文章编号:1000-0887(2017)11-1195-13
ⓒ 应用数学和力学编委会,ISSN 1000-0887

软物质准晶三维广义流体动力学*

范天佑， 唐志毅

(北京理工大学 物理学院, 北京 100081)

(我刊编委陈伟球推荐)

摘要： 讨论了软物质准晶三维广义流体动力学.主要是给出了已经发现的和可能发现的第一类二维软物质准晶的动力学方程组,也简单地讨论了一下它们的解,以及这些解和固体准晶解的结果的巨大差别.

关 键 词： 软物质； 第一类二维准晶； 第二类二维准晶； 广义流体动力学； 状态方程

中图分类号： O35；O753⁺.3 **文献标志码：** A doi：10.21656/1000-0887.380063

引 言

文献[1-2]讨论了自 2004 年开始发现的具有 12 次对称和 18 次对称的软物质准晶[3-7],给出了它们和可能发现的软物质准晶的动力学方程,只涉及其平面场方程.现今发现的软物质准晶都是二维准晶.二维准晶可以划分为第一类二维准晶和第二类二维准晶两类.第一类二维准晶是三维的, 第二类二维准晶只能是二维的,并且只有平面场.对于第一类二维准晶,除了平面场方程之外,讨论其三维场方程也非常重要.本文就讨论了第一类二维软物质准晶动力学三维场方程.有了这种场方程,就可以分析软物质准晶的物质分布、变形、运动和结构重组等物理和力学结构和性质,初步的结果揭示了软物质准晶和固体准晶存在巨大的差别.

1 软物质准晶广义流体动力学

在文献[1-2]中, 笔者指出,从凝聚态物理的角度看, 软物质准晶和固体准晶之间存在巨大的差别.但是从广义流体动力学的角度看, Lubensky 等[8-9]提出的固体准晶广义流体动力学的理论框架或者数学结构可以为软物质准晶借鉴,把它改造和补充之后,可以为建立软物质准晶广义流体动力学所用.因为两者的运动方程都是用 Poisson 括号推导的,存在许多相似之处.注意用 Poisson 括号的推导,仅限于运动方程,不包括状态方程.同时软物质准晶和固体准晶的广义流体动力学两者之间存在原则的不同,例如：

1) 文献[8-9] 的固体黏性本构方程

$$\sigma'_{ij} = \eta_{ijkl}\,\dot{\xi}_{kl}\,,\ \dot{\xi}_{kl} = \frac{1}{2}\left(\frac{\partial V_k}{\partial x_l} + \frac{\partial V_l}{\partial x_k}\right)$$

必须被软物质准晶的流体声子本构方程

* 收稿日期： 2017-03-16；修订日期： 2017-10-09
基金项目： 国家自然科学基金(11272053)
作者简介： 范天佑(1939—),男,教授(通讯作者. E-mail：tyfan2013@163.com).

$$p_{ij} = -p\delta_{ij} + \sigma'_{ij} = -p\delta_{ij} + \eta_{ijkl}\dot{\xi}_{kl}$$

所取代,其中

$$\sigma'_{ij} = \eta_{ijkl}\dot{\xi}_{kl},\ \dot{\xi}_{kl} = \frac{1}{2}\left(\frac{\partial V_k}{\partial x_l} + \frac{\partial V_l}{\partial x_k}\right),$$

流体声子是 Landau 学派提出的[10],引入准晶,为笔者第一次建议.这样对于软物质准晶,它的声子、相位子和流体声子的本构方程(顺便也列出变形几何关系)为

$$\begin{cases}\sigma_{ij} = C_{ijkl}\varepsilon_{ik} + R_{ijkl}w_{kl},\\ H_{ij} = K_{ijkl}w_{ij} + R_{klij}\varepsilon_{kl},\\ p_{ij} = -p\delta_{ij} + \sigma'_{ij},\ \sigma'_{ij} = \eta_{ijkl}\dot{\xi}_{kl},\\ \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),\ w_{ij} = \frac{\partial w_i}{\partial x_j},\ \dot{\xi}_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right),\end{cases} \tag{1}$$

其中 u_i 为声子位移矢量, σ_{ij} 为声子应力张量, ε_{ij} 为声子应变张量, w_i 为相位子位移矢量, H_{ij} 为相位应力张量, w_{ij} 为相位应变张量, V_i 为流体声子速度矢量, p_{ij} 为流体声子应力张量, p 为流体压力, η_{ijkl} 为流体黏性系数张量, $\dot{\xi}_{ij}$ 为流体变形速度张量, C_{ijkl},K_{ijkl} 和 R_{ijkl} 分别代表声子、相位子和声子-相位子耦合弹性常数张量.为了简单起见,这里仅讨论最简单的流体本构方程,也就是

$$p_{ij} = -p\delta_{ij} + \sigma'_{ij} = -p\delta_{ij} + 2\eta\left(\dot{\xi}_{ij} - \frac{1}{3}\dot{\xi}_{kk}\delta_{ij}\right) + \eta'\dot{\xi}_{kk}\delta_{ij},$$

$$\sigma'_{ij} = 2\eta\left(\dot{\xi}_{ij} - \frac{1}{3}\dot{\xi}_{kk}\delta_{ij}\right) + \eta'\dot{\xi}_{kk}\delta_{ij}\ \dot{\xi}_{kk} = \dot{\xi}_{11} + \dot{\xi}_{22} + \dot{\xi}_{33},$$

$$\dot{\xi}_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right),$$

其中 η 代表第一黏性系数; η' 代表第二黏性系数, 由于其值太小而被忽略(注意上面提到的文献[8-9]的固体黏性本构方程,和本文下面的讨论无关,文献[8-9]的本构方程丝毫没有说固体没有压力).

2) 因为流体声子的引进,多了一个场变量,这里必须补充状态方程 $p = f(\rho)$, 而在固体准晶则不需要这一方程.状态方程是热力学和统计物理的研究内容,已经超出了纯流体动力学研究范畴.文献[1-2]修改了 Wensink 的结果[11]用于软物质准晶研究,得到

$$p = f(\rho) = 3\frac{k_B T}{l^3\rho_0^3}(\rho_0^2\rho + \rho_0\rho^2 + \rho^3), \tag{2}$$

其中 k_B 是 Boltzmann 常数, T 为绝对温度, l 为软物质的特征尺寸, ρ_0 是质量密度 ρ 的初始值.

