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应用全新 $G'/(G+G')$ 展开方法求解广义非线性 Schrödinger 方程和耦合非线性 Schrödinger 方程组*

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摘要： 研究了一种全新的 $G'/(G+G')$ 展开方法,并应用这种方法讨论了广义非线性 Schrödinger 方程和一类耦合非线性 Schrödinger 方程组新形式的精确解,包括双曲余切函数解、余切函数解和有理函数解.全新 $G'/(G+G')$ 展开方法不但直接而有效地求出方程的新精确解,而且扩大了解的范围,这种新方法对于研究偏微分方程具有广泛的应用意义.

关 键 词： 全新 $G'/(G+G')$ 展开方法； 广义非线性 Schrödinger 方程； 耦合非线性 Schrödinger 方程组； 精确解

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引 言

非线性科学的研究是目前自然科学领域发展的前沿和热点,非线性现象可以用非线性微分方程进行描述,非线性偏微分方程以各种学科问题为背景,广泛应用于物理学、流体力学、大气科学、信息科学、生命科学、水系统科学等领域,因此对这些非线性偏微分方程的求解十分有意义.目前一些求解非线性偏微分方程的方法,如 Bäcklund 变换方法[1]、Hirota 双线性变换法[2]、双曲正切函数展开法[3-4]、F- 函数展开方法[5]、指数函数法[6]、反散射法[7]、投射方法[8]、辅助方程法[9-10]、Jacobi(雅可比)椭圆函数展开法[11]、$G'/(G+G')$ 展开法[12-13]、首次积分法[14]等准确求出了这类方程的精确孤立波解,笔者也运用广义变分迭代法[15]、泛函映射方法[16]、同伦映射法[17-19]、奇摄动法[20-21]求得一系列非线性方程的渐近解.

非线性 Schrödinger 方程是数学物理等领域中一类重要的非线性演化方程,随着现代科技的迅速发展,它在等离子物理学、量子力学、电磁学、非线性光学,特别是在光孤子通讯中广泛应用,所以研究非线性 Schrödinger 的解就更具有重大理论意义和实用价值.而 Schrödinger 方程中含有虚数单位,其解是复函数,对其进行求解较其他非线性偏微分方程复杂.近些年专家学

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者已经研究出一些方法成功得到非线性 Schrödinger 方程的精确孤立波解,如 Riccati 方程映射法[22]、推广的 Lie 群约化法[23]、半逆变分原理[24]、正弦余弦法[25]、分离变量法[26]、直接截断法[27]、复 tanh 展开法[28].本文在 $G'/(G+G')$ 展开法的基础上,提出了一种全新的 $G'/(G+G')$ 展开法,分别讨论了广义非线性 Schrödinger 方程和一类耦合非线性 Schrödinger 方程组,并成功求得方程的新形式的精确解,扩大了解的范围.

1 全新 $G'/(G+G')$ 展开法基本思想

给定含非线性偏微分方程:

$$F(u,u_t,u_{x_1},u_{x_2},\cdots,u_{tt},u_{x_1t},u_{x_2t},\cdots,u_{x_1},u_{x_2x_2},\cdots)=0, \tag{1}$$

其中 $u=u(x_1,x_2,\cdots,x_n,t)$ 是未知函数, F 为 u 及 u 关于 $x_1,x_2,\cdots,x_n,t$ 各阶偏导数的多项式.

为求得非线性偏微分方程的精确解,主要步骤如下:

步骤 1 对方程(1)作行波变换

$$\xi=k_1x_1+k_2x_2+\cdots+k_nx_n-st, \tag{2}$$

则方程(1)转化为只含变量 ξ 的常微分方程

$$H(u,-su',k_1u',k_2u',\cdots,s^2u'',-k_1su'',\cdots,k_1^2u'',k_2^2u'',\cdots)=0, \tag{3}$$

其中 $u'=\dfrac{du}{d\xi},u''=\dfrac{d^2u}{d\xi^2},\cdots,H$ 是含 u 及 u 关于 ξ 的各阶导数的多项式.

步骤 2 假设常微分方程(3)的解可以表示成

$$u(\xi)=a_0+\sum_{i=1}^{m}a_i\left(\frac{G'}{G+G'}\right)^i+\sum_{i=-m}^{-1}a_i\left(\frac{G'}{G+G'}\right)^i, \tag{4}$$

其中 $G=G(\xi)$ 满足二阶非线性常微分方程

$$A(G')^2-BGG''+CG^2=0, \tag{5}$$

$a_{-m},\cdots,a_0,\cdots,a_m$ 是待定常数,正整数 m 通过平衡方程(3)中的最高阶导数项和最高次非线性项确定.

步骤 3 将式(4)代入式(3),合并 $G'/(G+G')$ 的相同幂次项,方程(3)的左端变成一个关于 $G'/(G+G')$ 的多项式,令该多项式的 $G'/(G+G')$ 各阶幂次系数为0,导出关于 $a_{-m},\cdots,a_0,\cdots,a_m,k_1,\cdots,k_n,s$ 和 A,B,C 的代数方程组.

