简支边界条件下无穷维 Hamilton 算子的本征值,该本征值可分为单根和重根两种情形,并分别算出其对应的本征函数系,还分别证明了两类本征值所对应的本征函数系的辛正交性及完备性.之后利用辛叠加方法,推导出了集中荷载下四边固支正交各向异性矩形薄板弯曲问题的解析解.最后将计算结果与已有文献的结果进行比较,验证了本文所得解析解的正确性.

1 Hamilton 系统

考虑正交各向异性矩形薄板方程:

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2H\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}+Kw=q, \tag{1}$$

定义区域为

$$\left\{(x,y,z)\mid 0\leqslant x\leqslant a, 0\leqslant y\leqslant b, -\frac{h}{2}\leqslant z\leqslant\frac{h}{2}\right\},$$

其中 w 是挠度, q 是横向外荷载, K 为地基的反应模量, D_{11} 和 D_{22} 分别是板关于 y 轴和 x 轴的弯曲刚度,通过相互独立的弹性常数 E_1, E_2, G_{12}, Poisson 比 ν_{12} 和 ν_{21} 以及板厚 h 来定义:

$$D_{11}=\frac{E_1h^3}{12(1-\nu_{12}\nu_{21})},\ D_{22}=\frac{E_2h^3}{12(1-\nu_{12}\nu_{21})}, \tag{2}$$

H 称为板的有效扭转刚度

$$H=D_{12}+2D_{66}, \tag{3}$$

其中 $D_{12}=\nu_{12}D_{22}=\nu_{21}D_{11}, D_{66}=\dfrac{G_{12}h^3}{12}$ 为板的扭转刚度.

板内弯矩、扭矩、剪力以及等效剪力分别为

$$M_x=-\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right),\ M_y=-\left(D_{22}\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}\right),\ M_{xy}=-2D_{66}\frac{\partial^2 w}{\partial x\partial y}, \tag{4}$$

$$Q_x=-\frac{\partial}{\partial x}\left(D_{11}\frac{\partial^2 w}{\partial x^2}+H\frac{\partial^2 w}{\partial y^2}\right),\ Q_y=-\frac{\partial}{\partial y}\left(D_{22}\frac{\partial^2 w}{\partial y^2}+H\frac{\partial^2 w}{\partial x^2}\right), \tag{5}$$

$$V_x=Q_x+\frac{\partial M_{xy}}{\partial y},\ V_y=Q_y+\frac{\partial M_{xy}}{\partial x}. \tag{6}$$

令 $\partial w/\partial y=\theta$, 则由方程(4)~(6)可得到 Hamilton 系统为

$$\dot{\boldsymbol{U}}=\boldsymbol{H}\boldsymbol{U}+\boldsymbol{f}, \tag{7}$$

其中

$$\boldsymbol{H}=\begin{pmatrix} 0 & 1 & 0 & 0\\ -\dfrac{D_{12}}{D_{22}}\dfrac{\partial^2}{\partial x^2} & 0 & 0 & -\dfrac{1}{D_{22}}\\ -K-\left(D_{11}-\dfrac{D_{12}^2}{D_{22}}\right)\dfrac{\partial^4}{\partial x^4} & 0 & 0 & \dfrac{D_{12}}{D_{22}}\dfrac{\partial^2}{\partial x^2}\\ 0 & 4D_{66}\dfrac{\partial^2}{\partial x^2} & -1 & 0 \end{pmatrix},\ \boldsymbol{U}=\begin{pmatrix} w\\ \theta\\ -V_y\\ M_y\end{pmatrix},\ \boldsymbol{f}=\begin{pmatrix}0\\0\\q\\0\end{pmatrix},$$

$\dot{\boldsymbol{U}}$ 表示对 y 求偏导.通过计算可验证算子矩阵 $\boldsymbol{H}$ 满足 $\boldsymbol{H}^{\mathrm{T}}=\boldsymbol{JHJ}$, 即 $\boldsymbol{H}$ 是 Hamilton 算子矩阵,从而式(7)为薄板的 Hamilton 系统.

2 本征值和本征函数系

为了求解 Hamilton 系统(7),先求对应的齐次方程:

$$\dot{\boldsymbol{U}}=\boldsymbol{HU}. \tag{8}$$

利用分离变量法求解式(8),令

$$\boldsymbol{U}=\boldsymbol{X}(x)Y(y), \tag{9}$$

将式(9)代入式(8)可得

$$\frac{\mathrm{d}Y(y)}{\mathrm{d}y}=\mu Y(y),\ \boldsymbol{HX}(x)=\mu\boldsymbol{X}(x), \tag{10}$$

式中 μ 为本征值, $\boldsymbol{X}(x)$ 为相应的本征函数,记 $\boldsymbol{X}(x)=(X_1(x)\quad X_2(x)\quad X_3(x)\quad X_4(x))^{\mathrm{T}}$.

