

基于 S-R 和分解定理的几何非线性问题的 数值计算分析*

宋彦琦, 郝亮钧, 李向上

(中国矿业大学(北京)力学与建筑工程学院,北京 100083)

摘要: 为了探究几何非线性问题的数值求解方法,采用理论推导、MATLAB 编程计算、有限元模拟相结合的方法,基于 S-R 和分解定理及更新拖带坐标描述法,运用插值型无单元 Galerkin 方法对几何非线性问题的增量变分方程进行了推导,并通过四点 Gauss 积分法和不动点迭代法对其进行求解.最后以平面悬臂梁的大变形问题为例进行求解计算,发现与 ANSYS 的计算结果拟合相似度很高,说明了所采用的几何非线性力学理论及数值计算方法的正确性和合理性,为求解几何非线性问题提供了一种新的依据.

关键词: 几何非线性问题; S-R 和分解定理; 更新拖带坐标法;
插值型无单元 Galerkin 法

中图分类号: O241 **文献标志码:** A **doi:** 10.21656/1000-0887.370229

引 言

随着宇航结构最优化设计中的失稳大变形问题、地质力学中的岩层构造大变形过程、以及岩石爆破中的冲击波非线性现象等不断出现,人们不再仅仅局限于线性问题的研究,非线性问题的求解逐渐成为关注的焦点^[1-5].

非线性问题可分为 3 大类,即材料非线性问题、几何非线性问题和边界非线性问题^[6-7].其中,几何非线性问题是指结构在载荷作用过程中产生大的位移和转动.此时,几何方程不再是线性方程,使得其求解的正确性、可靠性、精度及计算效率都存在很大的问题.为此,不仅需要寻求更加合理的非线性大变形理论,而且需要改进和完善非线性大变形数值计算方法.

在文献[8]中,李平提出了解决几何非线性问题的更新拖带坐标法,使得运动描述更加合理,但是其数值求解采用有限元法,无法避免大变形过程中的网格畸变,影响了计算精度.在文献[9-10]中,罗丹等基于更新拖带坐标法,运用无网格 Galerkin 方法,虽然避免了网格重构,但是无法直接施加本质边界条件,降低了计算效率.

笔者基于 S-R 和分解定理,采用更新拖带坐标描述法,导出几何非线性问题的增量变分方程.运用插值型无单元 Galerkin 方法实现增量变分方程的矩阵化,并基于四点 Gauss 积分法和

* 收稿日期: 2016-07-22; 修订日期: 2016-09-01

基金项目: 国家自然科学基金(41430640);深部岩土力学与地下工程国家重点实验室开放基金(SKLGDUK1728)

作者简介: 宋彦琦(1969—),女,教授,博士,博士生导师(E-mail: yanqi_song@sina.com);
郝亮钧(1990—),男,博士生(通讯作者. E-mail: 276592704@qq.com).

不动点迭代法,通过 MATLAB 编程进行求解计算.特别地,对平面应力问题的求解过程进行了详细的推导,为程序实现提供了清晰可靠的依据.最后以弹性、弹塑性状态下平面悬臂梁的大变形问题为例,计算得到悬臂梁的变形位形及端部挠度,发现与 ANSYS 的计算结果拟合度很高,说明了所采用的几何非线性力学理论及数值计算方法的正确性和合理性,为求解几何非线性问题提供了一种新的依据.

1 理论基础

1.1 S-R 和分解定理

S-R 和分解定理证明^[11-12]:给定一个物理可能的位移函数,在变形内单值连续,具有一阶导数,则此运动变换总可以分解为正交和对称两个子变换之和,正交变换体现为转动,而对称变换体现为形变.

S-R 和分解定理给出了准确合理的转动和应变度量,纠正了 Cauchy 应变张量在度量大变形大转动问题时的偏差,弥补了 Green 应变张量无法反映刚体转动现象的缺陷,保证了理论基础的正确性和可靠性,为后面的编程计算提供了可靠的依据.

1.2 更新拖带坐标法

首先理解初始拖带系的含义^[8].如图 1,设质点初始时刻 t_0 位于 P 点, P 处初始拖带基矢为 ${}^0\mathbf{g}_i$,初始拖带坐标为 ${}^0x^i$.在 t 时刻,质点移至 P' 点, P' 处初始拖带基矢为 ${}^t\mathbf{g}_i$,初始拖带坐标为 ${}^t x^i$.

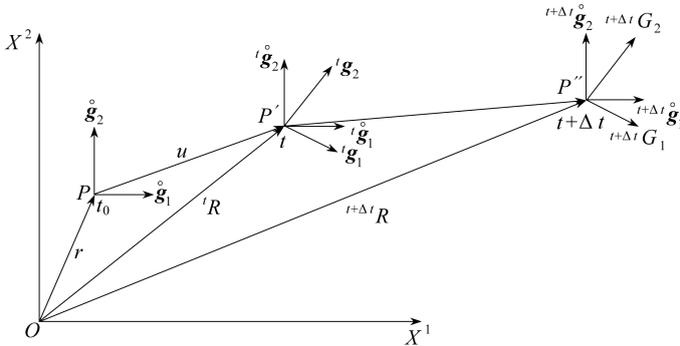


图 1 更新拖带坐标

Fig. 1 The updated co-moving coordinates

在由 t 时刻至 $t + \Delta t$ 时刻的增量步上,以 t 时刻初始系 ${}^t\mathbf{g}_i$ 作为参考基准.即在每一次迭代步结束后,重新选择初始拖带系与直线直角系同胚.不仅保证了应力速率的客观性,自动消除了转动的影响,使应力分量不随时间改变,而且大大简化了增量变分方程求解中的系数矩阵,降低了增量变分方程的求解难度,利于算法实现和推广应用.