步骤 4 用 Mathematica 软件求解步骤 3 中的代数方程组,得出相应的 $a_{-m},\cdots,a_0,\cdots,a_m,k_1,\cdots,k_n,s$, 再代入式(4)并利用式(2)和(5)得到方程(1)的精确解.

2 广义非线性 Schrödinger 方程的精确解

非线性 Schrödinger 方程在现代光孤子通信中有着极其重要的应用,标准的非线性 Schrödinger 方程是描述光孤子在理想无损耗单模光纤中传输的模型,满足该方程的光孤子在远距离光纤传播中波形、振幅、速度始终保持不变,不会产生信息失真和波形畸变,传输码率高.由于光纤是存在色散现象的,考虑色散项,所以光孤子在单模光纤中传输满足广义非线性 Schrödinger 方程[29]:

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{\partial^2u}{\partial x^2}+\alpha_0|u|^2u+\mathrm{i}[\gamma_1u_{xxx}+\gamma_2|u|^2u_x+\gamma_3(|u|^2)_xu]=0. \tag{6}$$

首先做一个变换

$$u=\varphi(\xi)\exp(\mathrm{i}(kx-\omega t)),\qquad \xi=p(x-ct),\tag{7}$$

将式(7)代入式(6),得

$$\begin{aligned}&\mathrm{i}(\gamma_1p^3\varphi'''-3\gamma_1k^2p\varphi'+\gamma_2p\varphi^2\varphi'+2\gamma_3p\varphi^2\varphi'-cp\varphi'+2kp\varphi')+\\&\quad(\omega\varphi+p^2\varphi''-k^2\varphi+\alpha_0\varphi^3-3\gamma_1kp^2\varphi''+\gamma_1k^3\varphi-\gamma_2k\varphi^3)=0.\end{aligned}\tag{8}$$

令式(8)中的实部与虚部分别为0,得

$$\gamma_1p^2\varphi'''+(-c-3\gamma_1k^2+2k)\varphi'+(\gamma_2+2\gamma_3)\varphi^2\varphi'=0,\tag{9}$$

$$p^2(1-3\gamma_1k)\varphi''+(\omega-k^2+\gamma_1k^3)\varphi+(\alpha_0-\gamma_2k)\varphi^3=0.\tag{10}$$

对式(9)进行一次积分,并令积分常数为0,得

$$\gamma_1p^2\varphi''+(-c-3\gamma_1k^2+2k)\varphi+\left(\frac{1}{3}\gamma_2+\frac{2}{3}\gamma_3\right)\varphi^3=0.\tag{11}$$

对比式(10)与(11),它们的解相同,则系数应满足

$$\frac{\gamma_1p^2}{p^2(1-3\gamma_1k)}=\frac{2k-c-3\gamma_1k^2}{\omega-k^2+\gamma_1k^3}=\frac{\frac{1}{3}\gamma_2+\frac{2}{3}\gamma_3}{\alpha_0-\gamma_2k}.\tag{12}$$

由式(12)可得

$$k=\frac{\gamma_2+2\gamma_3-3\gamma_1\alpha_0}{6\gamma_1\gamma_3},\ \omega=\frac{(1-3\gamma_1k)(2k-c-3\gamma_1k^2)}{\gamma_1}+k^2-\gamma_1k^3.\tag{13}$$

在式(12)成立情况下,解方程(11),令

$$\alpha=\gamma_1p^2,\ \eta=c+3\gamma_1k^2-2k,\ \beta=\frac{1}{3}\gamma_2+\frac{2}{3}\gamma_3,\tag{14}$$

则方程(11)可简化为

$$\alpha\varphi''-\eta\varphi+\beta\varphi^3=0.\tag{15}$$

运用齐次平衡法来平衡方程中的 φ'' 和 φ^3 项,得 $m+2=3m$,即 $m=1$.根据全新 $G'/(G+G')$ 展开法,可设方程(15)的解为

$$\varphi(\xi)=a_0+a_1\left(\frac{G'}{G+G'}\right)+a_{-1}\left(\frac{G'}{G+G'}\right)^{-1},\tag{16}$$

其中 $G=G(\xi)$,满足

$$A(G')^2-BGG''+CG^2=0,\tag{17}$$

a_0,a_1,a_{-1} 是待定常数.

