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Navier-Stokes 方程最优控制问题的一种非协调有限元局部稳定化方法*

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摘要: 基于局部 Gauss 积分和梯形外推公式, 速度/压力空间采用最低等阶非协调元 NCP_1-P_1 逼近, 针对非定常 Navier-Stokes 方程最优控制问题,建立了一种全离散的非协调有限元局部稳定化格式.该格式绕开了 inf-sup 条件的束缚,且在每一时间步上,只需要做线性计算,减少了计算量.证明了该格式是无条件稳定的,给出了详细的误差分析.误差结果表明,该线性格式在时间上具有二阶精度.

关 键 词: Navier-Stokes 方程; 最优控制; 稳定化方法; 外推公式
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引 言

粘性不可压缩 Navier-Stokes(简称 N-S)方程是流体运动的基本模型方程,有着极其重要的实际应用背景,其数值模拟一直是计算流体力学热点问题.N-S 方程最优控制问题数值解在海洋工程、天气预报、水利水电、地质勘探和航空航天等诸多领域同样具有重要物理意义,近年来也受到了广泛关注[1].

混合有限元方法是求解 N-S 方程的重要方法,由于 N-S 方程的复杂性决定了其数值模拟存在许多困难,主要体现在两个方面: 1) 要求速度/压力的逼近空间必须满足 inf-sup 稳定性条件,而工程上计算方便的等阶元或低阶线性常数组合却不满足 inf-sup 条件; 2) N-S 方程为非线性抛物方程,强烈的非线性容易引发非物理震荡,因此恰当的离散格式的选择尤为重要.N-S 方程最优控制问题的混合有限元求解同样会遇到这些问题,而且由于控制变量的引入会变得更加困难.

为了克服上述困难,近 30 多年来,许多学者做了大量研究.针对第一方面的困难,为了绕开 inf-sup 条件的束缚,许多稳定化方法相继被提出[2-11],如:最小二乘法、SUPG 方法、RFB 方

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法、涡旋子格粘性法、投影(局部投影)稳定化方法等.最近 Li 和 He[12-14]等基于局部 Gauss 积分,提出了一类新的局部稳定化方法.该类稳定化方法,由于简单、高效且不需要引入稳定化参数而受到关注[12-14].然而,这些研究主要是针对最低等阶协调有限元空间 P_1-P_1 或 Q_1-Q_1.相对于协调元,非协调有限元由于单元自由度较小,满足局部守恒条件,且变量之间的关联仅在相邻边的中点,所形成的方程未知数较少,进而更加利于并行求解.通常的非协调 Crouzeix-Raviart(C-R)元满足 inf-sup 条件,相比之下非协调 NCP_1-P_1 元在计算上,具有更加精确的优点,但是却不满足 inf-sup 条件.因此文献[15-16]分别针对 Stokes 方程和 N-S 方程,速度和压力离散空间采用非协调 NCP_1-P_1 元进行逼近,结合局部 Guass 积分,建立了新的非协调有限元局部稳定化格式,成功地绕开了 inf-sup 条件对非协调 NCP_1-P_1 元的束缚.在文献[15-16]的基础上,本文进一步将非协调有限元 NCP_1-P_1 进行拓展,结合局部 Guass 积分,针对非定常 N-S 方程最优控制问题,给出非协调有限元局部稳定化方法.

目前,有关 N-S 方程最优控制问题的有限元方法的研究工作主要是围绕数值解法和约束条件展开的[17-21],而相对来说其稳定化方法的研究较少.最近 Chen(陈刚)等[22]针对 Oseen 方程最优控制问题,利用 SUPG 方法,给出了一种全离散稳定化方法.针对 N-S 方程最优控制问题的有限元稳定化方法的研究较少,主要有 Braack[23]和 Yilmaz[24].Braack[23]用通常的局部投影稳定化方法,只对定常 N-S 最优控制问题建立了离散格式,而未做稳定性分析和误差估计.Yilmaz[24]则是对非定常的 N-S 最优控制问题只给出了半离散格式.特别地,文献[23-24]中,速度/压力空间都是采用协调有限元进行逼近.本文的目的在于:将非协调 NCP_1-P_1 元,应用于 N-S 方程最优控制问题,研究其非协调有限元稳定化方法.

针对第二方面的困难,许多精确高效的数值格式被给出,如:Crank-Nicolson 格式[25]、Crank-Nicolson 外推格式[26]和 Crank-Nicolson 外推两级格式[27]等.这些格式在时间上达到二阶精度,但这些格式仍是非线性的,在很多情况下仍然失效.本文首次将外推公式用于全离散的非定常 N-S 方程最优控制问题,建立的线性系统在时间上仍然具有二阶精度.