基于上面的考虑,通过 Poisson 括号方法推导得到软物质准晶运动方程补充状态方程之后得到软物质准晶的动力学控制方程如下:

$$\frac{\partial\rho}{\partial t} + \nabla_k(\rho V_k) = 0, \tag{3a}$$

$$\frac{\partial g_i(\boldsymbol{r},t)}{\partial t} = -\nabla_k(\boldsymbol{r})(V_k g_i) + \nabla_j(\boldsymbol{r})(-p\delta_{ij} + \eta_{ijkl}\nabla_k(\boldsymbol{r})V_l) -$$

$$(\delta_{ij} - \nabla_i u_j)\frac{\delta H}{\delta u_j(\boldsymbol{r},t)} + (\nabla_i w_j)\frac{\delta H}{\delta w_j(\boldsymbol{r},t)} -$$

$$\rho\nabla_i(\boldsymbol{r})\frac{\delta H}{\delta\rho(\boldsymbol{r},t)},\qquad g_j=\rho V_j,\tag{3b}$$

$$\begin{cases}\dfrac{\partial u_i(\boldsymbol{r},t)}{\partial t}=-V_j\nabla_j(\boldsymbol{r})u_i-\varGamma_{\boldsymbol{u}}\dfrac{\delta H}{\delta u_i(\boldsymbol{r},t)}+V_i,\\ \dfrac{\partial w_i(\boldsymbol{r},t)}{\partial t}=-V_j\nabla_j(\boldsymbol{r})w_i-\varGamma_{\boldsymbol{w}}\dfrac{\delta H}{\delta w_i(\boldsymbol{r},t)},\\ p=f(\rho)=3\dfrac{k_{\mathrm{B}}T}{l^3\rho_0^3}(\rho_0^2\rho+\rho_0\rho^2+\rho^3),\end{cases}\tag{3c}$$

其中 H 记为准晶系统的能量泛函,或者 Hamilton 量.对于第一类二维准晶, 该能量泛函或 Hamilton 量与 Lubensky 等[8-9]针对固体准晶提出的很相似:

$$\begin{cases}H=H[\varPsi(\boldsymbol{r},t)]=\displaystyle\int\frac{\boldsymbol{g}^2}{2\rho}\mathrm{d}^d\boldsymbol{r}+\int\left[\frac{1}{2}A\left(\frac{\delta\rho}{\rho_0}\right)^2+B\left(\frac{\delta\rho}{\rho_0}\right)\nabla\cdot u\right]\mathrm{d}^d\boldsymbol{r}+F_{\mathrm{el}}=\\ \qquad H_{\mathrm{kin}}+H_{\mathrm{density}}+F_{\mathrm{el}},\\ \boldsymbol{g}=\rho\boldsymbol{V},\ F_{\mathrm{el}}=F_{\boldsymbol{u}}+F_{\boldsymbol{w}}+F_{\boldsymbol{uw}},\end{cases}\tag{4}$$

这里 $\delta\rho=\rho-\rho_0$,A,B 是质量密度变化导致的两个材料常数, F_{el} 为弹性应变能, $F_{\boldsymbol{u}}$,$F_{\boldsymbol{w}}$,$F_{\boldsymbol{uw}}$ 分别代表声子、相位子和声子-相位子耦合应变能

$$\begin{cases}F_{\boldsymbol{u}}=\displaystyle\int\frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\mathrm{d}^d\boldsymbol{r},\\ F_{\boldsymbol{w}}=\displaystyle\int\frac{1}{2}K_{ijkl}w_{ij}w_{kl}\mathrm{d}^d\boldsymbol{r},\\ F_{\boldsymbol{uw}}=\displaystyle\int(R_{ijkl}\varepsilon_{ij}w_{kl}+R_{klij}w_{ij}\varepsilon_{kl})\mathrm{d}^d\boldsymbol{r}.\end{cases}\tag{5}$$

这里再重复一遍方程组(3)的前 4 个方程(即运动方程)是由凝聚态物理的 Poisson 括号方法[12]推导出来的, 该方法由 Lubensky 等[8-9]第一次用到固体准晶.从文献[1-2]中可以查到有关细节.Poisson 括号方法的应用在针对具体各准晶系的 Hamilton 量的正确表示,在以下各节中将一一给出.很显然,状态方程或物态方程是不可能由 Poisson 括号方法推导出来的,它是由热力学和统计力学的分析得到的[11].

方程组(3)是由固体准晶流体动力学修改和补充得来的,称其为广义流体动力学比较适宜,当然现在的工作是 Lubensky 工作的传承和发展.

2 12 次对称软物质准晶

文献[1-2]讨论的是软物质准晶的平面场动力学,没有详细讨论其三维问题,而方程组(3)适合一维、二维和三维情形.方程组(3)很紧凑,但是很难理解,而且不便于计算.现在实验极其缺乏,如果又没有计算, 研究无法开展.

首先,针对 12 次对称准晶化简方程组(3).如所熟知,12 次对称准晶,是目前软物质准晶中最重要的一种准晶.为了这个目的,必须首先列出 12 次对称准晶的本构方程,假设 xOy 平面为准周期平面, z 轴为 12 次对称轴,所以 $w_z\equiv0$, 这时声子、相位子和流体声子的三维本构关系如下[13-16]:

$$
\begin{cases}
\sigma_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz}, \\
\sigma_{yy} = C_{12}\varepsilon_{xx} + C_{11}\varepsilon_{yy} + C_{13}\varepsilon_{zz}, \\
\sigma_{zz} = C_{13}\varepsilon_{xx} + C_{13}\varepsilon_{yy} + C_{33}\varepsilon_{zz}, \\
\sigma_{yz} = \sigma_{zy} = 2C_{44}\varepsilon_{yz}, \\
\sigma_{zx} = \sigma_{xz} = 2C_{44}\varepsilon_{zx}, \\
\sigma_{xy} = \sigma_{yx} = 2C_{66}\varepsilon_{xy}, \\
H_{xx} = K_1 w_{xx} + K_2 w_{yy}, \\
H_{yy} = K_2 w_{xx} + K_1 w_{yy}, \\
H_{yz} = K_4 w_{yz}, \\
H_{xy} = (K_1 + K_2 + K_3) w_{xy} + K_2 w_{yz}, \\
H_{xz} = K_4 w_{xz}, \\
H_{yx} = K_3 w_{xy} + (K_1 + K_2 + K_3) w_{yx}, \\
p_{xx} = -p + 2\eta\dot{\xi}_{xx} - \dfrac{2}{3}\eta\dot{\xi}_{kk}, \\
p_{yy} = -p + 2\eta\dot{\xi}_{yy} - \dfrac{2}{3}\eta\dot{\xi}_{kk}, \\
p_{zz} = -p + 2\eta\dot{\xi}_{zz} - \dfrac{2}{3}\eta\dot{\xi}_{kk}, \\
p_{yz} = 2\eta\dot{\xi}_{yz}, \\
p_{zx} = 2\eta\dot{\xi}_{zx}, \\
p_{xy} = 2\eta\dot{\xi}_{xy}.
\end{cases}
\tag{6}
$$

由它们写出 Hamilton 量,进而代入方程组(3),忽略项

$$\nabla_i\left(u_j \frac{\delta H}{\delta u_j}\right),\ \nabla_i\left(w_j \frac{\delta H}{\delta w_j}\right)$$