由式(16)和(17)可得出 φ'' 与 φ^3 的表达式,详情见附录A,将式(16)及 φ'' 与 φ^3 的表达式代入方程(15),合并 $(G'/(G+G'))^i(i=-3,-2,-1,0,1,2,3)$ 的相同幂次系数,并令这些系数为0,得到一系列方程组,详情见附录A,然后运用 Mathematica 求解这些方程组,得到3组解,它们分别为

$$①\ \begin{cases}a_0=\mp\sqrt{\dfrac{2\alpha}{-\beta}}\dfrac{C}{B},\ a_1=\pm\sqrt{\dfrac{2\alpha}{-\beta}}\left(\dfrac{A}{B}-1+\dfrac{C}{B}\right),\ a_{-1}=0,\\ \dfrac{\eta}{\alpha}=2\dfrac{C}{B}\left(\dfrac{A}{B}-1+\dfrac{C}{B}\right)-2\dfrac{C}{B};\end{cases}\tag{18}$$

$$②\ \begin{cases}a_0=\mp\sqrt{\dfrac{2\alpha}{-\beta}}\dfrac{C}{B},\ a_1=0,\ a_{-1}=\pm\sqrt{\dfrac{2\alpha}{-\beta}}\left(\dfrac{A}{B}-1+\dfrac{C}{B}\right),\\ \dfrac{\eta}{\alpha}=2\dfrac{C}{B}\left(\dfrac{A}{B}-1+\dfrac{C}{B}\right)-2\dfrac{C}{B};\end{cases}\tag{19}$$

$$③\quad \begin{cases} a_0 = \mp \sqrt{\dfrac{2\alpha}{-\beta}}\dfrac{C}{B},\ a_1 = \pm\sqrt{\dfrac{2\alpha}{-\beta}}\left(\dfrac{A}{B} - 1 + \dfrac{C}{B}\right),\ a_{-1} = \mp \sqrt{\dfrac{2\alpha}{-\beta}}\dfrac{C}{B}, \\ \dfrac{\eta}{\alpha} = -10\dfrac{C}{B}\left(\dfrac{A}{B} - 1 + \dfrac{C}{B}\right) - 2\left(\dfrac{C}{B}\right)^2. \end{cases} \tag{20}$$

分别将式(18)、(19)、(20)和(16)代入式(7),并且利用方程(17)的3种情形下的解,得到方程(6)的3种类型精确解,它们分别为双曲余切函数解、余切函数解和有理函数解.

情形1 当 $\dfrac{C}{B}\left(\dfrac{A}{B} - 1\right) > 0$ 时,

$$u_{1,2}(x,t) = \left[\mp \sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B} \pm \sqrt{\frac{2\alpha}{-\beta}}\left(\frac{A}{B} - 1 + \frac{C}{B}\right) \times \frac{1}{\sqrt{(A-B)/C}\cot(D\xi + c_0) + 1}\right]\exp(\mathrm{i}(kx - wt)),$$

$$u_{3,4}(x,t) = \left\{\mp \sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B} \pm \sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\left[\sqrt{\frac{A-b}{C}}\cot(D\xi + c_0) + 1\right]\right\}\exp(\mathrm{i}(kx - wt)),$$

$$u_{5,6}(x,t) = \left\{\mp \sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B} \pm \sqrt{\frac{2\alpha}{-\beta}}\left(\frac{A}{B} - 1 + \frac{C}{B}\right)\cdot\frac{1}{\sqrt{(A-B)C}\cot(D\xi + c_0) + 1} \pm \sqrt{\frac{2\alpha}{-\beta}}\frac{B}{C}\cdot\frac{1}{\sqrt{(A-B)/C}\cot(D\xi + c_0) + 1}\right\}\exp(\mathrm{i}(kx - wt)),$$

其中 $D = \dfrac{C}{B}\sqrt{\dfrac{A/B - 1}{C/B}}$.

情形2 当 $\dfrac{C}{B}\left(\dfrac{A}{B} - 1\right) > 0$ 时,

$$u_{7,8}(x,t) = \left[\mp \sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B} \pm \sqrt{\frac{2\alpha}{-\beta}}\left(\frac{A}{B} - 1 + \frac{C}{B}\right) \times \frac{1}{\sqrt{(B-A)/C}\coth(E\xi + c_0) + 1}\right]\exp(\mathrm{i}(kx - wt)),$$

$$u_{9,10}(x,t) = \left\{\mp \sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B} \pm \sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\left[\sqrt{\frac{B-A}{C}}\coth(E\xi + c_0) + 1\right]\right\}\exp(\mathrm{i}(kx - wt)),$$

$$u_{11,12}(x,t) = \left\{\mp \sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B} \pm \sqrt{\frac{2\alpha}{-\beta}}\left(\frac{A}{B} - 1 + \frac{C}{B}\right) \times \frac{1}{\sqrt{(B-A)/C}\coth(E\xi + c_0) + 1} \pm \sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\left[\sqrt{\frac{B-A}{C}}\coth(E\xi + c_0) + 1\right]\right\}\exp(\mathrm{i}(kx - wt)),$$

其中 $E = \dfrac{C}{B}\sqrt{\dfrac{1 - A/B}{C/B}}$.