受上面讨论启发,本文结合局部 Guass 积分,对速度/压力空间上采用在计算上表现更为精确的非协调 NCP_1-P_1 元进行逼近,时间上首次采用外推公式进行离散,针对非定常 N-S 方程最优控制问题,建立了一种全离散的非协调有限元局部稳定化格式.该格式成功地绕开了 inf-sup 条件对非协调等阶元的限制,且在每一时间步上,只需要做线性计算,减少了计算量.给出了详细的稳定性分析和误差估计.结果表明:本文所建立的格式是无条件稳定的,且具有二阶精度.

1　预　　备

设 $\Omega \subset \mathbb{R}^2$,$\Omega_U \subset \mathbb{R}^2$是有界的多边形区域,边界分别为 $\Gamma = \partial\Omega$,$\Gamma_U = \partial\Omega_U$.用 $\|\cdot\|_m$ 和 $(\cdot)_m$ 分别表示空间 $H^m(\Omega)$ 的范数和内积,$\|\cdot\|_{m,U}$ 和 $(\cdot)_{m,U}$ 分别表示空间 $H^m(\Omega_U)$ 的范数和内积.特别地,$(\cdot,\cdot) = (\cdot,\cdot)_0$,$(\cdot,\cdot)_U = (\cdot,\cdot)_{0,U}$.

为了简单起见,记 $Y = (H_0^1(\Omega))^2$,$R = L_0^2(\Omega)$,$U = (L^2(\Omega_U))^2$,$V \subset U$ 为有界凸集.

本文考虑如下的最优控制问题:

$$\min_{(\boldsymbol{y},\boldsymbol{u})} J(\boldsymbol{y},\boldsymbol{u}) = \frac{1}{2}\int_Q \left((\boldsymbol{y}(\boldsymbol{x},t) - \boldsymbol{y}_{\mathrm{d}}(\boldsymbol{x},t))^2 + \alpha(\boldsymbol{u}(\boldsymbol{x},t))^2 \right)\mathrm{d}\boldsymbol{x}\mathrm{d}t, \tag{1}$$

其状态方程为非定常 N-S 方程

$$\begin{cases} \boldsymbol{y}_t - \lambda\Delta\boldsymbol{y} + (\boldsymbol{y}\cdot\nabla)\boldsymbol{y} + \nabla r = \boldsymbol{f} + B\boldsymbol{u}, & \text{in } Q, \\ \nabla\cdot\boldsymbol{y} = 0, & \text{in } Q, \\ \boldsymbol{y} = \boldsymbol{0}, & \text{on } [0,T]\times\Gamma, \\ \boldsymbol{y}(\boldsymbol{x},0) = \boldsymbol{y}_0(\boldsymbol{x}), & \text{in } \Omega, \end{cases} \tag{2}$$

其中 $Q=[0,T]\times\Omega, T>0$, α 为正常数, $\boldsymbol{y}_{\mathrm{d}}\in L^2(0,T;(L^2(\Omega))^2)$ 为目标函数, $\boldsymbol{y}(\boldsymbol{x},t)\in\mathbb{R}^d$ 为速度, $r(\boldsymbol{x},t)\in\mathbb{R}$ 为压力, $\boldsymbol{u}\in(L^2(\Omega_U))^2$ 为控制变量, $\boldsymbol{f}(\boldsymbol{x},t)\in\mathbb{R}^d$ 为体力, $\lambda=Re^{-1}$ 为黏性系数, Re 为 Reynolds(雷诺) 数, B 是 $(L^2(\Omega_U))^2$ 到 $(L^2(\Omega))^2$ 的连续线性算子.

仿照文献[24]的方法,引入连续的 Lagrange 函数

$$L(\boldsymbol{y},\boldsymbol{u},\boldsymbol{z}) = J(\boldsymbol{y},\boldsymbol{u}) - \langle \boldsymbol{y}_t - \lambda\Delta\boldsymbol{y} + (\boldsymbol{y}\cdot\nabla)\boldsymbol{y} + \nabla r - \boldsymbol{f} - B\boldsymbol{u}, \boldsymbol{z}\rangle$$

可得到一阶最优条件为

$$\begin{cases} -\boldsymbol{z}_t - \lambda\Delta\boldsymbol{z} - (\boldsymbol{y}\cdot\nabla)\boldsymbol{z} + (\nabla\boldsymbol{y})^{\mathrm{T}}\boldsymbol{z} + \nabla q = \boldsymbol{y} - \boldsymbol{y}_{\mathrm{d}}, & \text{in } Q, \\ \nabla\cdot\boldsymbol{z} = 0, & \text{in } Q, \\ \boldsymbol{z} = \boldsymbol{0}, & \text{on } [0,T]\times\Gamma, \\ \boldsymbol{z}(\boldsymbol{x},T) = \boldsymbol{0}, & \text{in } \Omega, \end{cases} \tag{3}$$