将式(10)中的第二式整理可得

$$D_{11}\frac{\mathrm{d}^4X_1(x)}{\mathrm{d}x^4}+2H\mu^2\frac{\mathrm{d}^2X_1(x)}{\mathrm{d}x^2}+(D_{22}\mu^4+K)X_1(x)=0. \tag{11}$$

令 $X_1(x)=\mathrm{e}^{\lambda x}$, 代入式(11)得其解为

$$X_1(x)=c_1\mathrm{e}^{\lambda_1x}+c_2\mathrm{e}^{-\lambda_1x}+c_3\mathrm{e}^{\lambda_2x}+c_4\mathrm{e}^{-\lambda_2x},$$

代入对边简支条件

$$\begin{cases}w(0,y)=w(a,y)=0,\\M_x(0,y)=M_x(a,y)=0\end{cases} \tag{12}$$

得

$$\begin{cases}\lambda_1=\sqrt{\dfrac{-H\mu^2+\sqrt{\mu^4(H^2-D_{11}D_{22})-KD_{11}}}{D_{11}}},\\\lambda_2=\sqrt{\dfrac{-H\mu^2-\sqrt{\mu^4(H^2-D_{11}D_{22})-KD_{11}}}{D_{11}}}.\end{cases} \tag{13}$$

通过式(13)可得 μ 为

$$\begin{cases}\mu_1=\pm\sqrt{\dfrac{a^2n^2\pi^2H+\sqrt{a^4[-a^4KD_{22}+n^4\pi^4(H^2-D_{11}D_{22})]}}{a^4D_{22}}},\\\mu_2=\pm\sqrt{\dfrac{a^2n^2\pi^2H-\sqrt{a^4[-a^4KD_{22}+n^4\pi^4(H^2-D_{11}D_{22})]}}{a^4D_{22}}}.\end{cases} \tag{14}$$

2.1 本征值为重根的情形

当 $K^2+(H^2-D_{11}D_{22})^2=0$, 根据式(14)可得 2 重根的本征值:

$$\mu_n=\alpha_n\sqrt{\frac{H}{D_{22}}},\qquad n=\pm1,\ \pm2,\ \pm3,\cdots, \tag{15}$$

其中 $\alpha_n=n\pi/a$.由式(10)中的第二式可得 μ_n 相应的本征函数为

$$\boldsymbol{X}_n^0(x)=\begin{pmatrix}1\\ \mu_n\\ \dfrac{a^2\mu_n^2D_{22}-n^2\pi^2(D_{12}+4D_{66})}{a^2}\mu_n\\ \dfrac{n^2\pi^2D_{12}-a^2\mu_n^2D_{22}}{a^2}\end{pmatrix}\sin(\alpha_n x).$$

根据 $\boldsymbol{HX}_n^1(x)=\mu_n\boldsymbol{X}_n^1(x)+\boldsymbol{X}_n^0(x)$[9]，得到 μ_n 对应的一阶 Jordan 型本征函数：

$$\boldsymbol{X}_n^1(x)=\begin{pmatrix}1\\ 1+\mu_n\\ \dfrac{-n^2\pi^2(1+\mu_n)D_{12}+a^2\mu_n^2(3+\mu_n)D_{22}-4n^2\pi^2(1+\mu_n)D_{66}}{a^2}\\ \dfrac{n^2\pi^2D_{12}-a^2\mu_n(2+\mu_n)D_{22}}{a^2}\end{pmatrix}\sin(\alpha_n x).$$

通过计算，可得到 μ_{-n} 对应的本征函数以及一阶 Jordan 型本征函数，分别为

$$\boldsymbol{X}_{-n}^0(x)=\begin{pmatrix}-1\\ \mu_n\\ \dfrac{a^2\mu_n^2D_{22}-n^2\pi^2(D_{12}+4D_{66})}{a^2}\mu_n\\ \dfrac{a^2\mu_n^2D_{22}-n^2\pi^2D_{12}}{a^2}\end{pmatrix}\sin(\alpha_n x),$$

$$\boldsymbol{X}_{-n}^1(x)=\begin{pmatrix}-1-\dfrac{1}{\mu_n}\\ \mu_n\\ \dfrac{a^2(-2+\mu_n)\mu_nD_{22}-n^2\pi^2(D_{12}+4D_{66})}{a^2}\mu_n\\ \dfrac{-n^2\pi^2(1+\mu_n)D_{12}+a^2(-1+\mu_n)\mu_n^2D_{22}}{a^2\mu_n}\end{pmatrix}\sin(\alpha_n x).$$

2.2 本征值为单根的情形

当 $K^2+(H^2-D_{11}D_{22})^2\neq 0$，根据式(14)可得单重本征值：

$$\tilde{\mu}_{n1}=\sqrt{\frac{\alpha_n^2H}{D_{22}}+\sqrt{\frac{\alpha_n^4(H^2-D_{11}D_{22})}{D_{22}^2}-\frac{K}{D_{22}}}},\ \tilde{\mu}_{n2}=-\tilde{\mu}_{n1},$$

$$\tilde{\mu}_{n3}=\sqrt{\frac{\alpha_n^2H}{D_{22}}-\sqrt{\frac{\alpha_n^4(H^2-D_{11}D_{22})}{D_{22}^2}-\frac{K}{D_{22}}}},\ \tilde{\mu}_{n4}=-\tilde{\mu}_{n3}. \tag{16}$$

对应的本征函数系为

$$\tilde{\boldsymbol{X}}_{ni}(x)=(1,\ \tilde{\mu}_{ni},\ \tilde{\mu}_{ni}^3D_{22}-\alpha_n^2(D_{12}+4D_{66})\tilde{\mu}_{ni},\ \alpha_n^2D_{12}-\tilde{\mu}_{ni}^2D_{22})^{\mathrm{T}}\sin(\alpha_n x), \tag{17}$$

其中 $n=1,2,3,\cdots,i=1,2,3,4.$

3 辛正交性及完备性

定义 1 设 $X = L^2[0,a] \times L^2[0,a] \times L^2[0,a] \times L^2[0,a]$, 在 X 上定义 $\boldsymbol{P},\boldsymbol{Q}$ 的辛内积为

$$\langle \boldsymbol{P},\boldsymbol{Q} \rangle = \int_0^a \boldsymbol{P}^{\mathrm{T}} \boldsymbol{J} \boldsymbol{Q} \mathrm{d}x,$$

其中 $\boldsymbol{P},\boldsymbol{Q} \in X, \boldsymbol{J} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I}_2 \\ -\boldsymbol{I}_2 & \boldsymbol{0} \end{pmatrix}$, $\boldsymbol{I}_2$ 为 2 阶单位矩阵.