之后(因为这两项很小),得到

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{V}) = 0, \tag{7a}$$

$$
\begin{aligned}
&\frac{\partial(\rho V_x)}{\partial t} + \frac{\partial(V_x \rho V_x)}{\partial x} + \frac{\partial(V_y \rho V_x)}{\partial y} + \frac{\partial(V_z \rho V_x)}{\partial z} = \\
&\quad -\frac{\partial p}{\partial x} + \eta\nabla^2(\rho V_x) + \frac{1}{3}\eta\frac{\partial}{\partial x}\nabla\cdot\boldsymbol{V} + \\
&\quad \left(C_{66}\frac{\partial^2}{\partial y^2} + C_{44}\frac{\partial^2}{\partial z^2}\right)u_x - C_{66}\frac{\partial^2 u_y}{\partial x \partial y} + (C_{13} + C_{44} - C_{11})\frac{\partial^2 u_z}{\partial x \partial z} + \\
&\quad (C_{11} - B)\frac{\partial}{\partial x}\nabla\cdot\boldsymbol{u} - (A - B)\frac{1}{\rho_0}\frac{\partial \delta\rho}{\partial x},
\end{aligned}
\tag{7b}
$$

$$\frac{\partial(\rho V_y)}{\partial t} + \frac{\partial(V_x \rho V_y)}{\partial x} + \frac{\partial(V_y \rho V_y)}{\partial y} + \frac{\partial(V_z \rho V_y)}{\partial z} = -\frac{\partial p}{\partial y} + \eta\nabla^2(\rho V_y) +$$

$$\frac{1}{3}\eta\frac{\partial}{\partial y}\nabla\cdot\boldsymbol{V}-C_{66}\frac{\partial^2 u_x}{\partial x\partial y}+\left(C_{66}\frac{\partial^2}{\partial x^2}+C_{11}\frac{\partial^2}{\partial y^2}+C_{44}\frac{\partial^2}{\partial z^2}\right)u_y+$$

$$(C_{13}+C_{44})\frac{\partial^2 u_z}{\partial y\partial z}+(C_{11}-B)\frac{\partial}{\partial y}\nabla\cdot\boldsymbol{u}-(A-B)\frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial y},\tag{7c}$$

$$\frac{\partial(\rho V_z)}{\partial t}+\frac{\partial(V_x\rho V_z)}{\partial x}+\frac{\partial(V_y\rho V_z)}{\partial y}+\frac{\partial(V_z\rho V_z)}{\partial z}=-\frac{\partial p}{\partial z}+\eta\nabla^2(\rho V_z)+$$

$$\frac{1}{3}\eta\frac{\partial}{\partial z}\nabla\cdot\boldsymbol{V}+\left(C_{44}\frac{\partial^2}{\partial x^2}+C_{44}\frac{\partial^2}{\partial y^2}+(C_{33}-C_{13}-C_{44})\frac{\partial^2}{\partial z^2}\right)u_z+$$

$$(C_{13}+C_{44}-B)\frac{\partial}{\partial z}\nabla\cdot\boldsymbol{u}-(A-B)\frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial z},\tag{7b}$$

$$\frac{\partial u_x}{\partial t}+V_x\frac{\partial u_x}{\partial x}+V_y\frac{\partial u_x}{\partial y}+V_z\frac{\partial u_x}{\partial z}=V_x+\Gamma_{\boldsymbol{u}}\left[\left(C_{11}\frac{\partial^2}{\partial x^2}+C_{66}\frac{\partial^2}{\partial y^2}+C_{44}\frac{\partial^2}{\partial z^2}\right)u_x+\right.$$

$$\left.(C_{11}-C_{66})\frac{\partial^2 u_y}{\partial x\partial y}+(C_{13}+C_{44})\frac{\partial^2 u_z}{\partial x\partial z}\right],\tag{7c}$$

$$\frac{\partial u_y}{\partial t}+V_x\frac{\partial u_y}{\partial x}+V_y\frac{\partial u_y}{\partial y}+V_z\frac{\partial u_y}{\partial z}=V_y+\Gamma_{\boldsymbol{u}}\left[(C_{11}-C_{66})\frac{\partial^2 u_x}{\partial x\partial y}+\right.$$

$$\left.\left(C_{66}\frac{\partial^2}{\partial x^2}+C_{11}\frac{\partial^2}{\partial y^2}+C_{44}\frac{\partial^2}{\partial z^2}\right)u_y+(C_{13}+C_{44})\frac{\partial^2 u_z}{\partial y\partial z}\right],\tag{7d}$$

$$\frac{\partial u_z}{\partial t}+V_x\frac{\partial u_z}{\partial x}+V_y\frac{\partial u_z}{\partial y}+V_z\frac{\partial u_z}{\partial z}=V_z+\Gamma_{\boldsymbol{u}}\left[(C_{13}+C_{44})\left(\frac{\partial^2 u_x}{\partial x\partial z}+\frac{\partial^2 u_y}{\partial y\partial z}\right)+\right.$$

$$\left.\left(C_{44}\frac{\partial^2}{\partial x^2}+C_{44}\frac{\partial^2}{\partial y^2}+C_{33}\frac{\partial^2}{\partial z^2}\right)u_z\right],\tag{7e}$$

$$\frac{\partial w_x}{\partial t}+V_x\frac{\partial w_x}{\partial x}+V_y\frac{\partial w_x}{\partial y}+V_z\frac{\partial w_x}{\partial z}=$$

$$\Gamma_{\boldsymbol{w}}\left[K_1\nabla_1^2 w_x+(K_2+K_3)\frac{\partial^2 w_x}{\partial y^2}+K_4\frac{\partial^2 w_x}{\partial z^2}+K_2\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial z}\right)w_y\right],\tag{7f}$$

$$\frac{\partial w_y}{\partial t}+V_x\frac{\partial w_y}{\partial x}+V_y\frac{\partial w_y}{\partial y}+V_z\frac{\partial w_y}{\partial z}=\Gamma_{\boldsymbol{w}}\left[(K_2+K_3)\frac{\partial^2 w_x}{\partial x\partial y}+\right.$$

$$\left.K_3\frac{\partial^2 w_x}{\partial y\partial z}+K_1\nabla_1^2 w_y+(K_2+K_3)\frac{\partial^2 w_y}{\partial x^2}+(K_1+K_2+K_3)\frac{\partial^2 w_y}{\partial x\partial z}\right],\tag{7g}$$

$$p=f(\rho)=3\frac{k_{\mathrm{B}}T}{l^3\rho_0^3}(\rho_0^2\rho+\rho_0\rho^2+\rho^3),\tag{7h}$$

其中

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\ \nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\ \nabla=\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z},$$

$$\boldsymbol{V}=\boldsymbol{i}V_x+\boldsymbol{j}V_y+\boldsymbol{k}V_z,\ \boldsymbol{u}=\boldsymbol{i}u_x+\boldsymbol{j}u_y+\boldsymbol{k}u_z,\ \boldsymbol{w}=\boldsymbol{i}w_x+\boldsymbol{j}w_y,$$

$C_{11},C_{12},C_{13},C_{33},C_{44},C_{66}=(C_{11}-C_{12})/2$ 为声子弹性常数，K_1,K_2,K_3,K_4 为相位子弹性常数，$\boldsymbol{\eta}$ 为流体动力黏性系数，$\Gamma_{\boldsymbol{u}}$ 和 $\Gamma_{\boldsymbol{w}}$ 为声子和相位子耗散系数，A 和 B 的意义在前文中已经交待.