情形 3 当 $\frac{C}{B}\left(\frac{A}{B}-1\right)=0$ 时：

(i) 当 $\frac{C}{B}=0,\frac{A}{B}-1\neq 0$ 时，

$$u_{13,14}(x,t)=\left[\mp\sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\pm\sqrt{\frac{2\alpha}{-\beta}}\left(\frac{A}{B}-1+\frac{C}{B}\right)\times\frac{1}{1-\left(\frac{A}{B}-1\right)\xi-c_0}\right]\exp(\mathrm{i}(kx-wt)),$$

$$u_{15,16}(x,t)=\left(\mp\sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\pm\sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\frac{1}{1-(A/B-1)\xi-c_0}\right)\exp(\mathrm{i}(kx-wt)),$$

$$u_{17,18}(x,t)=\left[\mp\sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\pm\sqrt{\frac{2\alpha}{-\beta}}\left(\frac{A}{B}-1+\frac{C}{B}\right)\cdot\frac{1}{1-(A/B-1)X-c_0}\pm\sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\left(1-\left(\frac{A}{B}-1\right)\xi-C_0\right)\right]\exp(\mathrm{i}(kx-wt)).$$

(ii) 当 $\frac{A}{B}-1=0,\frac{C}{B}\neq 0$ 时，

$$u_{19,20}(x,t)=\left[\mp\sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\pm\sqrt{\frac{2\alpha}{-\beta}}\left(\frac{A}{B}-1+\frac{C}{B}\right)\times\frac{(C/B)\xi+c_0}{1+(C/B)\xi+c_0}\right]\exp(\mathrm{i}(kx-wt)),$$

$$u_{21,22}(x,t)=\left(\mp\sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\pm\sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\frac{(C/B)\xi+c_0}{1+(C/B)\xi+c_0}\right)\exp(\mathrm{i}(kx-wt)),$$

$$u_{23,24}(x,t)=\left[\mp\sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\pm\sqrt{\frac{2\alpha}{-\beta}}\left(\frac{A}{B}-1+\frac{C}{B}\right)\cdot\frac{(C/B)\xi+c_0}{1+(C/B)\xi+c_0}\pm\sqrt{\frac{2\alpha}{-\beta}}\frac{C}{B}\left(\frac{(C/B)\xi+c_0}{1+(C/B)\xi+c_0}\right)\right]\exp(\mathrm{i}(kx-wt)),$$

其中 $k,\omega,\alpha,\eta,\beta$ 满足

$$k=\frac{\gamma_2+2\gamma_3-3\gamma_1\alpha_0}{6\gamma_1\gamma_3},\ \omega=\frac{(1-3\gamma_1k)(2k-c-3\gamma_1k^2)}{\gamma_1}+k^2-\gamma_1k^3,$$

$$\alpha=\gamma_1p^2,\ \eta=c+3\gamma_1k^2-2k,\ \beta=\frac{1}{3}\gamma_2+\frac{2}{3}\gamma_3.$$

3 耦合非线性 Schrödinger 方程组的精确解

以光孤子通信为应用背景的非线性 Schrödinger 方程成为当前科学发展的热点问题，本文继续研究了一类以光纤通信为背景的非线性耦合 Schrödinger 方程组：

$$\begin{cases}\mathrm{i}u_{1z}-\alpha\dfrac{\partial^2u_1}{\partial^2x}+u_1u_2\exp(-\mathrm{i}\beta z)=0,\\ \mathrm{i}u_{2z}-\dfrac{\alpha}{2}\dfrac{\partial^2u_2}{\partial^2x}+u_1^2\exp(-\mathrm{i}\beta z)=0,\end{cases}\tag{21}$$

其中 $u_1=u_1(x,z)$ 和 $u_2=u_1(x,z)$ 表示两耦合光纤中的孤子波包，x 表示距离，z 表示时间，α 和 β 是常数.这个方程组描述了在非临界相位匹配下，光纤通信中一些慢变包络波的传播，求得该方程组的精确解将为物理学家从深层次上了解慢变包络波传播提供更多的理论依据，意义重大[30].

首先，做一个变换：

$$u_1(x,z)=u_1(\xi)=g(\xi)\exp(\mathrm{i}\eta_1),\ u_2(x,z)=u_2(\xi)=h(\xi)\exp(\mathrm{i}\eta_2),\tag{22}$$

$$\xi=x+2\alpha k_1 z,\ \eta_i=k_i x+\omega_i z\qquad(i=1,2),\ k_2=2k_1,\ \omega_2=2\omega_1+\beta,\tag{23}$$

其中 $g(\xi)$ 和 $h(\xi)$ 是关于 ξ 的函数，k_1 和 ω_1 是待定常数.把式(22)、(23)代入方程组(21)，将实部与虚部分离，分别令其为 0，得

$$\begin{cases}\alpha g''+(\omega_1-\alpha k_1^2)g-gh=0,\\ \alpha h''+2(2\omega_1+\beta-2\alpha k_1^2)h-2g^2=0.\end{cases}\tag{24}$$

令

$$g(\xi)=\frac{1}{\sqrt{2}}h(\xi),\ \omega_1=-\frac{2}{3}\beta+\alpha k_1^2,\tag{25}$$

则方程组(24)转化为

$$\alpha h''-\frac{2}{3}\beta h-h^2=0.\tag{26}$$

运用齐次平衡法平衡方程(26)中最高阶导数项 h'' 和非线性项 h^2，得 $m+2=2m$，即 $m=2$.根据全新 $G'/(G+G')$ 展开法，可设方程(26)的解为

$$h(\xi)=a_0+a_1\left(\frac{G'}{G+G'}\right)+a_2\left(\frac{G'}{G+G'}\right)^2+a_{-1}\left(\frac{G'}{G+G'}\right)^{-1}+a_{-2}\left(\frac{G'}{G+G'}\right)^{-2},\tag{27}$$

其中 $G=G(\xi)$，满足

$$A(G')^2-BGG''+CG^2=0,\tag{28}$$

$a_0,a_1,a_{-1},a_2,a_{-2}$ 是待定常数.