3.1 本征值为重根时的辛正交性及完备性

根据定义 1, 有如下引理.

引理 1 在空间 X 中,无穷维 Hamilton 算子 $\boldsymbol{H}$ 的本征函数系 $\{\boldsymbol{X}_n^i(x) \mid i = 0, 1; n = \pm 1, \pm 2, \pm 3, \pm 4\}$ 是辛正交的.

证明 通过符号计算,可验证本征函数系 $\{\boldsymbol{X}_n^i(x) \mid i = 0,1; n = \pm 1, \pm 2, \pm 3, \pm 4\}$ 满足

$$\langle \boldsymbol{X}_m^0(x), \boldsymbol{X}_{-n}^1(x) \rangle = \begin{cases} -\dfrac{2n^2\pi^2 H}{a}, & m = n, \\ 0, & m \neq n, \end{cases}$$

$$\langle \boldsymbol{X}_m^1(x), \boldsymbol{X}_{-n}^0(x) \rangle = \begin{cases} \dfrac{2n^2\pi^2 H}{a}, & m = n, \\ 0, & m \neq n. \end{cases}$$

因此,引理得证.

定理 1 在空间 X 中,无穷维 Hamilton 算子 $\boldsymbol{H}$ 的本征函数系 $\{\boldsymbol{X}_n^i(x) \mid i = 0, 1; n = \pm 1, \pm 2, \pm 3, \pm 4\}$ 在 Cauchy(柯西)主值意义下是完备的.

证明 $\forall \boldsymbol{F}(x) = (f_1(x) \quad f_2(x) \quad f_3(x) \quad f_4(x))^{\mathrm{T}} \in X$, 在 Cauchy 主值意义下,利用辛本征函数 $\{\boldsymbol{X}_n^i(x) \mid i = 0,1; n = \pm 1, \pm 2, \pm 3, \pm 4\}$, $\boldsymbol{F}(x)$ 有如下辛-Fourier 表达:

$$\boldsymbol{F}(x) = \sum_{n=1}^{\infty} (g_n^0 \boldsymbol{X}_n^0(x) + g_n^1 \boldsymbol{X}_n^1(x) + g_{-n}^0 \boldsymbol{X}_{-n}^0(x) + g_{-n}^1 \boldsymbol{X}_{-n}^1(x)). \tag{18}$$

根据引理 1,有

$$g_n^0 = \frac{\langle \boldsymbol{F}(x), \boldsymbol{X}_{-n}^1(x) \rangle}{\langle \boldsymbol{X}_n^0(x), \boldsymbol{X}_{-n}^1(x) \rangle}, \quad g_n^1 = \frac{\langle \boldsymbol{F}(x), \boldsymbol{X}_{-n}^0(x) \rangle}{\langle \boldsymbol{X}_n^1(x), \boldsymbol{X}_{-n}^0(x) \rangle},$$

$$g_{-n}^0 = \frac{\langle \boldsymbol{F}(x), \boldsymbol{X}_n^1(x) \rangle}{\langle \boldsymbol{X}_{-n}^0(x), \boldsymbol{X}_n^1(x) \rangle}, \quad g_{-n}^1 = \frac{\langle \boldsymbol{F}(x), \boldsymbol{X}_n^0(x) \rangle}{\langle \boldsymbol{X}_{-n}^1(x), \boldsymbol{X}_n^0(x) \rangle}.$$

通过符号计算,有

$$\sum_{n=1}^{\infty} (g_n^0 \boldsymbol{X}_n^0(x) + g_n^1 \boldsymbol{X}_n^1(x) + g_{-n}^0 \boldsymbol{X}_{-n}^0(x) + g_{-n}^1 \boldsymbol{X}_{-n}^1(x)) =$$

$$\sum_{n=1}^{\infty} \begin{pmatrix} \dfrac{2}{a}\left(\displaystyle\int_0^a \sin(\alpha_n x) f_1(x) \mathrm{d}x\right) \sin(\alpha_n x) \\ \dfrac{2}{a}\left(\displaystyle\int_0^a \sin(\alpha_n x) f_2(x) \mathrm{d}x\right) \sin(\alpha_n x) \\ \dfrac{2}{a}\left(\displaystyle\int_0^a \sin(\alpha_n x) f_3(x) \mathrm{d}x\right) \sin(\alpha_n x) \\ \dfrac{2}{a}\left(\displaystyle\int_0^a \sin(\alpha_n x) f_4(x) \mathrm{d}x\right) \sin(\alpha_n x) \end{pmatrix}, \tag{19}$$

注意到 $\sum_{n=1}^{\infty}\frac{2}{a}\left(\int_0^a \sin(\alpha_n x)f_k(x)\mathrm{d}x\right)\sin(\alpha_n x)$[16] 是函数 $f_k(x)(k=1,2,3,4)$ 在 $L^2[0,a]$ 空间中完备正交基 $\{\sin(\alpha_n x)\}_{n=1}^{\infty}$ 下的 Fourier 级数展开.因此,定理 1 得证.

3.2　本征值为单根时的辛正交性及完备性

根据定义 1,有如下引理.

引理 2　在空间 X 中,无穷维 Hamilton 算子 $\boldsymbol{H}$ 的本征函数系 $\{\tilde{\boldsymbol{X}}_{ni}(x)\mid n=1,2,3,\cdots,i=1,2,3,4\}$ 是辛正交的.