2 增量变分方程的推导

由增量方法和势能率变分原理可得

$$\begin{aligned} {}^{t+\Delta t}\delta J = & \int_{\Omega(t+\Delta t)} {}^{t+\Delta t}\sigma_j^i {}^{t+\Delta t}\delta V^j \parallel_i d\Omega - \int_{S_p(t+\Delta t)} {}^{t+\Delta t}P_i {}^{t+\Delta t}\delta V^i dS - \\ & \int_{\Omega(t+\Delta t)} {}^{t+\Delta t}\rho f_i {}^{t+\Delta t}\delta V^i d\Omega = 0, \end{aligned} \quad (1)$$

其中, ${}^{t+\Delta t}V^j \parallel_i$ 表示 $t + \Delta t$ 时刻速度 ${}^{t+\Delta t}V^j$ 关于 $t + \Delta t$ 时刻拖带系协变基矢 ${}^{t+\Delta t}\mathbf{g}_i$ 的协变导数.

根据速度梯度和应力的线性近似逼近有

$${}^{t+\Delta t}V^k \parallel_i = (\delta_j^k - \Delta U^k \parallel_j) \cdot V^j \parallel_i, \quad (2)$$

$${}^{t+\Delta t}\sigma_j^i = {}^t\sigma_j^i + \Delta t \cdot \dot{\sigma}_j^i, \quad (3)$$

其中, $\Delta U^k \parallel_j$ 是位移增量 ΔU^k 在 ${}^t\mathbf{g}_j$ 系下的协变导数, Δt 为时间增量, δ_j^k 为 Kronecker(克罗内克)记号.

将式(2)、(3)代入式(1)可得

$$\Delta t \int \dot{\sigma}_j^i \delta V^j \parallel_i d\Omega - \Delta t \int \sigma_j^i V^j \parallel_k \delta V^k \parallel_i d\Omega = {}^{t+\Delta t}Q - \int \sigma_j^i \delta V^j \parallel_i d\Omega, \quad (4)$$

其中, ${}^{t+\Delta t}Q = \int_{S_0(t+\Delta t)} P_i \delta V^i dS + \int_{\Omega(t+\Delta t)} \rho f_i \delta V^i d\Omega$, $\dot{\sigma}_j^i$ 为应力速率张量.

由拖带系下的物性方程 $\dot{\sigma}_{sj}^i = C_{jl}^{ik} \dot{S}_k^l$, 以及增量步足够小时: $\dot{\sigma}_{aj}^i - \sigma_k^i \dot{S}_j^k + \sigma_j^l \dot{S}_i^l = 0$, 可得

$$\begin{aligned} & \Delta t \int C_{jl}^{ik} \dot{S}_k^l \delta \dot{S}_i^j d\Omega - \Delta t \int \sigma_j^i \dot{S}_k^j \delta \dot{S}_i^k d\Omega - \\ & \Delta t \int (\sigma_k^i \delta \dot{R}_j^k \dot{S}_i^j + \sigma_k^i \dot{R}_j^k \delta \dot{S}_i^j) d\Omega - \Delta t \int \sigma_j^i \dot{R}_k^j \delta \dot{R}_i^k d\Omega = \\ & {}^{t+\Delta t}R - \int \sigma_j^i \delta \dot{S}_i^j d\Omega, \end{aligned} \quad (5)$$

其中, $\dot{\sigma}_{sj}^i, \dot{\sigma}_{aj}^i$ 分别为应力速率张量 $\dot{\sigma}_j^i$ 的对称和反对称分量; \dot{S}_i^j, \dot{R}_i^j 分别为速度梯度 $V^i \parallel_j$ 的对称和反对称分量; C_{jl}^{ik} 为一般形式物性张量, 具体随材料而变.

在更新拖带坐标下, 应力、速度梯度和物性系数转换分别为

$$\bar{\sigma}_j^i = \sigma_l^k \frac{\partial x^l}{\partial x^j} \frac{\partial x^i}{\partial x^k}, \quad (6)$$

$${}^tV^i \parallel_j = {}^t\bar{V}^k \parallel_l \frac{\partial x^i}{\partial x^k} \frac{\partial x^l}{\partial x^j}, \quad (7)$$

$$\bar{C}_{jl}^{ik} = C_{nr}^{ms} \frac{\partial x^n}{\partial x^j} \frac{\partial x^i}{\partial x^m} \frac{\partial x^r}{\partial x^l} \frac{\partial x^k}{\partial x^s}. \quad (8)$$

将式(6)~(8)代入式(5)可得, 更新拖带坐标系下的增量变分方程为

$$\begin{aligned} & \Delta t \int \bar{C}_{jl}^{ik} \dot{S}_k^l \delta \dot{S}_i^j d\Omega - \Delta t \int \bar{\sigma}_j^i \dot{S}_k^j \delta \dot{S}_i^k d\Omega - \\ & \Delta t \int (\bar{\sigma}_k^i \delta \dot{R}_j^k \dot{S}_i^j + \bar{\sigma}_k^i \dot{R}_j^k \delta \dot{S}_i^j) d\Omega - \Delta t \int \bar{\sigma}_j^i \dot{R}_k^j \delta \dot{R}_i^k d\Omega = \\ & {}^{t+\Delta t}R - \int \bar{\sigma}_j^i \delta \dot{S}_i^j d\Omega. \end{aligned} \quad (9)$$

3 增量变分方程的求解

3.1 基于插值型无单元 Galerkin 方法的矩阵化

插值型无单元 Galerkin 方法^[13-15], 能够直接施加本质边界条件, 提高了计算效率. 现推导增量变分方程式(9)在该方法下的矩阵形式.

将速度作如下插值:

$${}^tV^i = \sum_{k=1}^n \Phi_k \cdot {}^tV_k^i, \quad i = 1, 2, 3, \quad (10)$$

其中, ${}^tV_k^i$ 为 t 时刻节点 k 对应于 i 方向的速度, Φ_k 为对应于节点 k 的形函数, n 为单元的节点数.