由式(27)和(28)，可得出 h'' 与 h^2 的表达式，详情见附录 B，将式(27)及 h'' 与 h^2 的表达式代入方程(26)，合并 $(G'/(G+G'))^i(i=-3,-2,-1,0,1,2,3)$ 的相同幂次系数，并令这些系数为 0，得到一系列方程组，详情见附录 B，然后运用 Mathematica 求解这些方程组，得到两组解，分别为

$$①\quad\begin{cases}a_0=36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right],\\ a_1=-12\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right),\ a_2=6\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)^2,\ a_{-1}=0,\ a_{-2}=0;\end{cases}\tag{29}$$

$$②\quad\begin{cases}a_1=0,\ a_2=0,\\ a_0=36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right],\\ a_{-1}=-12\alpha\left(\frac{C}{B}\right)^2,\ a_{-2}=6\alpha\left(\frac{C}{B}\right)^2.\end{cases}\tag{30}$$

分别将式(29)、(30)和(27)代入式(22)，并且利用方程(28)的 3 种情形下的解，与本文第 2 部分 3 种情形相同，得到方程组(21)的 3 种类型精确解，分别为双曲余切函数解、余切函数解和有理函数解.

情形 1 当 $\frac{C}{B}\left(\frac{A}{B}-1\right)>0$ 时,

$$u_1(x,z)=\frac{1}{\sqrt{2}}\left\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-\right.$$
$$12\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)\frac{1}{\sqrt{(A-B)/C}\cot(D\xi+c_0)+1}+$$
$$\left.6\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2\right\}\exp(\mathrm{i}\eta_1),$$

$$u_2(x,z)=\left\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-\right.$$
$$12\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)\frac{1}{\sqrt{(A-B)/C}\cot(D\xi+c_0)+1}+$$
$$\left.6\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2\right\}\exp(\mathrm{i}\eta_2),$$

或

$$u_1(x,z)=\frac{1}{\sqrt{2}}\left\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-\right.$$
$$12\alpha\left(\frac{C}{B}\right)^2\left[\sqrt{\frac{A-B}{C}}\cot(D\xi+c_0)+1\right]+$$
$$\left.6\alpha\left(\frac{C}{B}\right)^2\left[\sqrt{\frac{A-B}{C}}\cot(D\xi+c_0)+1\right]^2\right\}\exp(\mathrm{i}\eta_1),$$

$$u_2(x,z)=\left\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-\right.$$
$$12\alpha\left(\frac{C}{B}\right)^2\left[\sqrt{\frac{A-B}{C}}\cot(D\xi+c_0)+1\right]+$$
$$\left.6\alpha\left(\frac{C}{B}\right)^2\left[\sqrt{\frac{A-B}{C}}\cot(D\xi+c_0)+1\right]^2\right\}\exp(\mathrm{i}\eta_2),$$

其中 $D=\frac{C}{B}\sqrt{\frac{A/B-1}{C/B}}$.

情形 2 当 $\frac{C}{B}\left(\frac{A}{B}-1\right)<0$ 时,

$$u_1(x,t)=\frac{1}{\sqrt{2}}\left\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-\right.$$
$$12\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)\frac{1}{\sqrt{(B-A)C}\coth(E\xi+c_0)+1}+$$
$$\left.6\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left[\frac{1}{\sqrt{(B-A)C}\coth(E\xi+c_0)+1}\right]^2\right\}\exp(\mathrm{i}\eta_1),$$

$$u_2(x,t)=\left\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-\right.$$

$$12\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)\frac{1}{\sqrt{(B-A)/C}\coth(E\xi+c_0)+1}+$$
$$6\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left[\frac{1}{\sqrt{(B-A)C}\coth(E\xi+c_0)+1}\right]^2\Bigg\}\exp(\mathrm{i}\eta_2),$$

或

$$u_1(x,z)=\frac{1}{\sqrt{2}}\Bigg\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-$$
$$12\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)\left[\sqrt{\frac{B-A}{C}}\coth(E\xi+c_0)+1\right]+$$
$$6\alpha\left(\frac{C}{B}\right)^2\left[\sqrt{\frac{B-A}{C}}\coth(E\xi+c_0)+1\right]^2\Bigg\}\exp(\mathrm{i}\eta_1),$$

$$u_2(x,z)=\Bigg\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-$$
$$12\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)\left[\sqrt{\frac{B-A}{C}}\coth(E\xi+c_0)+1\right]+$$
$$6\alpha\left(\frac{C}{B}\right)^2\left[\sqrt{\frac{B-A}{C}}\coth(E\xi+c_0)+1\right]^2\Bigg\}\exp(\mathrm{i}\eta_2),$$

其中 $E=\frac{C}{B}\sqrt{\frac{1-A/B}{C/B}}$.