证明　根据辛正交性的定义,通过符号计算,可验证 $\boldsymbol{H}$ 的本征函数系 $\{\tilde{\boldsymbol{X}}_{ni}(x)\mid n=1,2,3,\cdots,i=1,2,3,4\}$ 满足

$$\langle\tilde{\boldsymbol{X}}_{n1}(x),\tilde{\boldsymbol{X}}_{m2}(x)\rangle=\begin{cases}2a(\alpha_n^2H\tilde{\mu}_{n1}-D_{22}\tilde{\mu}_{n1}^2), & m=n,\\ 0, & m\neq n,\end{cases}$$

$$\langle\tilde{\boldsymbol{X}}_{n3}(x),\tilde{\boldsymbol{X}}_{m2}(x)\rangle=\begin{cases}2a(\alpha_n^2H\tilde{\mu}_{n3}-D_{22}\tilde{\mu}_{n3}^2), & m=n,\\ 0, & m\neq n.\end{cases}$$

故该本征函数系满足辛正交性.

定理 2　在空间 X 中,无穷维 Hamilton 算子 $\boldsymbol{H}$ 的本征函数系 $\{\tilde{\boldsymbol{X}}_{ni}(x)\mid n=1,2,3,\cdots,i=1,2,3,4\}$ 在 Cauchy 主值意义下是完备的.

证明　$\forall\,\tilde{\boldsymbol{F}}(x)=(f_1(x)\quad f_2(x)\quad f_3(x)\quad f_4(x))^{\mathrm{T}}\in X$, 在 Cauchy 主值意义下,利用辛本征函数系 $\{\tilde{\boldsymbol{X}}_{ni}(x)\mid n=1,2,3,\cdots,i=1,2,3,4\}$, $\tilde{\boldsymbol{F}}(x)$ 有如下辛-Fourier 表达:

$$\tilde{\boldsymbol{F}}(x)=\sum_{n=1}^{\infty}(a_{n1}\tilde{\boldsymbol{X}}_{n1}(x)+a_{n2}\tilde{\boldsymbol{X}}_{n2}(x)+a_{n3}\tilde{\boldsymbol{X}}_{n3}(x)+a_{n4}\tilde{\boldsymbol{X}}_{n4}(x)).\tag{20}$$

根据引理 2,得

$$a_{n1}=\frac{\langle\tilde{\boldsymbol{X}}_{n2}(x),\tilde{\boldsymbol{F}}(x)\rangle}{\langle\tilde{\boldsymbol{X}}_{n2}(x),\tilde{\boldsymbol{X}}_{n1}(x)\rangle},\ a_{n2}=\frac{\langle\tilde{\boldsymbol{X}}_{n1}(x),\tilde{\boldsymbol{F}}(x)\rangle}{\langle\tilde{\boldsymbol{X}}_{n1}(x),\tilde{\boldsymbol{X}}_{n2}(x)\rangle},$$
$$a_{n3}=\frac{\langle\tilde{\boldsymbol{X}}_{n4}(x),\tilde{\boldsymbol{F}}(x)\rangle}{\langle\tilde{\boldsymbol{X}}_{n4}(x),\tilde{\boldsymbol{X}}_{n3}(x)\rangle},\ a_{n4}=\frac{\langle\tilde{\boldsymbol{X}}_{n3}(x),\tilde{\boldsymbol{F}}(x)\rangle}{\langle\tilde{\boldsymbol{X}}_{n3}(x),\tilde{\boldsymbol{X}}_{n4}(x)\rangle},$$

直接计算得

$$\sum_{n=1}^{\infty}(a_{n1}\tilde{\boldsymbol{X}}_{n1}(x)+a_{n2}\tilde{\boldsymbol{X}}_{n2}(x)+a_{n3}\tilde{\boldsymbol{X}}_{n3}(x)+a_{n4}\tilde{\boldsymbol{X}}_{n4}(x))=$$
$$\sum_{n=1}^{\infty}\begin{pmatrix}\frac{2}{a}\left(\int_0^a\sin(\alpha_n x)f_1(x)\mathrm{d}x\right)\sin(\alpha_n x)\\ \frac{2}{a}\left(\int_0^a\sin(\alpha_n x)f_2(x)\mathrm{d}x\right)\sin(\alpha_n x)\\ \frac{2}{a}\left(\int_0^a\sin(\alpha_n x)f_3(x)\mathrm{d}x\right)\sin(\alpha_n x)\\ \frac{2}{a}\left(\int_0^a\sin(\alpha_n x)f_4(x)\mathrm{d}x\right)\sin(\alpha_n x)\end{pmatrix}.\tag{21}$$

故定理 2 得证.

4　辛叠加解析解

4.1　本征值为重根情形下的辛叠加解

根据定理 1,假设在边界条件(12)下 Hamilton 系统(7)的通解为

$$\boldsymbol{U}(x,y)=\sum_{n=1}^{\infty}\left(Y_n^0(y)\boldsymbol{X}_n^0(x)+Y_n^1(y)\boldsymbol{X}_n^1(x)+Y_{-n}^0(y)\boldsymbol{X}_{-n}^0(x)+Y_{-n}^1(y)\boldsymbol{X}_{-n}^1(x)\right), \tag{22}$$

其中 $\boldsymbol{U}(x,y)$ 的第一分量为所求的挠度 $w(x,y)$.