则速度梯度的插值,由

$$V^i \parallel_j = \frac{\partial V^i}{\partial x^j} + \Gamma_{jl}^i V^l$$

得

$$V^i \parallel_j = \sum_{k=1}^n \frac{\partial \Phi_k}{\partial x^j} V_k^i + \Gamma_{jl}^i \sum_{k=1}^n \Phi_k V_k^l, \quad (11)$$

其中, Γ_{jl}^i 为 Christoffel(克里斯托费尔)符号.

由式(11)可得,速度梯度分量的插值为

$$\begin{cases} 2\dot{S}_j^i = \sum_{k=1}^n \frac{\partial \Phi_k}{\partial x^j} V_k^i + \sum_{k=1}^n \frac{\partial \Phi_k}{\partial x^i} V_k^j + \Gamma_{jl}^i \sum_{k=1}^n \Phi_k V_k^l + \Gamma_{il}^j \sum_{k=1}^n \Phi_k V_k^l, \\ 2\dot{R}_j^i = \sum_{k=1}^n \frac{\partial \Phi_k}{\partial x^j} V_k^i - \sum_{k=1}^n \frac{\partial \Phi_k}{\partial x^i} V_k^j + \Gamma_{jl}^i \sum_{k=1}^n \Phi_k V_k^l - \Gamma_{il}^j \sum_{k=1}^n \Phi_k V_k^l. \end{cases} \quad (12)$$

则增量变分方程(9)的矩阵形式为

$$\mathbf{K}_L - \mathbf{K}_N \cdot \Delta u = {}^{t+\Delta t} \mathbf{Q} - {}^t \mathbf{F}, \quad (13)$$

其中

$$\mathbf{K}_L = \int_{\Omega(t)} \mathbf{B}^{sT} \mathbf{D} \mathbf{B}^s d\Omega,$$

$$\mathbf{K}_N = \int_{\Omega(t)} \mathbf{B}^{sT} \boldsymbol{\sigma}_1 \mathbf{B}^s d\Omega + \int_{\Omega(t)} \mathbf{B}^{rT} \boldsymbol{\sigma}_2 \mathbf{B}^r d\Omega + \int_{\Omega(t)} \mathbf{B}^{sT} \boldsymbol{\sigma}_3 \mathbf{B}^r d\Omega + \int_{\Omega(t)} \mathbf{B}^{rT} \boldsymbol{\sigma}_3^T \mathbf{B}^s d\Omega,$$

$${}^{t+\Delta t} \mathbf{Q} = \int_{s_p} \boldsymbol{\Phi}^T \mathbf{p} dS + \int_{\Omega} \boldsymbol{\Phi}^T \boldsymbol{\rho} f d\Omega,$$

$${}^t \mathbf{F} = \int_{\Omega} \mathbf{B}^{sT} \boldsymbol{\sigma} d\Omega,$$

式中, \mathbf{D} 为物性系数矩阵,

$$\boldsymbol{\sigma} = \{ \sigma_1^1, \sigma_2^2, \sigma_3^3, \sigma_2^1, \sigma_3^2, \sigma_3^1 \}^T,$$

$$\boldsymbol{\sigma}_1 = \begin{bmatrix} \sigma_1^1 & 0 & 0 & \frac{\sigma_2^1}{2} & 0 & \frac{\sigma_3^1}{2} \\ 0 & \sigma_2^2 & 0 & \frac{\sigma_1^2}{2} & \frac{\sigma_3^2}{2} & 0 \\ 0 & 0 & \sigma_3^3 & 0 & \frac{\sigma_2^3}{2} & \frac{\sigma_1^3}{2} \\ \frac{\sigma_1^2}{2} & \frac{\sigma_2^1}{2} & 0 & \frac{1}{4}(\sigma_1^1 + \sigma_2^2) & \frac{\sigma_3^1}{4} & \frac{\sigma_3^2}{4} \\ 0 & \frac{\sigma_2^3}{2} & \frac{\sigma_3^2}{2} & \frac{\sigma_1^3}{4} & \frac{1}{4}(\sigma_2^2 + \sigma_3^3) & \frac{\sigma_2^1}{4} \\ \frac{\sigma_1^3}{2} & 0 & \frac{\sigma_3^1}{2} & \frac{\sigma_2^3}{4} & \frac{\sigma_2^1}{4} & \frac{1}{4}(\sigma_1^1 + \sigma_3^3) \end{bmatrix},$$

$$\boldsymbol{\sigma}_2 = -\frac{1}{4} \begin{bmatrix} \sigma_1^1 + \sigma_2^2 & -\sigma_3^1 & \sigma_3^2 \\ -\sigma_3^1 & \sigma_2^2 + \sigma_3^3 & \sigma_1^2 \\ \sigma_3^2 & \sigma_2^1 & \sigma_1^1 + \sigma_3^3 \end{bmatrix},$$

$$\sigma_3 = \begin{bmatrix} -\frac{1}{2}\sigma_2^1 & 0 & -\frac{1}{2}\sigma_3^1 \\ \frac{1}{2}\sigma_1^2 & -\frac{1}{2}\sigma_3^2 & 0 \\ 0 & \frac{1}{2}\sigma_2^3 & \frac{1}{2}\sigma_1^3 \\ \frac{1}{4}(\sigma_1^1 - \sigma_2^2) & -\frac{1}{4}\sigma_3^1 & -\frac{1}{4}\sigma_3^2 \\ -\frac{1}{4}\sigma_1^3 & \frac{1}{4}(\sigma_2^2 - \sigma_3^3) & \frac{1}{4}\sigma_1^2 \\ -\frac{1}{4}\sigma_2^3 & \frac{1}{4}\sigma_2^1 & \frac{1}{4}(\sigma_1^1 - \sigma_3^3) \end{bmatrix},$$