方程组(7)是 12 次对称软物质准晶三维动力学方程组，它包含 10 个场变量 u_x,u_y,u_z,w_x，

w_y, V_x, V_y, V_z, ρ 和 p, 方程也是 10 个：第 1 个方程是质量守恒方程, 第 2~4 个方程是动量守恒方程, 第 5~7 个方程为对称性破缺导致的声子运动方程,第 8、第 9 个方程为相位子耗散方程, 第 10 个方程是状态方程或物态方程.现在方程组相容,在数学上可解.如果没有状态方程或物态方程,则方程组不封闭,在物理上和数学上无意义.可见状态方程或物态方程极其重要.

这些方程揭示了声子场 $\boldsymbol{u}$ 和流体声子场 $\boldsymbol{V}$ 代表波传播, 前者传播速度为 $c_1 = \sqrt{(2A + C_{12} - B)/\rho}$, $c_2 = c_3\sqrt{(C_{11} - C_{12})/(2\rho)}$,后者传播速度为 $c_4 = \sqrt{(\partial p/\partial \rho)_s}$,也揭示了相位子场 $\boldsymbol{w}$ 的扩散特性, 扩散系数为 $D = 1/\Gamma_w$.

3 8 次对称软物质准晶

除去已经观测到的 12 次和 18 次对称软物质准晶之外,8 次对称软物质准晶可能在近期被发现.8 次对称固体准晶非常稳定,展现了其重要性,特别是它的声子与相位子强耦合,在物理学性质和数学解上很有趣,使得需要推导其动力学方程组.假设 xOy 平面是准周期平面, z 轴是 8 次对称轴, $w_z \equiv 0$, 所以这时声子、相位子和流体声子的三维本构关系如下[13-16]：

$$\begin{cases}\sigma_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz} + R(w_{xx} + w_{yy}),\\ \sigma_{yy} = C_{12}\varepsilon_{xx} + C_{11}\varepsilon_{yy} + C_{13}\varepsilon_{zz} - R(w_{xx} + w_{yy}),\\ \sigma_{zz} = C_{13}\varepsilon_{xx} + C_{13}\varepsilon_{yy} + C_{33}\varepsilon_{zz},\ \sigma_{yz} = \sigma_{zy} = 2C_{44}\varepsilon_{yz},\ \sigma_{zx} = \sigma_{xz} = 2C_{44}\varepsilon_{zx},\\ \sigma_{xy} = \sigma_{yx} = 2C_{66}\varepsilon - Rw_{xy} + Rw_{yx},\\ H_{xx} = K_1 w_{xx} + K_2 w_{yy} + R(\varepsilon_{xx} - \varepsilon_{yy}),\\ H_{yy} = K_2 w_{xx} + K_1 w_{yy} + R(\varepsilon_{xx} - \varepsilon_{yy}),\\ H_{yz} = K_4 w_{yz},\ H_{xy} = (K_1 + K_2 + K_3)w_{xy} + K_2 w_{yz} - 2R\varepsilon_{xy},\\ H_{xz} = K_4 w_{xz},\ H_{yx} = K_3 w_{xy} + (K_1 + K_2 + K_3)w_{yx} + 2R\varepsilon_{xy},\\ p_{xx} = -p + 2\eta\dot{\xi}_{xx} - \dfrac{2}{3}\eta\dot{\xi}_{kk},\\ p_{yy} = -p + 2\eta\dot{\xi}_{yy} - \dfrac{2}{3}\eta\dot{\xi}_{kk},\\ p_{zz} = -p + 2\eta\dot{\xi}_{zz} - \dfrac{2}{3}\eta\dot{\xi}_{kk},\\ p_{yz} = 2\eta\dot{\xi}_{yz},\ p_{zx} = 2\eta\dot{\xi}_{zx},\ p_{xy} = 2\eta\dot{\xi}_{xy}.\end{cases} \tag{8}$$

根据方程组(8)写出相应的 Hamilton 量,进而代入方程组(3),忽略项

$$\nabla_i\left(u_j\frac{\delta H}{\delta u_j}\right),\ \nabla_i\left(w_j\frac{\delta H}{\delta w_j}\right)$$

之后(因为这两项很小),得到 8 次对称软物质准晶广义动力学三维方程组如下：

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{V}) = 0, \tag{9a}$$

$$\frac{\partial(\rho V_x)}{\partial t} + \frac{\partial(V_x\rho V_x)}{\partial x} + \frac{\partial(V_y\rho V_x)}{\partial y} + \frac{\partial(V_z\rho V_x)}{\partial z} = -\frac{\partial p}{\partial x} + \eta\nabla^2(\rho V_x) +$$
$$\frac{1}{3}\eta\frac{\partial}{\partial x}\nabla\cdot\boldsymbol{V} + \left(C_{66}\frac{\partial^2}{\partial y^2} + C_{44}\frac{\partial^2}{\partial z^2}\right)u_x +$$

$$
(C_{12}+C_{66})\frac{\partial^2 u_y}{\partial x\partial y}+(C_{13}+C_{44}-C_{11})\frac{\partial^2 u_z}{\partial x\partial z}+(C_{11}-B)\frac{\partial}{\partial x}\nabla\cdot\boldsymbol{u}+
R\frac{\partial}{\partial x}\nabla_1\cdot\boldsymbol{w}-R\frac{\partial}{\partial y}\left(\frac{\partial w_x}{\partial y}-\frac{\partial w_y}{\partial x}\right)-(A-B)\frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial x},\tag{9b}
$$

$$
\frac{\partial(\rho V_y)}{\partial t}+\frac{\partial(V_x\rho V_y)}{\partial x}+\frac{\partial(V_y\rho V_y)}{\partial y}+\frac{\partial(V_z\rho V_y)}{\partial z}=-\frac{\partial p}{\partial y}+\eta\nabla^2(\rho V_y)+
\frac{1}{3}\eta\frac{\partial}{\partial y}\nabla\cdot\boldsymbol{V}+(C_{12}+C_{66})\frac{\partial^2 u_x}{\partial x\partial y}+\left(C_{66}\frac{\partial^2}{\partial x^2}+C_{11}\frac{\partial^2}{\partial y^2}+C_{44}\frac{\partial^2}{\partial z^2}\right)u_y+
(C_{13}+C_{44})\frac{\partial^2 u_z}{\partial y\partial z}+(C_{11}-B)\frac{\partial}{\partial y}\nabla\cdot\boldsymbol{u}-
R\frac{\partial}{\partial x}\left(\frac{\partial w_x}{\partial y}-\frac{\partial w_y}{\partial x}\right)-R\frac{\partial}{\partial y}\nabla_1\cdot\boldsymbol{w}-(A-B)\frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial y},\tag{9c}
$$