集中荷载作用下四边固支正交各向异性矩形薄板弯曲问题分为 3 个子问题[17],如下:(a) 集中荷载作用下四边简支薄板弯曲问题;(b) 在 $x=0$ 和 $x=a$ 边简支,在 $y=0$ 和 $y=b$ 边的弯矩分别为 $\sum_{n=1}^{\infty}E_n\sin(\alpha_n x)$ 和 $\sum_{n=1}^{\infty}F_n\sin(\alpha_n x)$ 的问题;(c) 在 $y=0$ 和 $y=b$ 边简支,在 $x=0$ 和 $x=a$ 边的弯矩分别为 $\sum_{n=1}^{\infty}G_n\sin(\beta_n y)$ 和 $\sum_{n=1}^{\infty}H_n\sin(\beta_n y)$ 的问题,其中 $\beta_n=n\pi/b$.

将上述 3 个子问题的通解叠加后可得到集中荷载作用下四边固支正交各向异性矩形薄板弯曲问题的通解.

经计算,可得子问题(a)的解为

$$\begin{aligned}w_1(x,y)=&\frac{1}{4n^2\pi^2H}aP\sin(\alpha_n x)\sin(\alpha_n x_0)\Bigg(\frac{1}{\mu_n}2\mathrm{e}^{-y\mu_n}(\mathrm{e}^{(b+2y)\mu_n}(-1+\coth(b\mu_n))-\\&(1+(1+\mathrm{e}^{2y\mu_n})y\mu_n)\operatorname{csch}(b\mu_n))\sinh((b-y_0)\mu_n)+\\&\frac{1}{\mu_n}4\mathscr{H}(y-y_0)((y-y_0)\mu_n\cosh((y-y_0)\mu_n)-\sinh((y-y_0)\mu_n))+\\&2\operatorname{csch}(b\mu_n)^2\sinh(y\mu_n)(y_0\sinh((2b-y_0)\mu_n)+(-2b+y_0)\sinh(y_0\mu_n))\Bigg),\end{aligned}$$

其中 (x_0,y_0) 为集中荷载的作用点, $\mathscr{H}(y-y_0)$ 为 MATHEMATICA 软件中 Heaviside theta 函数.

子问题(b)的通解为

$$\begin{aligned}w_2(x,y)=&\frac{1}{4\mu_n D_{22}}\operatorname{csch}(b\mu_n)^2\sin(\alpha_n x)((y\sinh((2b-y)\mu_n)+\\&(-2b+y)\sinh(y\mu_n))E_n+\\&2(-y\cosh(y\mu_n)\sinh(b\mu_n)+b\cosh(b\mu_n)\sinh(y\mu_n))F_n).\end{aligned}$$

子问题(c)的通解为

$$\begin{aligned}w_3(x,y)=&\frac{1}{4D_{11}\xi_n}\operatorname{csch}(a\xi_n)^2\sin(\beta_n y)((x\sinh((2a-x)\xi_n)+\\&(-2a+x)\sinh(x\xi_n))G_n+\\&2(-x\cosh(x\xi_n)\sinh(a\xi_n)+a\cosh(a\xi_n)\sinh(x\xi_n))H_n),\end{aligned}$$

其中 $\xi_n=\beta_n\sqrt{\dfrac{H}{D_{11}}},n=\pm1,\ \pm2,\ \pm3,\cdots$.

为了满足四边固支边界条件,在边界 $y=0$ 处有

$$\begin{aligned}&\frac{aP\operatorname{csch}(b\mu_i)^2\sin(\alpha_i x_0)(y_0\sinh((2b-y_0)\mu_i)+(-2b+y_0)\sinh(y_0\mu_i))\mu_i}{2i^2\pi^2H}+\\&\frac{\operatorname{csch}(b\mu_i)^2(E_i(\sinh(2b\mu_i)-2b\mu_i)+2F_i(-\sinh(b\mu_i)+b\operatorname{csch}(b\mu_i)\mu_i))}{4D_{22}\mu_i}+\\&\sum_{n=1}^{\infty}\frac{2a^2in\pi^2(G_n-\cos(i\pi)H_n)}{bD_{11}(i^2\pi^2+a^2\xi_n^2)^2}=0,\qquad i=1,2,3,\cdots.\end{aligned}\tag{23}$$

在边界 $y=b$ 处,有

$$\frac{aP\operatorname{csch}(b\mu_i)\sin(\alpha_i x_0)(y_0\cosh(y_0\mu_i)-b\coth(b\mu_i)\sinh(y_0\mu_i))\mu_i}{i^2\pi^2H}-$$
$$\frac{\operatorname{csch}(b\mu_i)^2(F_i(\sinh(2b\mu_i)-2b\mu_i)+2E_i(-\sinh(b\mu_i)+b\operatorname{csch}(b\mu_i)\mu_i))}{4D_{22}\mu_i}+$$
$$\sum_{n=1}^{\infty}\frac{2a^2in\pi^2\cos(n\pi)(G_n-\cos(i\pi)H_n)}{bD_{11}\,(i^2\pi^2+a^2\xi_n^2)^2}=0,\qquad i=1,2,3,\cdots. \tag{24}$$

在边界 $x=0$ 处,有

$$\frac{bP\operatorname{csch}(a\xi_j)^2\sin(\beta_jy_0)(x_0\sinh((2a-x_0)\xi_j)+(-2a+x_0)\sinh(x_0\xi_j))\xi_j}{2j^2\pi^2H}+$$
$$\frac{\operatorname{csch}(a\xi_j)^2(G_j(\sinh(2a\xi_j)-2a\xi_j)+2H_j(-\sinh(a\xi_j)+a\operatorname{csch}(a\xi_j)\xi_j))}{4D_{11}\xi_j}+$$
$$\sum_{m=1}^{\infty}\frac{2b^2jm\pi^2(E_m-\cos(j\pi)F_m)}{aD_{22}\,(j^2\pi^2+b^2\mu_m{}^2)^2}=0,\qquad j=1,2,3,\cdots. \tag{25}$$