$$B^s = \{B_1^s, \dots, B_k^s, \dots, B_n^s\},$$

$$B^r = \{B_1^r, \dots, B_k^r, \dots, B_n^r\},$$

$$B_k^s = \begin{bmatrix} \frac{\partial \Phi_k}{\partial x^1} + \Gamma_{11}^1 \Phi_k & \Gamma_{12}^1 \Phi_k & \Gamma_{13}^1 \Phi_k \\ \Gamma_{21}^2 \Phi_k & \frac{\partial \Phi_k}{\partial x^2} + \Gamma_{22}^2 \Phi_k & \Gamma_{23}^2 \Phi_k \\ \Gamma_{13}^3 \Phi_k & \Gamma_{32}^3 \Phi_k & \frac{\partial \Phi_k}{\partial x^3} + \Gamma_{33}^3 \Phi_k \\ \frac{\partial \Phi_k}{\partial x^2} + (\Gamma_{21}^1 + \Gamma_{11}^2) \Phi_k & \frac{\partial \Phi_k}{\partial x^1} + (\Gamma_{22}^1 + \Gamma_{12}^2) \Phi_k & (\Gamma_{23}^1 + \Gamma_{13}^2) \Phi_k \\ (\Gamma_{31}^2 + \Gamma_{21}^3) \Phi_k & \frac{\partial \Phi_k}{\partial x^3} + (\Gamma_{32}^2 + \Gamma_{22}^3) \Phi_k & \frac{\partial \Phi_k}{\partial x^2} + (\Gamma_{33}^2 + \Gamma_{22}^3) \Phi_k \\ \frac{\partial \Phi_k}{\partial x^3} + (\Gamma_{31}^1 + \Gamma_{11}^3) \Phi_k & (\Gamma_{32}^1 + \Gamma_{12}^3) \Phi_k & \frac{\partial \Phi_k}{\partial x^1} + (\Gamma_{33}^1 + \Gamma_{13}^3) \Phi_k \end{bmatrix},$$

$$B_k^r = \begin{bmatrix} \frac{\partial \Phi_k}{\partial x^2} + (\Gamma_{21}^1 - \Gamma_{11}^2) \Phi_k & -\frac{\partial \Phi_k}{\partial x^1} + (\Gamma_{22}^1 - \Gamma_{12}^2) \Phi_k & (\Gamma_{23}^1 - \Gamma_{13}^2) \Phi_k \\ (\Gamma_{31}^2 - \Gamma_{21}^3) \Phi_k & \frac{\partial \Phi_k}{\partial x^3} + (\Gamma_{32}^2 - \Gamma_{22}^3) \Phi_k & -\frac{\partial \Phi_k}{\partial x^2} + (\Gamma_{33}^2 - \Gamma_{22}^3) \Phi_k \\ \frac{\partial \Phi_k}{\partial x^3} + (\Gamma_{31}^1 - \Gamma_{11}^3) \Phi_k & (\Gamma_{32}^1 - \Gamma_{12}^3) \Phi_k & -\frac{\partial \Phi_k}{\partial x^1} + (\Gamma_{33}^1 - \Gamma_{13}^3) \Phi_k \end{bmatrix}.$$

3.2 求解增量变分方程的迭代收敛准则

不动点迭代法是求解非线性方程组最基本的方法:

$$x^{(k+1)} = \Phi(x^{(k)}), \quad k = 0, 1, 2, \dots. \tag{14}$$

为了保证增量变分方程计算求解的收敛性,在每个迭代步末端,进行一次收敛性检测,看迭代解是否收敛到误差容许范围内,从而判定是否中断迭代过程.其中,收敛性检测的两个准则为

① 位移准则

$$\frac{\|\Delta \mathbf{u}^{(i)}\|}{\|{}^{t+\Delta t}\mathbf{u}^{(i)}\|} \leq \varepsilon_D; \quad (15)$$

② 力准则

$$\|{}^{t+\Delta t}\mathbf{Q} - {}^{t+\Delta t}\mathbf{F}^{(i)}\| \leq \varepsilon_F, \quad (16)$$

其中, $\varepsilon_D, \varepsilon_F$ 分别为位移和外载荷的容许误差。

3.3 平面应力问题增量变分方程的求解过程

3.1 小节介绍了求解几何非线性问题增量变分方程的一般方法, 现以平面应力问题为例, 对求解过程进行详细的推导, 为 MATLAB 编程实现提供清晰的依据。

① 物性系数矩阵

对于弹性变形

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad (17)$$

其中, E 为弹性模量, ν 为 Poisson(泊松)比。

对于弹塑性变形

$$\mathbf{D} = \frac{E}{S} \begin{bmatrix} \sigma'_{22}{}^2 + 2P & -\sigma'_{11}\sigma'_{22} + 2\nu P & -\frac{\sigma'_{11} + \nu\sigma'_{22}}{1+\nu}\sigma_{12} \\ -\sigma'_{11}\sigma'_{22} + 2\nu P & \sigma'_{11}{}^2 + 2P & -\frac{\sigma'_{22} + \nu\sigma'_{11}}{1+\nu}\sigma_{12} \\ -\frac{\sigma'_{11} + \nu\sigma'_{22}}{1+\nu}\sigma_{12} & -\frac{\sigma'_{22} + \nu\sigma'_{11}}{1+\nu}\sigma_{12} & \frac{R}{2(1+\nu)} + \frac{2H'}{9E}(1-\nu)\hat{\sigma}^2 \end{bmatrix}, \quad (18)$$

其中, $\sigma'_{11}, \sigma'_{22}$ 为应力偏量, $\hat{\sigma}$ 为等效应力, $H' = EE'/(E - E')$, E' 为切线模量, $P = (2H'/(9E))\hat{\sigma}^2 + \sigma_{12}^2/(1+\nu)$, E 为弹性模量,

$$R = \sigma'_{11}{}^2 + 2\nu\sigma'_{11}\sigma'_{22} + \sigma'_{22}{}^2, S = R + 2(1-\nu^2)P.$$