情形3 当 $\frac{C}{B}\left(\frac{A}{B}-1\right)=0$ 时:

(i) 当 $\frac{C}{B}=0,\frac{A}{B}-1\neq 0$ 时,

$$u_1(x,z)=\frac{1}{\sqrt{2}}\Bigg\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-$$
$$12\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)\frac{1}{1-(A/B-1)\xi-c_0}+$$
$$6\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\frac{1}{1-(A/B-1)\xi-c_0}\Bigg\}\exp(\mathrm{i}\eta_1),$$

$$u_2(x,z)=\Bigg\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-$$
$$12\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)\frac{1}{1-(A/B-1)\xi-c_0}+$$
$$6\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\frac{1}{1-(A/B-1)\xi-c_0}\Bigg\}\exp(\mathrm{i}\eta_2),$$

或

$$u_1(x,z)=\frac{1}{\sqrt{2}}\Bigg\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-$$
$$12\alpha\left(\frac{C}{B}\right)^2\frac{1}{1-(A/B-1)\xi-c_0}+6\alpha\left(\frac{C}{B}\right)^2\left[\frac{1}{1-(A/B-1)\xi-c_0}\right]^2\Bigg\}\exp(\mathrm{i}\eta_1),$$

$$u_2(x,z)=\left\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-\right.$$

$$12\alpha\left(\frac{C}{B}\right)^2\frac{1}{1-(A/B-1)\xi-c_0}+$$

$$\left.6\alpha\left(\frac{C}{B}\right)^2\left[\frac{1}{1-(A/B-1)\xi-c_0}\right]^2\right\}\exp(\mathrm{i}\eta_2).$$

(ii) 当 $\frac{A}{B}-1=0,\frac{C}{B}\neq 0$ 时，

$$u_1(x,z)=\frac{1}{\sqrt{2}}\left\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-\right.$$

$$12\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)\frac{(C/B)\xi+c_0}{1+(C/B)\xi+c_0}+$$

$$\left.6\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{(C/B)\xi+c_0}{1+(C/B)\xi+c_0}\right)^2\right\}\exp(\mathrm{i}\eta_1),$$

$$u_2(x,z)=36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-$$

$$12\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)\frac{(C/B)\xi+c_0}{1+(C/B)\xi+c_0}+$$

$$6\alpha\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{(C/B)\xi+c_0}{1+(C/B)\xi+c_0}\right)^2\exp(\mathrm{i}\eta_1),$$

或

$$u_1(x,z)=\frac{1}{\sqrt{2}}\left\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-\right.$$

$$\left.12\alpha\left(\frac{C}{B}\right)^2\frac{(C/B)x+c_0}{1+(C/B)x+c_0}+6\alpha\left(\frac{C}{B}\right)^2\left(\frac{(C/B)x+c_0}{1+(C/B)x+c_0}\right)^2\right\}\exp(\mathrm{i}\eta_1),$$

$$u_2(x,z)=\left\{36\alpha^2\left[\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{C}{B}\right)^2-2\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{C}{B}\right)^3+\left(\frac{C}{B}\right)^4\right]-\right.$$

$$\left.12\alpha\left(\frac{C}{B}\right)^2\frac{(C/B)x+c_0}{1+(C/B)x+c_0}+6\alpha\left(\frac{C}{B}\right)^2\left(\frac{(C/B)x+c_0}{1+(C/B)x+c_0}\right)^2\right\}\exp(\mathrm{i}\eta_2),$$

其中 $\xi=x+2\alpha k_1 z,\ \eta_i=k_i x+\omega_i z,\qquad i=1,2,\ k_2=2k_1,\ \omega_2=2\omega_1+\beta.$

4 结 论

本文通过全新 $G'/(G+G')$ 展开方法成功求解出广义非线性 Schrödinger 方程和一类耦合非线性 Schrödinger 方程组的精确解，这种新方法求解此类方程直接有效，并求得了方程的多组新形式精确解，扩大了解的范围，而且这种新方法还能推广应用到许多其他非线性偏微分方程的求解中.