$$
\frac{\partial(\rho V_z)}{\partial t}+\frac{\partial(V_x\rho V_z)}{\partial x}+\frac{\partial(V_y\rho V_z)}{\partial y}+\frac{\partial(V_z\rho V_z)}{\partial z}=-\frac{\partial p}{\partial z}+\eta\nabla^2(\rho V_z)+
\frac{1}{3}\eta\frac{\partial}{\partial z}\nabla\cdot\boldsymbol{V}+\left(C_{44}\frac{\partial^2}{\partial x^2}+C_{44}\frac{\partial^2}{\partial y^2}+(C_{33}-C_{13}-C_{44})\frac{\partial^2}{\partial z^2}\right)u_z+
(C_{13}+C_{44}-B)\frac{\partial}{\partial z}\nabla\cdot\boldsymbol{u}-(A-B)\frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial z},\tag{9d}
$$

$$
\frac{\partial u_x}{\partial t}+V_x\frac{\partial u_x}{\partial x}+V_y\frac{\partial u_x}{\partial y}+V_z\frac{\partial u_x}{\partial z}=
V_x+\Gamma_{\boldsymbol{u}}\left[\left(C_{11}\frac{\partial^2}{\partial x^2}+C_{66}\frac{\partial^2}{\partial y^2}+C_{44}\frac{\partial^2}{\partial z^2}\right)u_x+(C_{12}+C_{66})\frac{\partial^2 u_y}{\partial x\partial y}+
(C_{13}+C_{44})\frac{\partial^2 u_z}{\partial x\partial z}+R\frac{\partial}{\partial x}\nabla_1\cdot\boldsymbol{w}-R\frac{\partial}{\partial y}\left(\frac{\partial w_x}{\partial y}-\frac{\partial w_y}{\partial x}\right)\right],\tag{9e}
$$

$$
\frac{\partial u_y}{\partial t}+V_x\frac{\partial u_y}{\partial x}+V_y\frac{\partial u_y}{\partial y}+V_z\frac{\partial u_y}{\partial z}=
V_y+\Gamma_{\boldsymbol{u}}\left[(C_{12}+C_{66})\frac{\partial^2 u_x}{\partial x\partial y}+\left(C_{66}\frac{\partial^2}{\partial x^2}+C_{11}\frac{\partial^2}{\partial y^2}+C_{44}\frac{\partial^2}{\partial z^2}\right)u_y+
(C_{13}+C_{44})\frac{\partial^2 u_z}{\partial y\partial z}-R\frac{\partial}{\partial x}\left(\frac{\partial w_x}{\partial y}-\frac{\partial w_y}{\partial x}\right)-R\frac{\partial}{\partial y}\nabla_1\cdot\boldsymbol{w}\right],\tag{9f}
$$

$$
\frac{\partial u_z}{\partial t}+V_x\frac{\partial u_z}{\partial x}+V_y\frac{\partial u_z}{\partial y}+V_z\frac{\partial u_z}{\partial z}=
V_z+\Gamma_{\boldsymbol{u}}\left[(C_{13}+C_{44})\left(\frac{\partial^2 u_x}{\partial x\partial z}+\frac{\partial^2 u_y}{\partial y\partial z}\right)+\left(C_{44}\frac{\partial^2}{\partial x^2}+C_{44}\frac{\partial^2}{\partial y^2}+C_{33}\frac{\partial^2}{\partial z^2}\right)u_z\right],\tag{9g}
$$

$$
\frac{\partial w_x}{\partial t}+V_x\frac{\partial w_x}{\partial x}+V_y\frac{\partial w_x}{\partial y}+V_z\frac{\partial w_x}{\partial z}=\Gamma_{\boldsymbol{w}}\left[K_1\nabla_1^2 w_x+(K_2+K_3)\frac{\partial^2 w_x}{\partial y^2}+
K_4\frac{\partial^2 w_x}{\partial z^2}+K_2\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial z}\right)w_y+R\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial y}\right)-R\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\right],\tag{9h}
$$

$$
\frac{\partial w_y}{\partial t}+V_x\frac{\partial w_y}{\partial x}+V_y\frac{\partial w_y}{\partial y}+V_z\frac{\partial w_y}{\partial z}=\Gamma_{\boldsymbol{w}}\left[(K_2+K_3)\frac{\partial^2 w_x}{\partial x\partial y}+K_3\frac{\partial^2 w_x}{\partial y\partial z}+\right.
$$

$$K_1\nabla_1^2 w_y + (K_2+K_3)\frac{\partial^2 w_y}{\partial x^2} + (K_1+K_2+K_3)\frac{\partial^2 w_y}{\partial x\partial z} +$$

$$R\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) + R\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial y}\right)\Bigg], \tag{9i}$$

$$p=f(\rho)=3\frac{k_B T}{l^3\rho_0^3}(\rho_0^2\rho+\rho_0\rho^2+\rho^3), \tag{9j}$$

其中

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\ \nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\ \nabla=\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z},\ \nabla_1=\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y},$$

$$\boldsymbol{V}=\boldsymbol{i}V_x+\boldsymbol{j}V_y+\boldsymbol{k}V_z,\ \boldsymbol{u}=\boldsymbol{i}u_x+\boldsymbol{j}u_y+\boldsymbol{k}u_z,\ \boldsymbol{w}=\boldsymbol{i}w_x+\boldsymbol{j}w_y,$$

$C_{11},C_{12},C_{13},C_{33},C_{44},C_{66}=(C_{11}-C_{12})/2$ 为声子弹性常数,K_1,K_2,K_3,K_4 为相位子弹性常数,R 为声子-相位子耦合弹性常数, η 为流体体动力黏性系数,Γ_u 和 Γ_w 为声子和相位子耗散系数, A 和 B 的意义在前文中已经交待.

方程组(9)是 8 次对称软物质准晶三维动力学方程组,它包含 10 个场变量 $u_x,u_y,u_z,w_x,w_y,V_x,V_y,V_z,\rho$ 和 p, 方程也是 10 个: 第 1 个方程是质量守恒方程, 第 2~4 个方程是动量守恒方程, 第 5~7 个方程是对称性破缺导致的声子运动方程, 第 8、第 9 个方程是相位子耗散方程, 第 10 个方程是状态方程或物态方程.现在方程组在物理上自洽,在数学上可解.如果没有状态方程或物态方程,则方程组不封闭,在物理上无意义和数学上无法求解.可见状态方程或物态方程极其重要.

这些方程刻画了声子场 $\boldsymbol{u}$ 和流体声子场 $\boldsymbol{V}$ 代表波传播, 前者传播速度为 $c_1=\sqrt{(2A+C_{12}-B)/\rho}$, $c_2=c_3\sqrt{(C_{11}-C_{12})/(2\rho)}$,后者传播速度为 $c_4=\sqrt{(\partial p/\partial\rho)_s}$,也揭示了相位子场 $\boldsymbol{w}$ 的扩散特性, 扩散系数为 $D=1/\Gamma_w$.