在边界 $x=a$ 处,有

$$\frac{4b\mathrm{e}^{2a\xi_j}P\sin(\beta_jy_0)(x_0\cosh(x_0\xi_j)\sinh(a\xi_j)-a\cosh(a\xi_j)\sinh(x_0\xi_j))\xi_j}{(-1+\mathrm{e}^{2a\xi_j})^2j^2\pi^2H}-$$
$$\frac{\operatorname{csch}(a\xi_j)^2(H_j(\sinh(2a\xi_j)-2a\xi_j)+2G_j(-\sinh(a\xi_j)+a\operatorname{csch}(a\xi_j)\xi_j))}{4D_{11}\xi_j}+$$
$$\sum_{m=1}^{\infty}\frac{2b^2jm\pi^2\cos(m\pi)(E_m-\cos(j\pi)F_m)}{aD_{22}(j^2\pi^2+b^2\mu_m^2)^2}=0,\qquad j=1,2,3,\cdots. \tag{26}$$

通过求解方程(23)~(26),可得到对应的系数 E_n,F_n,G_n,H_n, 这样便得到通解

$$w(x,y)=w_1(x,y)+w_2(x,y)+w_3(x,y). \tag{27}$$

4.2 本征值为单根情形下的辛叠加解

根据定理 2,假设边界条件(12)下 Hamilton 体系(7)的解有如下形式:

$$\tilde{\boldsymbol{U}}(x,y)=\sum_{n=1}^{\infty}(T_{n1}\tilde{\boldsymbol{X}}_{n1}(x)+T_{n2}\tilde{\boldsymbol{X}}_{n2}(x)+T_{n3}\tilde{\boldsymbol{X}}_{n3}(x)+T_{n4}\tilde{\boldsymbol{X}}_{n4}(x)). \tag{28}$$

经计算,得到子问题(a)的弯曲解析解

$$w_1(x,y)=aP\sum_{n=1}^{\infty}\sin(\alpha_nx)\sin(\alpha_nx_0)\left\{\frac{\operatorname{csch}(b\mu_{n3})\sinh(\mu_{n3}y)\sinh(\mu_{n3}(b-y_0))}{n^2\pi^2H\mu_{n3}-a^2D_{22}\mu_{n3}^3}+\right.$$
$$\frac{\operatorname{csch}(b\mu_{n1})\sinh(\mu_{n1}y)\sinh(\mu_{n1}(b-y_0))}{n^2\pi^2H\mu_{n1}-a^2D_{22}\mu_{n1}^3}+$$
$$\left.\mathscr{H}(y-y_0)\left[\frac{\sinh(\mu_{n1}(y-y_0))}{-n^2\pi^2H\mu_{n1}+a^2D_{22}\mu_{n1}^3}+\frac{\sinh(\mu_{n3}(y-y_0))}{-n^2\pi^2H\mu_{n3}+a^2D_{22}\mu_{n3}^3}\right]\right\}. \tag{29}$$

子问题(b)和(c)的弯曲解析解,分别如下:

$$w_2(x,y)=\sum_{n=1}^{\infty}\frac{\sin(\alpha_nx)}{D_{22}(\mu_{n1}^2-\mu_{n3}^2)}(E_n(\cosh(y\mu_{n3})-\cosh(y\mu_{n1})+$$
$$\coth(b\mu_{n1})\sinh(y\mu_{n1})-\coth(b\mu_{n3})\sinh(y\mu_{n3}))+$$
$$F_n(\operatorname{csch}(b\mu_{n3})\sinh(y\mu_{n3})-\operatorname{csch}(b\mu_{n1})\sinh(y\mu_{n1}))), \tag{30}$$

$$w_3(x,y)=\sum_{n=1}^{\infty}\frac{\sin(\beta_n y)}{D_{22}(\xi_{n1}^2-\xi_{n3}^2)}(G_n(\cosh(x\xi_{n3})-\cosh(x\xi_{n1})+\coth(a\xi_{n1})\sinh(x\xi_{n1})-\coth(a\xi_{n3})\sinh(x\xi_{n3}))+H_n(\operatorname{csch}(a\xi_{n3})\sinh(x\xi_{n3})-\operatorname{csch}(a\xi_{n1})\sinh(x\xi_{n1}))),\tag{31}$$

其中

$$\xi_{n1}=\sqrt{\frac{\beta_n^2H}{D_{11}}+\sqrt{\frac{\beta_n^4(H^2-D_{11}D_{22})}{D_{11}^2}-\frac{K}{D_{11}}}},\ \xi_{n3}=\sqrt{\frac{\beta_n^2H}{D_{11}}-\sqrt{\frac{\beta_n^4(H^2-D_{11}D_{22})}{D_{11}^2}-\frac{K}{D_{11}}}}.$$

为了满足固支边界条件,3 个子问题之和在边界 $y=0$ 处,有

$$aP\sin(\alpha_i x_0)\left\{\frac{\operatorname{csch}(b\mu_{i1})\sinh((b-y_0)\mu_{i1})}{i^2\pi^2H-a^2D_{22}\mu_{i1}^2}+\frac{\operatorname{csch}(b\mu_{i3})\sinh((b-y_0)\mu_{i3})}{i^2\pi^2H-a^2D_{22}\mu_{i3}^2}\right\}+\frac{1}{D_{22}(\mu_{i1}^2-\mu_{i3}^2)}(E_i(\coth(b\mu_{i1})\mu_{i1}-\coth(b\mu_{i3})\mu_{i3})+F_i(\operatorname{csch}(b\mu_{i3})\mu_{i3}-\operatorname{csch}(b\mu_{i1})\mu_{i1}))+\sum_{n=1}^{\infty}\left\{\frac{2a^2i\pi(G_n-\cos(i\pi)H_n)\beta_n}{D_{11}(i^2\pi^2+a^2\xi_{n1}^2)(i^2\pi^2+a^2\xi_{n3}^2)}\right\}=0,\qquad i=1,2,3,\cdots.\tag{32}$$