附录 A 本文第 2 节相关推导过程

$$\varphi''=2a_1\left(\frac{A}{B}-1+\frac{C}{B}\right)^2\left(\frac{G'}{G+G'}\right)^3-6a_1\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)\left(\frac{G'}{G+G'}\right)^2+$$

$$\left[2a_1\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)+4a_1\left(\frac{C}{B}\right)^2\right]\left(\frac{G'}{G+G'}\right)-2a_1\left(\frac{C}{B}\right)^2-$$

$$2a_{-1}\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)+2a_{-1}\left(\frac{C}{B}\right)^2\left(\frac{G'}{G+G'}\right)^{-3}-6a_{-1}\left(\frac{C}{B}\right)^2\left(\frac{G'}{G+G'}\right)^{-2}+$$

$$\left[4a_{-1}\left(\frac{C}{B}\right)^2+2a_{-1}\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right]\left(\frac{G'}{G+G'}\right)^{-1},$$

$$\varphi^3=a_1^3\left(\frac{G'}{G+G'}\right)^3+3a_1^2a_0\left(\frac{G'}{G+G'}\right)^2+(3a_0^2a_1+3a_1^2a_{-1})\left(\frac{G'}{G+G'}\right)+$$

$$6a_0a_1a_{-1}+a_0^3+a_{-1}^3\left(\frac{G'}{G+G'}\right)^{-3}+3a_{-1}^2a_0\left(\frac{G'}{G+G'}\right)^{-2}+$$

$$(3a_0^2a_{-1}+3a_{-1}^2a_1)\left(\frac{G'}{G+G'}\right)^{-1}.$$

$$\left(\frac{G'}{G+G'}\right)^3:\quad \alpha\cdot 2a_1\left(\frac{A}{B}-1+\frac{C}{B}\right)^2+a_1^3\beta=0,$$

$$\left(\frac{G'}{G+G'}\right)^2:\quad -\alpha\cdot 6a_1\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)+3a_1^2a_0\beta=0,$$

$$\left(\frac{G'}{G+G'}\right)^1:\quad \alpha\cdot\left[2a_1\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)+4a_1\left(\frac{C}{B}\right)^2\right]-\eta a_1+\beta(3a_0^2a_1+3a_1^2a_{-1})=0,$$

$$\left(\frac{G'}{G+G'}\right)^0:\quad \alpha\cdot\left[-2a_1\left(\frac{C}{B}\right)^2-2a_{-1}\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right]-\eta a_0+\beta(6a_0a_1a_{-1}+a_0^3)=0,$$

$$\left(\frac{G'}{G+G'}\right)^{-3}:\quad \alpha\cdot 2a_{-1}\left(\frac{C}{B}\right)^2+\beta a_{-1}^3=0,$$

$$\left(\frac{G'}{G+G'}\right)^{-2}:\quad \alpha\cdot\left[-6a_{-1}\left(\frac{C}{B}\right)^2\right]+\beta\cdot 3a_{-1}^2a_0=0,$$

$$\left(\frac{G'}{G+G'}\right)^{-1}:\quad \alpha\cdot\left[4a_{-1}\left(\frac{C}{B}\right)^2+2a_{-1}\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right]-\eta a_{-1}+\beta(3a_0^2a_{-1}+3a_{-1}^2a_1)=0.$$

附录 B 本文第 3 节相关推导过程

$$h''(\xi)=6a_2\left(\frac{A}{B}-1+\frac{C}{B}\right)2\left(\frac{G'}{G+G'}\right)^4+\left\{-12a_2\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)+\right.$$

$$\left.2\left[a_1\left(\frac{A}{B}-1+\frac{C}{B}\right)-4a_2\frac{C}{B}\right]\left(\frac{A}{B}-1+\frac{C}{B}\right)\right\}\left(\frac{G'}{G+G'}\right)^3+$$

$$\left\{6a_2\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)-4\frac{C}{B}\left[a_1\left(\frac{A}{B}-1+\frac{C}{B}\right)-4a_2\frac{C}{B}\right]+\right.$$

$$\left.2(a_2-a_1)\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right\}\left(\frac{G'}{G+G'}\right)^2+\left\{2\frac{C}{B}\left[a_1\left(\frac{A}{B}-1+\frac{C}{B}\right)-4a_2\frac{C}{B}\right]-\right.$$

$$\left.4(a_2-a_1)\left(\frac{C}{B}\right)^2\right\}\left(\frac{G'}{G+G'}\right)+2(a_2-a_1)\left(\frac{C}{B}\right)^2-$$

$$\left[2a_{-1}\frac{C}{B}-2a_{-2}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right]\left(\frac{A}{B}-1+\frac{C}{B}\right)+$$

$$6a_{-2}\left(\frac{C}{B}\right)^2\left(\frac{G'}{G+G'}\right)^{-4}+\left[-12a_{-2}\left(\frac{C}{B}\right)^2-2(4a_{-2}-a_{-1})\left(\frac{C}{B}\right)^2\right]\left(\frac{G'}{G+G'}\right)^{-3}+$$

$$\left\{6a_{-2}\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)+4(4a_{-2}-a_{-1})\left(\frac{C}{B}\right)^2-\right.$$

$$\left.\frac{C}{B}\left[2a_{-1}\frac{C}{B}-2a_{-2}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right]\right\}\left(\frac{G'}{G+G'}\right)^{-2}+$$

$$\left\{2\frac{C}{B}\left[2a_{-1}\frac{C}{B}-2a_{-2}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right]-2(4a_{-2}-a_{-1})\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right\}\left(\frac{G'}{G+G'}\right)^{-1},$$