4 10 次对称软物质准晶

除去已经观测到的 12 次和 18 次对称软物质准晶之外,10 次对称软物质准晶可能在不远的将来被发现.10 次对称固体准晶非常稳定,其重要性仅仅弱于二十面体准晶,特别是它的声子与相位子强耦合,在物理学性质和数学解上很有趣,使得需要推导其动力学方程组.假设 xOy 平面是准周期平面, z 轴是10次对称轴,所以 $w_z\equiv 0$, 这时声子、相位子和流体声子的三维本构关系如下[13-16]:

$$\begin{cases}\sigma_{xx}=C_{11}\varepsilon_{xx}+C_{12}\varepsilon_{yy}+C_{13}\varepsilon_{zz}+R(w_{xx}+w_{yy}),\\ \sigma_{yy}=C_{12}\varepsilon_{xx}+C_{11}\varepsilon_{yy}+C_{13}\varepsilon_{zz}-R(w_{xx}+w_{yy}),\\ \sigma_{zz}=C_{13}\varepsilon_{xx}+C_{13}\varepsilon_{yy}+C_{33}\varepsilon_{zz},\\ \sigma_{yz}=\sigma_{zy}=2C_{44}\varepsilon_{yz},\ \sigma_{zx}=\sigma_{xz}=2C_{44}\varepsilon_{zx},\\ \sigma_{xy}=\sigma_{yx}=2C_{66}\varepsilon_{xy}-R(w_{xy}-w_{yx}),\\ H_{xx}=K_1w_{xx}+K_2w_{yy}+R(\varepsilon_{xx}-\varepsilon_{yy}),\\ H_{yy}=K_2w_{xx}+K_1w_{yy}+R(\varepsilon_{xx}-\varepsilon_{yy}),\\ H_{yz}=K_4w_{yz},\ H_{xy}=K_1w_{xy}-K_2w_{yx},\ H_{xz}=K_4w_{xz},\\ H_{yx}=-K_2w_{xy}+K_1w_{yx}+2R\varepsilon_{xy},\end{cases} \tag{10a}$$

$$\begin{cases} p_{xx} = -p + 2\eta\dot{\xi}_{xx} - \dfrac{2}{3}\eta\dot{\xi}_{kk}, \\ p_{yy} = -p + 2\eta\dot{\xi}_{yy} - \dfrac{2}{3}\eta\dot{\xi}_{kk}, \\ p_{zz} = -p + 2\eta\dot{\xi}_{zz} - \dfrac{2}{3}\eta\dot{\xi}_{kk}, \\ p_{yz} = 2\eta\dot{\xi}_{yz},\ p_{zx} = 2\eta\dot{\xi}_{zx},\ p_{xy} = 2\eta\dot{\xi}_{xy}. \end{cases} \tag{10b}$$

和前面两节相类似,得到 10 次对称软物质准晶动力学方程组:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{V}) = 0, \tag{11a}$$

$$\begin{aligned} &\frac{\partial(\rho V_x)}{\partial t} + \frac{\partial(V_x\rho V_x)}{\partial x} + \frac{\partial(V_y\rho V_x)}{\partial y} + \frac{\partial(V_z\rho V_x)}{\partial z} = -\frac{\partial p}{\partial x} + \eta\nabla^2(\rho V_x) + \\ &\quad \frac{1}{3}\eta\frac{\partial}{\partial x}\nabla\cdot\boldsymbol{V} + \left(C_{66}\frac{\partial^2}{\partial y^2} + C_{44}\frac{\partial^2}{\partial z^2}\right)u_x + \\ &\quad (C_{12} + C_{66})\frac{\partial^2 u_y}{\partial x\partial y} + (C_{13} + C_{44} - C_{11})\frac{\partial^2 u_z}{\partial x\partial z} + (C_{11} - B)\frac{\partial}{\partial x}\nabla\cdot\boldsymbol{u} + \\ &\quad R\frac{\partial}{\partial x}\nabla_1\cdot\boldsymbol{w} - R\frac{\partial}{\partial y}\left(\frac{\partial w_x}{\partial y} - \frac{\partial w_y}{\partial x}\right) - (A - B)\frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial x}, \end{aligned} \tag{11b}$$

$$\begin{aligned} &\frac{\partial(\rho V_y)}{\partial t} + \frac{\partial(V_x\rho V_y)}{\partial x} + \frac{\partial(V_y\rho V_y)}{\partial y} + \frac{\partial(V_z\rho V_y)}{\partial z} = -\frac{\partial p}{\partial y} + \eta\nabla^2(\rho V_y) + \\ &\quad \frac{1}{3}\eta\frac{\partial}{\partial y}\nabla\cdot\boldsymbol{V} + (C_{12} + C_{66})\frac{\partial^2 u_x}{\partial x\partial y} + \left(C_{66}\frac{\partial^2}{\partial x^2} + C_{11}\frac{\partial^2}{\partial y^2} + C_{44}\frac{\partial^2}{\partial z^2}\right)u_y + \\ &\quad (C_{13} + C_{44})\frac{\partial^2 u_z}{\partial y\partial z} + (C_{11} - B)\frac{\partial}{\partial y}\nabla\cdot\boldsymbol{u} - \\ &\quad R\frac{\partial}{\partial x}\left(\frac{\partial w_x}{\partial y} - \frac{\partial w_y}{\partial x}\right) - R\frac{\partial}{\partial y}\nabla_1\cdot\boldsymbol{w} - (A - B)\frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial y}, \end{aligned} \tag{11c}$$

$$\begin{aligned} &\frac{\partial(\rho V_z)}{\partial t} + \frac{\partial(V_x\rho V_z)}{\partial x} + \frac{\partial(V_y\rho V_z)}{\partial y} + \frac{\partial(V_z\rho V_z)}{\partial z} = -\frac{\partial p}{\partial z} + \eta\nabla^2(\rho V_z) + \\ &\quad \frac{1}{3}\eta\frac{\partial}{\partial z}\nabla\cdot\boldsymbol{V} + \left(C_{44}\frac{\partial^2}{\partial x^2} + C_{44}\frac{\partial^2}{\partial y^2} + (C_{33} - C_{13} - C_{44})\frac{\partial^2}{\partial z^2}\right)u_z + \\ &\quad (C_{13} + C_{44} - B)\frac{\partial}{\partial z}\nabla\cdot\boldsymbol{u} - (A - B)\frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial z}, \end{aligned} \tag{11d}$$

$$\begin{aligned} &\frac{\partial u_x}{\partial t} + V_x\frac{\partial u_x}{\partial x} + V_y\frac{\partial u_x}{\partial y} + V_z\frac{\partial u_x}{\partial z} = \\ &\quad V_x + \Gamma_{\boldsymbol{u}}\left[\left(C_{11}\frac{\partial^2}{\partial x^2} + C_{66}\frac{\partial^2}{\partial y^2} + C_{44}\frac{\partial^2}{\partial z^2}\right)u_x + (C_{12} + C_{66})\frac{\partial^2 u_y}{\partial x\partial y} + \right. \\ &\quad \left.(C_{13} + C_{44})\frac{\partial^2 u_z}{\partial x\partial z} + R\frac{\partial}{\partial x}\nabla_1\cdot\boldsymbol{w} - R\frac{\partial}{\partial y}\left(\frac{\partial w_x}{\partial y} - \frac{\partial w_y}{\partial x}\right)\right], \end{aligned} \tag{11e}$$

$$\frac{\partial u_y}{\partial t} + V_x\frac{\partial u_y}{\partial x} + V_y\frac{\partial u_y}{\partial y} + V_z\frac{\partial u_y}{\partial z} =$$