在边界 $y=b$ 处,有

$$aP\sin(\alpha_i x_0)\left\{-\frac{\operatorname{csch}(b\mu_{i1})\sinh(y_0\mu_{i1})}{i^2\pi^2H-a^2D_{22}\mu_{i1}^2}-\frac{\operatorname{csch}(b\mu_{i3})\sinh(y_0\mu_{i3})}{i^2\pi^2H-a^2D_{22}\mu_{i3}^2}\right\}+\frac{1}{D_{22}(\mu_{i1}^2-\mu_{i3}^2)}(F_i(\coth(b\mu_{i3})\mu_{i3}-\coth(b\mu_{i1})\mu_{i1})+E_i(\operatorname{csch}(b\mu_{i1})\mu_{i1}-\operatorname{csch}(b\mu_{i3})\mu_{i3}))+\sum_{n=1}^{\infty}\left\{\frac{2a^2i\pi\cos(n\pi)(G_n-\cos(i\pi)H_n)\beta_n}{D_{11}(i^2\pi^2+a^2\xi_{n1}^2)(i^2\pi^2+a^2\xi_{n3}^2)}\right\}=0,\qquad i=1,2,3,\cdots.\tag{33}$$

在边界 $x=0$ 处,有

$$bP\sin(\beta_j y_0)\left\{\frac{\operatorname{csch}(a\xi_{j1})\sinh((a-x_0)\xi_{j1})}{j^2\pi^2H-b^2D_{11}\xi_{j1}^2}+\frac{\operatorname{csch}(a\xi_{j3})\sinh((a-x_0)\xi_{j3})}{j^2\pi^2H-b^2D_{11}\xi_{j3}^2}\right\}+\frac{1}{D_{11}(\xi_{j1}^2-\xi_{j3}^2)}(G_j(\coth(a\xi_{j1})\xi_{j1}-\coth(a\xi_{j3})\xi_{j3})+H_j(\operatorname{csch}(a\xi_{j3})\xi_{j3}-\operatorname{csch}(a\xi_{j1})\xi_{j1}))+\sum_{m=1}^{\infty}\left\{\frac{2b^2j\pi(E_m-\cos(j\pi)F_m)\alpha_m}{D_{22}(j^2\pi^2+b^2\mu_{m1}^2)(j^2\pi^2+b^2\mu_{m3}^2)}\right\}=0,\qquad j=1,2,3,\cdots.\tag{34}$$

在边界 $x=a$ 处,有

$$bP\sin(\beta_j y_0)\left\{-\frac{\operatorname{csch}(a\xi_{j1})\sinh(x_0\xi_{j1})}{j^2\pi^2H-b^2D_{11}\xi_{j1}^2}-\frac{\operatorname{csch}(a\xi_{j3})\sinh(x_0\xi_{j3})}{j^2\pi^2H-b^2D_{11}\xi_{j3}^2}\right\}+\frac{1}{D_{11}(\xi_{j1}^2-\xi_{j3}^2)}(H_j(\coth(a\xi_{j3})\xi_{j3}-\coth(a\xi_{j1})\xi_{j1})+G_j(\operatorname{csch}(a\xi_{j1})\xi_{j1}-\operatorname{csch}(a\xi_{j3})\xi_{j3}))+\sum_{m=1}^{\infty}\left\{\frac{2b^2j\pi\cos(m\pi)(E_m-\cos(j\pi)F_m)\alpha_m}{D_{22}(j^2\pi^2+b^2\mu_{m1}^2)(j^2\pi^2+b^2\mu_{m3}{}^2)}\right\}=0,\qquad j=1,2,3,\cdots.\tag{35}$$

根据方程(32)~(35),解得系数 E_m,F_m,G_n 和 H_n($m,n = 1,2,3,\cdots$).将所得结果代入式(29)~(31),得到本征单根情形下正交各向异性矩形薄板问题的弯曲解析解如下:

$$w(x,y) = w_1(x,y) + w_2(x,y) + w_3(x,y). \tag{36}$$

5 算 例

例 1 集中荷载下四边固支正交各向异性矩形薄板弯曲问题中,取

$$(x_0,y_0) = \left(\frac{a}{2},\frac{b}{2}\right),\ D_{12} = 0.3D_{11},\ D_{22} = 4D_{11},\ D_{66} = 0.85D_{11},\ b = a,$$

将其代入式(23)~(26),解得相应的 E_n,F_n,G_n,H_n,把所得结果代入式(27)得到对应的解析解,应用该解析解计算了一些具体点处的挠度值并与文献[3]的结果进行比较,具体如表 1.

例 2 计算在 $Ka^4/D = 10^2$ 弹性地基上,点 $(a/2,a/2)$ 处受到集中载荷 P, Poisson(泊松)比 $\nu = 0.3$ 的四边固支正交各向同性薄板弯曲解,此时在正交各向异性薄板中的对应参数取为

$$\nu_{12} = \nu_{21} = \nu,\ D_{11} = D_{22} = H = D,\ D_{12} = \nu D,\ D_{66} = \frac{D(1-\nu)}{2},$$

所得结果列于表 2.