② 插值矩阵

更新拖带坐标系下, $\Gamma_{jk}^i = \overset{\circ}{\Gamma}_{jk}^i = 0$, 可得

$$\mathbf{B}_k^s = \begin{bmatrix} \frac{\partial \Phi_k}{\partial x^1} & 0 \\ 0 & \frac{\partial \Phi_k}{\partial x^2} \\ \frac{\partial \Phi_k}{\partial x^2} & \frac{\partial \Phi_k}{\partial x^1} \end{bmatrix},$$

$$\mathbf{B}_k^r = \begin{bmatrix} \frac{\partial \Phi_k}{\partial x^2} & -\frac{\partial \Phi_k}{\partial x^2} \end{bmatrix}. \quad (19)$$

③ 应力

$$\boldsymbol{\sigma} = \{\sigma_1^1, \sigma_2^2, \sigma_2^1\}^T,$$

$$\boldsymbol{\sigma}_1 = \begin{bmatrix} \sigma_1^1 & 0 & \frac{\sigma_2^1}{2} \\ 0 & \sigma_2^2 & \frac{\sigma_1^2}{2} \\ \frac{\sigma_1^2}{2} & \frac{\sigma_2^1}{2} & \frac{1}{4}(\sigma_1^1 + \sigma_2^2) \end{bmatrix},$$

$$\boldsymbol{\sigma}_2 = -\frac{1}{4}(\sigma_1^1 + \sigma_2^2), \boldsymbol{\sigma}_3 = \left[-\frac{1}{2}\sigma_1^1, \frac{1}{2}\sigma_2^2, \frac{1}{4}(\sigma_1^1 - \sigma_2^2) \right]^T.$$

因为力产生的应力分量在拖带系下表示,其应力分量不随时间改变,即拖带系下普通应力速率自动消除了转动的影响,代表了应力变化的实质部分,故

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{B}^s \mathbf{u}, \tag{20}$$

其中, $\mathbf{u} = [u_1^1, u_1^2, \dots, u_k^1, u_k^2, \dots, u_n^1, u_n^2]^T$.

④ 系数矩阵的计算

由式(13)可知,增量变分方程系数矩阵的求解,关键在于插值矩阵 \mathbf{B}^s 和 \mathbf{B}^r 的计算,由式(19)可以看出,需对形函数 $\boldsymbol{\Phi}(\mathbf{x})$ 进行求导.

形函数:

$$\boldsymbol{\Phi}^T(\mathbf{x}) = [\Phi_1(\mathbf{x}), \Phi_2(\mathbf{x}), \dots, \Phi_n(\mathbf{x})] = \mathbf{V}^T(\mathbf{x}) + \mathbf{P}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{F}_\omega(\mathbf{x}), \tag{21}$$

其中

$$\mathbf{V}^T(\mathbf{x}) = [v(\mathbf{x}, \mathbf{x}_1), v(\mathbf{x}, \mathbf{x}_2), \dots, v(\mathbf{x}, \mathbf{x}_n)],$$

$$\mathbf{P}^T(\mathbf{x}) = [\tilde{p}_1(\mathbf{x}), \tilde{p}_2(\mathbf{x}), \dots, \tilde{p}_m(\mathbf{x})],$$

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} \tilde{p}_1(\mathbf{x}; \mathbf{x}_1) & \tilde{p}_1(\mathbf{x}; \mathbf{x}_2) & \dots & \tilde{p}_1(\mathbf{x}; \mathbf{x}_n) \\ \vdots & \vdots & & \vdots \\ \tilde{p}_m(\mathbf{x}; \mathbf{x}_1) & \tilde{p}_m(\mathbf{x}; \mathbf{x}_2) & \dots & \tilde{p}_m(\mathbf{x}; \mathbf{x}_n) \end{bmatrix},$$

$$\mathbf{F}_\omega(\mathbf{x}) = (\omega_{kl}(\mathbf{x}))_{m \times n},$$

$$\omega_{kl}(\mathbf{x}) = \begin{cases} \omega(\mathbf{x}, \mathbf{x}_j) \tilde{p}_k(\mathbf{x}; \mathbf{x}_j), & \mathbf{x} \neq \mathbf{x}_j, \\ \sum_{l \in \tau_x, l \neq j} \omega(\mathbf{x}, \mathbf{x}_l) [P_k(\mathbf{x}) - P_k(\mathbf{x}_l)], & \mathbf{x} = \mathbf{x}_j, \end{cases}$$

$$\mathbf{A}(\mathbf{x}) = \mathbf{F}_\omega(\mathbf{x})\mathbf{F}^T(\mathbf{x}) = \mathbf{F}(\mathbf{x})\mathbf{W}(\mathbf{x})\mathbf{F}^T(\mathbf{x}),$$

式中, $v(\mathbf{x}, \mathbf{x}_l) = \frac{\omega(\mathbf{x}, \mathbf{x}_l)}{\sum_{J \in \tau_x} \omega(\mathbf{x}, \mathbf{x}_J)}$ 为插值形函数,

$\tilde{p}_i(\mathbf{x}) = p_i(\mathbf{x}) - S_{P_i(\mathbf{x})} = p_i(\mathbf{x}) - \sum_{l \in \tau_x} v(\mathbf{x}, \mathbf{x}_l) p_i(\mathbf{x}_l)$ 为新基函数,

$\omega(\mathbf{x}, \mathbf{x}_l) = \omega(\mathbf{x} - \mathbf{x}_l) = m_l(\mathbf{x} - \mathbf{x}_l) \left\| \frac{\mathbf{x}_1 - \mathbf{x}_l}{\rho_l} \right\|^{-\alpha}$, $\|\mathbf{x}_1 - \mathbf{x}_l\| \leq \rho_l$ 为权函数.