$$V_y+\boldsymbol{\Gamma_u}\left[(C_{12}+C_{66})\frac{\partial^2u_x}{\partial x\partial y}+\left(C_{66}\frac{\partial^2}{\partial x^2}+C_{11}\frac{\partial^2}{\partial y^2}+C_{44}\frac{\partial^2}{\partial z^2}\right)u_y+\right.$$

$$\left.(C_{13}+C_{44})\frac{\partial^2u_z}{\partial y\partial z}-R\frac{\partial}{\partial x}\left(\frac{\partial w_x}{\partial y}-\frac{\partial w_y}{\partial x}\right)-R\frac{\partial}{\partial y}\nabla_1\cdot\boldsymbol{w}\right], \tag{11f}$$

$$\frac{\partial u_z}{\partial t}+V_x\frac{\partial u_z}{\partial x}+V_y\frac{\partial u_z}{\partial y}+V_z\frac{\partial u_z}{\partial z}=$$

$$V_z+\boldsymbol{\Gamma_u}\left[(C_{13}+C_{44})\left(\frac{\partial^2u_x}{\partial x\partial z}+\frac{\partial^2u_y}{\partial y\partial z}\right)+\left(C_{44}\frac{\partial^2}{\partial x^2}+C_{44}\frac{\partial^2}{\partial y^2}+C_{33}\frac{\partial^2}{\partial z^2}\right)u_z\right], \tag{11g}$$

$$\frac{\partial w_x}{\partial t}+V_x\frac{\partial w_x}{\partial x}+V_y\frac{\partial w_x}{\partial y}+V_z\frac{\partial w_x}{\partial z}=\boldsymbol{\Gamma_w}\left[K_1\nabla_1^2w_x+K_4\frac{\partial^2w_x}{\partial z^2}+\right.$$

$$\left.K_2\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)w_y+R\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial y}\right)-R\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\right], \tag{11h}$$

$$\frac{\partial w_y}{\partial t}+V_x\frac{\partial w_y}{\partial x}+V_y\frac{\partial w_y}{\partial y}+V_z\frac{\partial w_y}{\partial z}=$$

$$\boldsymbol{\Gamma_w}\left[K_1\nabla_1^2w_y+K_4\frac{\partial^2w_y}{\partial z^2}+R\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+R\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial y}\right)\right], \tag{11i}$$

$$p=f(\rho)=3\frac{k_BT}{l^3\rho_0^3}(\rho_0^2\rho+\rho_0\rho^2+\rho^3), \tag{11j}$$

其中

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\ \nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\ \nabla=\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z},$$

$$\boldsymbol{V}=\boldsymbol{i}V_x+\boldsymbol{j}V_y+\boldsymbol{k}V_z,\ \boldsymbol{u}=\boldsymbol{i}u_x+\boldsymbol{j}u_y+\boldsymbol{k}u_z,\ \boldsymbol{w}=\boldsymbol{i}w_x+\boldsymbol{j}w_y,$$

$C_{11},C_{12},C_{13},C_{33},C_{44},C_{66}=(C_{11}-C_{12})/2$ 为声子弹性常数,K_1,K_2,K_3,K_4 为相位子弹性常数,R 为声子-相位子耦合弹性常数,η 为流体体动力黏性系数,$\boldsymbol{\Gamma_u}$ 和 $\boldsymbol{\Gamma_w}$ 为声子和相位子耗散系数,A 和 B 的意义在前文中已经交待.

方程组(11)是 10 次对称软物质准晶三维动力学方程组,它包含 10 个场变量 $u_x,u_y,u_z,w_x,w_y,V_x,V_y,V_z,\rho$ 和 p, 方程也是 10 个: 第 1 个方程是质量守恒方程, 第 2~4 个方程是动量守恒方程, 第 5~7 个方程是对称性破缺导致的声子运动方程, 第 8、第 9 个方程是相位子耗散方程, 第 10 个方程是状态方程或物态方程.现在方程组在物理上自洽,在数学上可解.如果没有状态方程或物态方程,则方程组不封闭,在物理上无意义和数学上无法求解.可见状态方程或物态方程极其重要.

这些方程刻画了声子场 $\boldsymbol{u}$ 和流体声子场 $\boldsymbol{V}$ 代表波传播, 前者传播速度为 $c_1=\sqrt{(2A+C_{12}-B)/\rho}$,$c_2=c_3\sqrt{(C_{11}-C_{12})/(2\rho)}$,后者传播速度为 $c_4=\sqrt{(\partial p/\partial\rho)_s}$,也揭示了相位子场 $\boldsymbol{w}$ 的扩散特性, 扩散系数为 $D=1/\boldsymbol{\Gamma_w}$.

5 求解程序和初步结果

未知函数 $u_x,u_y,u_z,w_x,w_y,V_x,V_y,V_z$ 代表各个元激发的场, 它们和其他的流体动力学量 p 和 ρ 遵循不同准晶系的方程组(7)、(9)和(11),注意温度 T 是给定的.为了确定这些流体动力学量必须在适宜的初始条件和边界条件下求解这些方程组.因而凝聚态物理问题和数学物理

及计算物理紧密相联系.

耦合初始条件和边界条件的方程组(7)(或方程组(9)、或方程组(11))极其复杂,求解极其困难,除了个别例外[17],只能用数值方法.

用有限差分方法求解,没有任何商业软件可用,只有自己编写计算机程序.从数学上研究这些有限差分方法的稳定性很重要,迄今的计算都是稳定的.通过物理意义分析,或者在特殊情形下的经典问题的解对比,判断方程组和所提初始条件和边界条件的正确性,因为从数学上证明解的存在性和唯一性在目前是不可能的.

初步的计算表明,软物质准晶的物理和力学性能与固体准晶的相差很大.例如对软物质准晶, $\delta\rho/\rho_0 \sim (10^{-4} \sim 10^{-3})$①~④,而固体准晶, $\delta\rho/\rho_0 \sim 10^{-13}$[18].同时, 对软物质准晶 $p_{ij}/\sigma_{ij} \sim 1$①~④,而固体准晶, $\sigma'_{ij}/\sigma_{ij} \sim 10^{-15}$[18].这些量上的巨大差别,显示了两者质上的根本不同.

6 结论和讨论

8 次、10 次和 12 次对称二维软物质准晶的广义动力学三维方程推导出来,改进了笔者以前得到的二维方程[1-2].

以前得到的二维方程已经做了详细求解,得到丰富的结果.三维方程的解意义更大,虽然求解更困难.

我们承认现在软物质准晶的广义动力学得益于 Lubensky 等[8-9]关于固体准晶的流体动力学方程的启发,但是计算发现它的理论存在巨大困难,因为它对固体准晶用了可压缩假定,这是导致其理论体系困难的根本原因[18].该理论必须重建.现在的软物质准晶广义动力学在理论上自洽,结果物理意义鲜明.

18 次对称准晶和其他第二类二维准晶只有平面场,这是由六维镶嵌空间理论,也就是群表示理论决定的,所以它们没有三维方程.