表 1 集中荷载下四边固支正交各向异性矩形薄板的挠度 $w(Pa^2/D_{11})$

Table 1 Deflections of the clamped orthotropic rectangular thin plate under concentrated load $w(Pa^2/D_{11})$

y		x	$a/4$	$a/2$	$3a/4$
$a/4$	present	$n = 10$	0.001 073	0.000 817 755	0.001 073
		$n = 20$	0.000 818	0.001 450 14	0.000 817 756
$a/2$	present	$n = 10$	0.001 196 73	0.002 275 71	0.001 196 73
		$n = 20$	0.001 224 81	0.002 311 61	0.001 224 81
	ref. [3]			0.002 407	
$3a/4$	present	$n = 10$	0.000 875 087	0.001 494 96	0.000 875 088
		$n = 20$	0.000 903 426	0.001 564 24	0.000 903 473

表 2 弹性地基上正交各向同性方板的挠度 $w(Pa^2/D)$

Table 2 Deflections of the isotropic square plate on elastic foundation $w(Pa^2/D)$

y		x	$a/8$	$a/4$	$3a/8$	$a/2$
$a/8$	present	$n = 10$	0.000 096 387 4	0.000 343 767	0.000 597 343	0.000 705 809
		$n = 20$	0.000 096 366 8	0.000 343 766	0.000 597 344	0.000 705 805
	ref. [17]		0.000 096 4	0.000 344	0.000 597	0.000 706
$a/4$	present	$n = 10$	0.000 343 751	0.001 122 36	0.001 924 65	0.002 279 98
		$n = 20$	0.000 343 766	0.001 122 34	0.001 924 65	0.002 280 01
	ref. [17]		0.000 344	0.001 12	0.001 92	0.002 28
$3a/8$	present	$n = 10$	0.000 596 762	0.001 925 37	0.003 378 25	0.004 103 66
		$n = 20$	0.000 597 341	0.001 924 65	0.003 378 14	0.004 104 92
	ref. [17]		0.000 597	0.001 92	0.003 38	0.004 10
$a/2$	present	$n = 10$	0.000 699 416	0.002 289 92	0.004 100 69	0.005 233 27
		$n = 20$	0.000 705 735	0.002 280 2	0.004 104 2	0.005 263 14
	ref. [17]		0.000 706	0.002 28	0.004 10	0.005 27

例 3 计算四边固支正交各向异性方板的解,取材料属性为

$$\frac{E_{\mathrm{L}}}{E_{\mathrm{T}}}=25,\ \frac{G_{\mathrm{LT}}}{E_{\mathrm{T}}}=0.5,\ \nu_{\mathrm{LT}}=0.25,$$

其中 L 和 T 分别表示纤维和横向方向.根据文献[18],弯曲刚度系数 D_{11},D_{12},D_{22} 和 D_{66} 可按如下选取:

$$D_{12}=0.01D_{11},\ D_{22}=0.04D_{11},\ D_{66}=0.019\ 95D_{11},$$

所得结果列于表 3.

表 3 集中荷载下正交各向异性方板的挠度($E_{\mathrm{T}}h^3w/Pa^2$)

Table 3 Deflections of the orthotropic square plate under concentrated load ($E_{\mathrm{T}}h^3w/Pa^2$)

y		x	$a/8$	$a/4$	$3a/8$	$a/2$
$a/8$	present	$n=10$	−0.000 262 021	−0.000 079 017 9	−0.000 352 576	−0.000 584 225
		$n=20$	−0.000 030 144 5	−0.000 234 132	−0.000 525 33	−0.000 695 576
$a/4$	present	$n=10$	0.000 196 253	0.000 502 36	0.000 787 042	0.000 897 372
		$n=20$	0.000 119 839	0.000 450 425	0.000 727 582	0.000 832 515
$3a/8$	present	$n=10$	0.000 896 115	0.002 532 45	0.004 103 29	0.004 741 54
		$n=20$	0.000 760 537	0.002 438 26	0.004 000 07	0.004 625 98
$a/2$	present	$n=10$	0.001 107 29	0.004 101 13	0.007 307 59	0.009 137 6
		$n=20$	0.001 267 38	0.004 119 53	0.007 320 72	0.009 183 02
	ref. [18]					0.009 167

6 结　　论

辛叠加方法广泛应用于各向同性矩形板问题中,此方法没有依靠任何试验函数,通过逐步的推导过程解决了不同边界条件下的各向同性矩形板问题,但至今仍没有运用此种方法解决正交各向异性矩形板问题的文献.本文第一次计算出对边简支正交各向异性薄板问题对应的无穷维 Hamilton 算子的本征值和本征函数系,证明了本征函数系的辛正交性及完备性,得到无穷维 Hamilton 系统的通解.之后用辛叠加方法得到了四边固支正交各向异性薄板问题的辛叠加解.最后将本文所得的弯曲解析解与已有文献结果的比较,说明了本文所得结果的有效性和准确性.本文运用辛叠加方法只解决了四边固支正交各向异性矩形薄板这一种特殊情况,但该方法可推广到其他边界条件的情形.

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Analytical Bending Solutions of Clamped Orthotropic Rectangular Thin Plates With the Symplectic Superposition Method

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Abstract: The orthotropic rectangular thin plate equations were transformed into the Hamiltonian system, and the corresponding infinite dimensional Hamiltonian operator was obtained with the method of separation of variables. Then the eigenvalues and corresponding eigenfunctions of the Hamiltonian operator were calculated, and the eigenfunction system was proved to be of symplectic orthogonality and completeness. Finally, with the symplectic superposition method, the analytical bending solutions of fully clamped orthotropic rectangular thin plates were presented. The comparison between the analytical solutions and the numerical examples shows the correctness of the proposed method.

Key words: orthotropic thin plate; infinite dimensional Hamiltonian operator; eigenfunction; analytical solution

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