根据式(21),进一步可求得形函数的导数为

$$\begin{cases} (d\boldsymbol{\Phi}d\mathbf{x}_{(1)})^T = (d\mathbf{V}d\mathbf{x}_{(1)})^T + (d\mathbf{P}d\mathbf{x}_{(1)})^T \mathbf{A}^{-1} \mathbf{F}_\omega + \\ \quad \mathbf{P}^T dn \mathbf{A} d\mathbf{x}_{(1)} \mathbf{F}_\omega + \mathbf{P}^T \mathbf{A}^{-1} d\mathbf{F}_\omega d\mathbf{x}_{(1)}, \\ (d\boldsymbol{\Phi}d\mathbf{x}_{(2)})^T = (d\mathbf{V}d\mathbf{x}_{(2)})^T + (d\mathbf{P}d\mathbf{x}_{(2)})^T \mathbf{A}^{-1} \mathbf{F}_\omega + \\ \quad \mathbf{P}^T dn \mathbf{A} d\mathbf{x}_{(2)} \mathbf{F}_\omega + \mathbf{P}^T \mathbf{A}^{-1} d\mathbf{F}_\omega d\mathbf{x}_{(2)}. \end{cases} \tag{22}$$

综合①、②、③、④,采用 Gauss 二重积分法求解系数矩阵,进一步运用不动点迭代法求解增量变分方程式(13),可得几何非线性问题的数值解。

4 数值算例求解

4.1 弹性状态下悬臂梁的大变形计算分析

如图 2 所示,受均布载荷作用的悬臂梁.已知梁的长度 $L = 8\text{ m}$,高度 $h = 1\text{ m}$,弹性模量 $E = 10^5\text{ Pa}$,Poisson 比 $\nu = 0.25$, 载荷集度 $q = 100\text{ N/m}$.

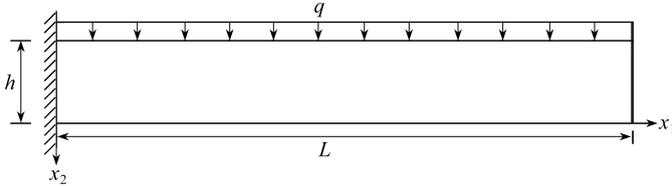


图 2 受均布载荷作用的平面悬臂梁

Fig. 2 A planar cantilever beam subjected to uniformly distributed load

用插值型无网格 Galerkin 法进行数值求解,在悬臂梁区域内均匀布置 21×5 个节点,如图 3 所示,图中同时显示了节点的影响域。

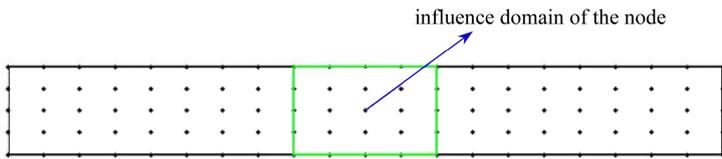
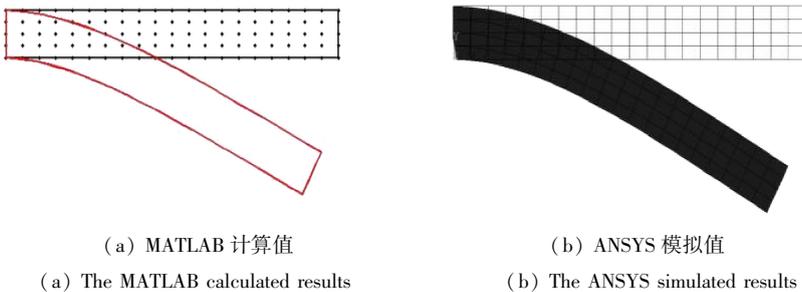


图 3 悬臂梁区域内的节点分布

Fig. 3 The distribution of nodes in the cantilever beam area



(a) MATLAB 计算值

(b) ANSYS 模拟值

(a) The MATLAB calculated results

(b) The ANSYS simulated results

图 4 弹性状态下平面悬臂梁的变形位形图

Fig. 4 The deformation configurations in the elastic state

计算可得弹性状态下平面悬臂梁的变形位形,如图 4(a) 所示;同时用有限元软件 ANSYS 模拟计算可得弹性状态下平面悬臂梁的变形位形,如图 4(b) 所示。

由图 4 可以看出,本文计算得到的悬臂梁变形位形与 ANSYS 模拟得到的变形位形相吻合.具体地,计算可得不同载荷作用下平面悬臂梁端部的挠度以及 ANSYS 的模拟值,见表 1.并将两者结果进行对比,如图 5。

根据表 1 和图 5 可知,弹性状态下平面悬臂梁右端中点处的挠度数值计算结果与 ANSYS 模拟结果基本吻合,两者的总体误差为 7.42%。

表 1 不同载荷集度下,端部挠度的计算值和模拟值

Table 1 The calculated values and ANSYS values of the end deflection under different loads

load $q/(N/m)$	end deflection d/mm	
	calculated value	ANSYS value
0	0	0
10.9	500.4	372.15
20.8	899.9	711.47
30.7	1 240.6	1 045.69
40.6	1 547.2	1 370.97
50.5	1 826.5	1 684.38
60.4	2 080.8	1 930.75
70.3	2 346.9	2 161.16
80.2	2 535.1	2 447.31
90.1	2 739.9	2 659.81
100	2 929.5	2 851.00