$$[h(\xi)]^2=a_2^2\left(\frac{G'}{G+G'}\right)^4+2a_1a_2\left(\frac{G'}{G+G'}\right)^3+(a_1^2+2a_0a_2)\left(\frac{G'}{G+G'}\right)^2+$$
$$(2a_0a_1+2a_2a_{-1})\left(\frac{G'}{G+G'}\right)+a_0^2+2a_1a_{-1}+2a_2a_{-2}+a_{-2}^2\left(\frac{G'}{G+G'}\right)^{-4}+$$
$$2a_{-1}a_{-2}\left(\frac{G'}{G+G'}\right)^{-3}+(a_{-1}^2+2a_0a_2)\left(\frac{G'}{G+G'}\right)^{-2}+(2a_0a_{-1}+2a_1a_{-2})\left(\frac{G'}{G+G'}\right)^{-1},$$

$$\left(\frac{G'}{G+G'}\right)^4:\quad \alpha\cdot 6a_2\left(\frac{A}{B}-1+\frac{C}{B}\right)^2-a_2^2=0,$$

$$\left(\frac{G'}{G+G'}\right)^3:\quad \alpha\cdot\left\{-12a_2\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)+\right.$$
$$\left.2\left[a_1\left(\frac{A}{B}-1+\frac{C}{B}\right)-4a_2\frac{C}{B}\right]\left(\frac{A}{B}-1+\frac{C}{B}\right)\right\}-2a_1a_2=0,$$

$$\left(\frac{G'}{G+G'}\right)^2:\quad \alpha\cdot\left\{6a_2\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)-4\frac{C}{B}\left[a_1\left(\frac{A}{B}-1+\frac{C}{B}\right)-4a_2\frac{C}{B}\right]+\right.$$
$$\left.2(a_2-a_1)\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right\}-\frac{2}{3}\beta a_2-(a_1^2+2a_0a_2)=0,$$

$$\left(\frac{G'}{G+G'}\right)^1:\quad \alpha\cdot\left\{2\frac{C}{B}\left[a_1\left(\frac{A}{B}-1+\frac{C}{B}\right)-4a_2\frac{C}{B}\right]-4(a_2-a_1)\left(\frac{C}{B}\right)^2\right\}-$$
$$\frac{2}{3}\beta a_1-(2a_0a_1+2a_2a_{-1})=0,$$

$$\left(\frac{G'}{G+G'}\right)^0:\quad \alpha\cdot\left\{2(a_2-a_1)\left(\frac{C}{B}\right)^2-\left[2a_{-1}\frac{C}{B}-2a_{-2}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right]\left(\frac{A}{B}-1+\frac{C}{B}\right)\right\}-$$
$$\frac{2}{3}\beta a_0-(a_0^2+2a_1a_{-1}+2a_2a_{-2})=0,$$

$$\left(\frac{G'}{G+G'}\right)^{-4}:\quad \alpha\cdot 6a_{-2}\left(\frac{C}{B}\right)^2-a_{-2}^2=0,$$

$$\left(\frac{G'}{G+G'}\right)^{-3}:\quad \alpha\cdot\left[-12a_{-2}\left(\frac{C}{B}\right)^2-2(4a_{-2}-a_{-1})\left(\frac{C}{B}\right)^2\right]-2a_{-1}a_{-2}=0,$$

$$\left(\frac{G'}{G+G'}\right)^{-2}:\quad \alpha\cdot\left\{6a_{-2}\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)+4(4a_{-2}-a_{-1})\left(\frac{C}{B}\right)^2-\right.$$
$$\left.\frac{C}{B}\left[2a_{-1}\frac{C}{B}-2a_{-2}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right]\right\}-\frac{2}{3}\beta a_{-2}-(a_{-1}^2+2a_0a_2)=0,$$

$$\left(\frac{G'}{G+G'}\right)^{-1}:\quad \alpha\cdot\left\{2\frac{C}{B}\left[2a_{-1}\frac{C}{B}-2a_{-2}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right]-2(4a_{-2}-a_{-1})\frac{C}{B}\left(\frac{A}{B}-1+\frac{C}{B}\right)\right\}-$$
$$\frac{2}{3}\beta a_{-1}-(2a_0a_{-1}+2a_1a_{-2})=0.$$

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Solutions to the Nonlinear Schrödinger Equation and Coupled Nonlinear Schrödinger Equations With a New $G'/(G+G')$-Expansion Method

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Abstract: A new $G'/(G+G')$-expansion method was proposed. Exact solutions to a class of Schrödinger equations and coupled nonlinear Schrödinger equations were obtained with this new method. The solutions can be expressed with the hyperbolic cotangent functions, the cotangent functions and the rational functions. This new $G'/(G+G')$-expansion method not only help gets new exact solutions to the equations directly and effectively, but also expands the scope of the solutions. This new method promises a very wide range of application for the study of related partial differential equations.

Key words: new $G'/(G+G')$-expansion method; generalized nonlinear Schrödinger equation; coupled nonlinear Schrödinger equations; exact solution

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