特别是最近的文献[19]报道了对层状液晶 B 的分子动力学摸拟,对模拟的 12 次对称软物质准晶的相位子自由度的存在提出质疑, 很有趣, 值得注意.不过在计算中①~②发现 12 次对称软物质准晶的相位子效应非常弱, 因为相位子-声子在 12 次对称软物质准晶不耦合(见方程(6)).

致谢 本文作者衷心感谢美国 Pennsylvania 大学的 Lubensky T C 教授、Akron 大学的 Stephen Z D 教授和荷兰 Utrecht 大学的 Wensink H H 教授的有益的讨论.

参考文献(References):

[1] 范天佑. 软物质准晶广义流体动力学方程组[J]. 应用数学和力学, 2016, **37**(4): 331-344.(FAN Tian-you. Equation systems of generalized hydrodynamics for soft-matter quasicrystals[J]. *Ap-*

① Cheng H, Fan T Y. Flow of soft-matter quasicrystals past a circular cylinder[J]. *Computer Methods of Applied Mechanics and Engineering*, 2017, submitted.

② Cheng H, Fan T Y, Wei H. Characters of deformation and motion of possible 5- and 10-fold symmetry soft-matter quasicrystals[J]. *Results in Physics*, 2016, in reviewing.

③ Wang F, Fan T Y, Cheng H. Hydrodynamic analysis of possible soft-matter octagonal quasicrystals[J]. *Applied Mathematics and Mechanics*(*English Edition*), 2017, accepted.

④ Cheng H, Fan T Y, Hu H Y, et al. Is the crack opened or closed in soft-matter pentagonal and decagonal quasicrystals?[J]. *Engineering Fracture Mechanics*, 2017, submitted.

plied Mathematics and Mechanics, 2016, **37**(4): 331-344.(in Chinese))

[2] 范天佑. 软物质第二类二维准晶广义流体动力学[J]. 应用数学和力学, 2017, **38**(2): 189-199.(FAN Tian-you. Generalized hydrodynamics for second 2D soft-matter quasicrystals[J]. *Applied Mathematics and Mechanics*, 2017, **38**(2): 189-199.(in Chinese))

[3] ZENG Xiang-bing, Ungar G, LIU Yong-song, et al. Supramolecular dendritic liquid quasicrystals[J]. *Nature*, 2004, **428**: 157-160.

[4] Takano K, Kawashima W, Noro A, et al. A mesoscopic archimedian tiling having a complexity in polymeric stars[J]. *Journal of Polymer Science Part B: Polymer Physics*, 2005, **43**(18): 2427-2432.

[5] Talapin D V, Shevechenko E V, Bodnarchuk M I, et al. Quasicrystalline order in self-assembled binary nanoparticle superlattices[J]. *Nature*, 2009, **461**: 964-967.

[6] Fischer S, Exner A, Zielske K, et al. Colloidal quasicrystals with 12-fold and 18-fold diffraction symmetry[J]. *Proceedings of the National Academy of Sciences of the United States of America*, 2011, **108**(5): 1810-1814.

[7] YUE Kan, HUANG Ming-jun, Marson R, et al. Geometry induced sequence of nanoscale Frank-Kasper and quasicrystal mesophases in giant surfactants[J]. *Proceedings of the National Academy of Sciences of the United States of America*, 2016, **113**(50): 14195-14200.

[8] Lubensky T C, Ramaswamy S, Toner J. Hydrodynamics of icosahedral quasicrystals[J]. *Physical Review B: Condensed Matter*, 1985, **32**(11): 7444-7452.

[9] Lubensky T C. Symmetry, elasticity and hydrodynamics of quasicrystals[M]//*Introduction to Quasicrystals*. Jaric V M, ed. Boston: Academic Press, 1988: 199-280.

[10] Lifshitz E M, Petaevskii L P. *Statistical Physics Part 2: Landau and Lifshitz Course of Theoretical Physics*[M]. Vol 9. Oxford: Pergamon Press, 1980.

[11] Wensink H H. Equation of state of a dense columnar liquid crystal[J]. *Physical Review Letters*, 2004, **93**: 157801.

[12] Dzyaloshinskii I E, Volovick G E. Poisson brackets in condensed matter physics[J]. *Annals of Physics*, 1980, **125**(1): 67-97.

[13] HU Cheng-zheng, WANG Ren-hui, DING Di-hua. Symmetry groups, physical property tensors, elasticity and dislocations in quasicrystals[J]. *Reports on Progress in Physics*, 2000, **63**(1): 1-39.

[14] 杨顺华, 丁棣华. 晶体位错理论基础[M]. 第2卷. 北京: 科学出版社, 1998.(YANG Shun-hua, DING Di-hua. *Theoretical Foundations of Crystal Dislocation*[M]. Vol 2. Beijing: Science Press, 1998.(in Chinese))

[15] FAN Tian-you. *Mathematical Theory of Elasticity of Quasicrystals and Its Applications*[M]. 2nd ed. Beijing: Science Press, 2016.

[16] 范天佑. 固体与软物质准晶数学弹性与相关理论及应用[M]. 北京: 北京理工大学出版社, 2014.(FAN Tian-you. *Mathematical Theory of Elasticity and Relevant Topics of Solid and Softmatter Quasicrystals and Its Applications*[M]. Beijing: Beijing Institute of Technology Press, 2014.(in Chinese))

[17] Li X F, Fan T Y. Dislocations in the second kind two-dimensional quasicrystals of soft matter[J]. *Physica B: Condensed Matter*, 2016, **502**(1): 175-180.

[18] CHENG Hui, FAN Tian-you, WEI Hao. Solutions for hydrodynamics of solid quasicrystals with 5- and 10-fold symmetry[J]. *Applied Mathematics and Mechanics*(*English Edition*), 2016, **37**(10): 1393-1404.

[19] Metere A, Oleynikov P, Dzugutov M, et al. A smectic dodecagonal quasicrystal[J]. *Soft Matter*, 2016, **12**(43): 8869-8876.

3D Generalized Hydrodynamics of Soft-Matter Quasicrystals

FAN Tian-you, TANG Zhi-yi
(*School of Physics*, *Beijing Institute of Technology*, *Beijing* 100081, *P.R.China*)

(Recommended by CHEN Wei-qiu, M. AMM Editorial Board)

Abstract: The 3D generalized hydrodynamics of soft-matter quasicrystals was investigated, and the governing equations for observed and possibly observed soft-matter quasicrystals were derived. The solving procedure for the equations was discussed briefly. Some results obtained reveal the gigantic dissimilarities between soft-matter quasicrystals and solid ones.

Key words: soft matter; 1st-kind 2D quasicrystal; 2nd-kind 2D quasicrystal; generalized hydrodynamics; state equation

Foundation item: The National Natural Science Foundation of China(11272053)

引用本文/Cite this paper:
范天佑, 唐志毅. 软物质准晶三维广义流体动力学[J]. 应用数学和力学, 2017, **38**(11): 1195-1207.
FAN Tian-you, TANG Zhi-yi. 3D generalized hydrodynamics of soft-matter quasicrystals[J]. *Applied Mathematics and Mechanics*, 2017, **38**(11): 1195-1207.