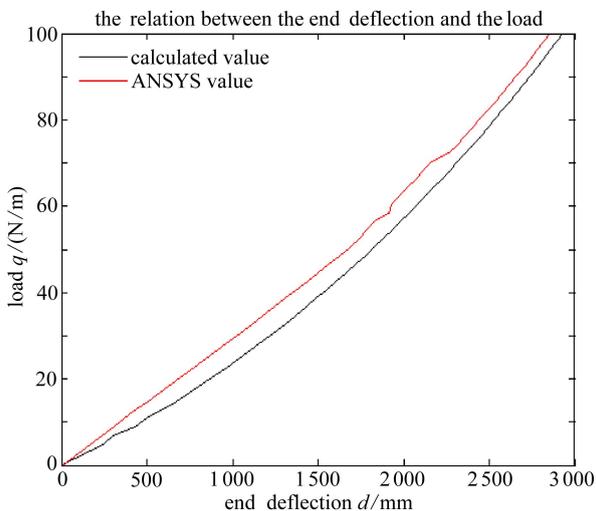
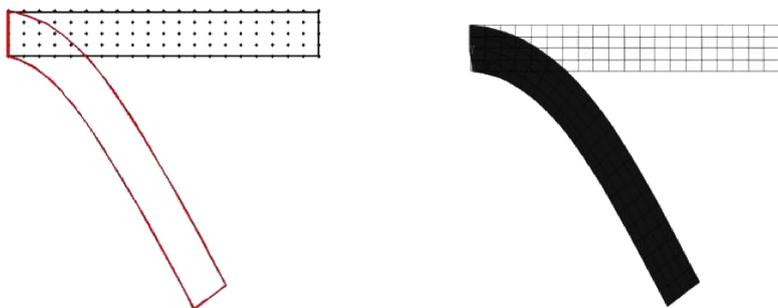


图 5 不同载荷下,端部挠度的计算值和模拟值的对比

Fig. 5 The comparison between the calculated values and the ANSYS values of the end deflection under different loads



(a) MATLAB 计算值

(b) ANSYS 模拟值

(a) The MATLAB calculated results

(b) The ANSYS simulated results

图 6 弹塑性状态下,平面悬臂梁的变形位形图

Fig. 6 The deformation configurations in the elastoplastic state

4.2 弹塑性状态下悬臂梁的大变形计算分析

以图 2 所示的悬臂梁为例,进一步进行弹塑性状态下的变形求解.已知屈服极限 $\sigma_s = 25$ Pa,计算可得弹塑性状态下平面悬臂梁的变形位形,如图 6(a)所示,同时用有限元软件 ANSYS 模拟计算可得弹塑性状态下平面悬臂梁的变形位形,如图 6(b)所示.

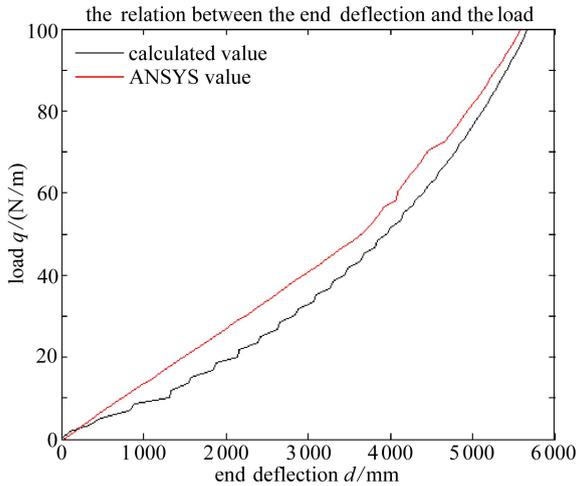


图 7 不同载荷下,端部挠度的计算值和模拟值的对比

Fig. 7 The comparison between the calculated values and the analog values of end deflection under different loads

由图 6 可以看出,本文计算得到的弹塑性状态下悬臂梁的变形位形与 ANSYS 模拟得到的变形位形相吻合.同样地,计算可得不同载荷作用下,平面悬臂梁端部的挠度以及 ANSYS 的模拟值,见表 2.并将两者结果进行对比,如图 7.

表 2 不同载荷集度下,端部挠度的计算值和模拟值

Table 2 The calculated values and the ANSYS values of the end deflection under different loads

load $q/(N/m)$	end deflection d/mm	
	calculated value	ANSYS value
0	0	0
10.15	1 312	1 433.87
20.14	2 145	2 155.8
30.12	2 853.8	2 668.1
40.10	3 456.8	3 163
50.08	3 971.7	3 634
60.07	4 411.2	4 071.3
70.05	4 790.5	4 478.2
80.03	5 123.2	4 992.5
90.12	5 416.6	5 362.6
100	5 669.1	5 690.2

根据表 2 和图 7 所示可知,弹塑性状态下平面悬臂梁端部处的挠度数值计算结果与 ANSYS 模拟结果基本吻合,两者的总体误差为 3.83%.

5 结 论

本文采用更新拖带坐标描述法,将 S-R 和分解定理下的应变度量运用到插值型无单元 Galerkin 方法中,基于平面悬臂梁的大变形数值算例,将 MATLAB 编程计算结果与有限元软件

的模拟结果进行比较,证明了 S-R 和分解定理和插值型无单元 Galerkin 方法在求解几何非线性问题时的合理性和有效性,为求解几何非线性问题提供了一种新的依据.理论上,该方法在求解更为复杂结构的大变形计算时同样有效,但还有待进一步完善和验证.

参考文献 (References):

- [1] 张龙飞, 胡全星. 固体火箭发动机试车架中板簧弹阻力有限元计算分析方法[J]. 火炮发射与控制学报, 2015, **36**(4): 50-54. (ZHANG Long-fei, HU Quan-xing. The finite element method for calculating elastic resistance of plate spring in SRM test frame[J]. *Journal of Gun Launch & Control*, 2015, **36**(4): 50-54. (in Chinese))
- [2] 李明霞, 董联杰. 层状反倾边坡变形特征及影响因素分析[J]. 计算力学学报, 2015, **32**(6): 831-837. (LI Ming-xia, DONG Lian-jie. Analysis on influential factors and deformation characteristics of toppling slope[J]. *Chinese Journal of Computational Mechanics*, 2015, **32**(6): 831-837. (in Chinese))
- [3] 秦勇, 邱爱慈, 张永民. 高聚能重复强脉冲波煤储层增渗新技术试验与探索[J]. 煤炭科学与技术, 2014, **42**(6): 1-7, 70. (QIN Yong, QIU Ai-ci, ZHANG Yong-min. Experiment and discovery on permeability improved technology of coal reservoir based on repeated strong pulse waves of high energy accumulation[J]. *Coal Science and Technology*, 2014, **42**(6): 1-7, 70. (in Chinese))
- [4] 肖舒敏, 闫云聚, 姜波澜. 基于小波神经网络方法的桥梁结构损伤识别研究[J]. 应用数学和力学, 2016, **37**(2): 149-159. (XIAO Shu-min, YAN Yun-ju, JIANG Bo-lan. Damage identification for bridge structures based on the wavelet neural network method[J]. *Applied Mathematics and Mechanics*, 2016, **37**(2): 149-159. (in Chinese))
- [5] 许进升, 杨晓红, 赵磊, 等. 聚合物时温等效模型有限元应用研究[J]. 应用数学和力学, 2015, **36**(5): 539-547. (XU Jin-sheng, YANG Xiao-hong, ZHAO Lei, et al. Finite element application of the time-temperature superposition principle (TTSP) to polymer[J]. *Applied Mathematics and Mechanics*, 2015, **36**(5): 539-547. (in Chinese))
- [6] 宋天霞, 邹时智, 杨文兵. 非线性结构有限元计算[M]. 武汉: 华中理工大学出版社, 1996. (SONG Tian-xia, ZOU Shi-zhi, YANG Wen-bing. *The Finite Element Calculation of Nonlinear Structure*[M]. Wuhan: Huazhong University of Science and Technology Press, 1996. (in Chinese))
- [7] 沈亚鹏, 薛奇. 平面粘弹性大变形问题的研究[J]. 上海交通大学学报, 1990, **24**(5/6): 7-15. (SHEN Ya-peng, XUE Qi. Research of viscoelastic large deformation plane problems[J]. *Journal of Shanghai Jiaotong University*, 1990, **24**(5/6): 7-15. (in Chinese))
- [8] 李平. 非线性大变形有限元分析的更新拖带坐标法及其应用[D]. 博士学位论文. 北京: 中国矿业大学(北京), 1991. (LI Ping. The updated co-moving coordinate formulation for the nonlinear large deformation finite element analysis and application[D]. PhD Thesis. Beijing: China University of Mining and Technology (Beijing), 1991. (in Chinese))
- [9] 罗丹. 基于 S-R 和分解定理的几何非线性问题的无网格 Galerkin 法分析[D]. 硕士学位论文. 长沙: 湖南大学, 2011. (LUO Dan. Based on S-R decomposition theorem analysis element free Galerkin method on geometric nonlinear problems[D]. Master Thesis. Changsha: Hunan University, 2011. (in Chinese))
- [10] 陈芳祖, 罗丹. 基于 S-R 和分解定理的无网格 Galerkin 法求解几何非线性问题[J]. 湖南大学学报(自然科学版), 2012, **39**(1): 42-46. (CHEN Fang-zu, LUO Dan. Element free Galerkin method for geometrically nonlinear problems based on the S-R decomposition theorem[J]. *Journal of Hunan University (Natural Sciences)*, 2012, **39**(1): 42-46. (in Chinese))

- [11] 陈至达. 理性力学[M]. 重庆: 重庆出版社, 1999. (CHEN Zhi-da. *Rational Mechanics* [M]. Chongqing: Chongqing Press, 1999. (in Chinese))
- [12] 陈至达. 有限变形力学基础[M]. 徐州: 中国矿业大学出版社, 2000. (CHEN Zhi-da. *The Foundation of the Finite Deformation Mechanics* [M]. Xuzhou: The University of Mining and Technology Press, 2000. (in Chinese))
- [13] FENG De-shan, GUO Rong-wen, WANG Hong-hua. An element-free Galerkin method for ground penetrating radar numerical simulation[J]. *Journal of Central South University*, 2015, **22**: 261-269.
- [14] JIANG Chen, LIU Gui-rong, HAN Xu, et al. A smoothed finite element method for analysis of anisotropic large deformation of passive rabbit ventricles in diastole[J]. *International Journal for Numerical Methods in Biomedical Engineering*, 2015, **31**(1): 1-25.
- [15] Ju S H, Hsu H H. Solving numerical difficulties for element-free Galerkin analyses[J]. *Computational Mechanics*, 2014, **53**(2): 273-281.

Numerical Analysis of Geometrically Nonlinear Problems Based on the S-R Decomposition Theorem

SONG Yan-qi, HAO Liang-jun, LI Xiang-shang

(School of Mechanics & Civil Engineering, China University of Mining and
Technology(Beijing), Beijing 100083, P.R.China)

Abstract: To explore the numerical solution method for geometrically nonlinear problems, the theoretical derivation, the MATLAB programming and the finite element simulation were used together. Based on the S-R decomposition theorem, the interpolated element-free Galerkin method was applied to deduce the incremental variational equations through the updated co-moving coordinate formulation, which were solved with the 4-point Gauss integration method and the fixed point iteration method. Finally, the large deformations of exemplary elastic and elastoplastic planar cantilever beams were calculated and the results agreed well with those from the ANSYS simulation. The examples illustrate the correctness and rationality of the proposed geometrically nonlinear mechanics theory and the present numerical calculation method. The work provides a new basis for the solutions to geometrically nonlinear problems.

Key words: geometrically nonlinear problem; S-R decomposition theorem; updated co-moving coordinate formulation; interpolated element-free Galerkin method

Foundation item: The National Natural Science Foundation of China (41430640)

引用本文/Cite this paper:

宋彦琦, 郝亮钧, 李向上. 基于 S-R 和分解定理的几何非线性问题的数值计算分析[J]. 应用数学和力学, 2017, **38**(9): 1029-1040.

SONG Yan-qi, HAO Liang-jun, LI Xiang-shang. Numerical analysis of geometrically nonlinear problems based on the S-R decomposition theorem[J]. *Applied Mathematics and Mechanics*, 2017, **38**(9): 1029-1040.