且 $(\boldsymbol{u},\boldsymbol{z})$ 满足

$$(\alpha\boldsymbol{u} + B^*\boldsymbol{z}, \tilde{\boldsymbol{u}} - \boldsymbol{u})_U \geqslant 0, \qquad \forall\, \tilde{\boldsymbol{u}}\in V, \tag{4}$$

其中 B^* 是 B 的伴随算子,且 $\boldsymbol{z}$ 有界.

根据以上记号和一阶最优条件(3)、(4),最优控制问题(1)、(2)的解,可以转化为求解如下最优系统:求 $(\boldsymbol{y},r,\boldsymbol{u})\in Y\times R\times V$, $(\boldsymbol{z},q)\in Y\times R$, 对任意的 $(\boldsymbol{w},\phi,\tilde{\boldsymbol{u}})\in Y\times R\times V$, 使得

$$\begin{cases} (\boldsymbol{y}_{\mathrm{t}},\boldsymbol{w}) + A(\boldsymbol{y},r;\boldsymbol{w},\phi) + b(\boldsymbol{y};\boldsymbol{y},\boldsymbol{w}) = (\boldsymbol{f} + B\boldsymbol{u},\boldsymbol{w}), \\ -(\boldsymbol{z}_{\mathrm{t}},\boldsymbol{w}) + A(\boldsymbol{z},q;\boldsymbol{w},\phi) + b(\boldsymbol{z};\boldsymbol{y},\boldsymbol{w}) + b(\boldsymbol{y};\boldsymbol{z},\boldsymbol{w}) = (\boldsymbol{y} - \boldsymbol{y}_{\mathrm{d}},\boldsymbol{w}), \\ (\alpha\boldsymbol{u} + B^*\boldsymbol{z}, \tilde{\boldsymbol{u}} - \boldsymbol{u})_U \geqslant 0, \\ (\boldsymbol{y}(\boldsymbol{x},0),\boldsymbol{w}) = (\boldsymbol{y}_0,\boldsymbol{w}),\ (\boldsymbol{z}(\boldsymbol{x},T),\boldsymbol{w}) = 0, \end{cases} \tag{5}$$

其中 $b(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{w}\cdot\nabla\boldsymbol{u},\boldsymbol{v})/2 - (\boldsymbol{w}\cdot\nabla\boldsymbol{v},\boldsymbol{u})/2$, 且

$$A(\boldsymbol{y},r;\boldsymbol{w},\phi) = a(\boldsymbol{y},\boldsymbol{w}) + d(\boldsymbol{w},r) - d(\boldsymbol{y},\phi),$$

$$a(\boldsymbol{y},\boldsymbol{w}) = \lambda(\nabla\boldsymbol{y},\nabla\boldsymbol{w}),\ d(\boldsymbol{w},r) = -(\nabla\cdot\boldsymbol{w},r).$$

引理 1 对任意的 $\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}\in Y$, 有

$$\begin{cases} |b(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})| \leqslant C\|\nabla\boldsymbol{w}\|\ \|\nabla\boldsymbol{u}\|_0\|\nabla\boldsymbol{v}\|_0, \\ |b(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})| \leqslant C\|\nabla\boldsymbol{w}\|_0^{1/2}\|\boldsymbol{w}\|_0^{1/2}\|\nabla\boldsymbol{u}\|_0^{1/2}\|\boldsymbol{u}\|_0^{1/2}\|\nabla\boldsymbol{v}\|_0. \end{cases} \tag{6}$$

由 Taylor 公式容易得到

引理 2 令 $t_i = i\Delta t$, $\Delta t = t_{i+1} - t_i$, $i = 0,1,\cdots,N-1, N = T/\Delta t$, 记 $t_{i+1/2} = (t_{i+1} + t_i)/2$. 若 $\psi(\cdot,t)$ 满足 $\psi_t\in L^2(0,T;L^2(\Omega))$, 则存在 $\theta_1\in(0,1)$ 使得

$$\left\|\frac{\psi(\cdot,t_{i+1}) - \psi(\cdot,t_i)}{\Delta t}\right\|_0 \leqslant c\|\psi_t(\cdot,t_{n+\theta_1})\|_0.$$

若 $\psi_{tt}\in L^2(0,T;L^2(\Omega))$, 则存在 $\theta_2,\theta_3\in(0,1)$, 使得

$$\left\|\frac{\psi(\cdot,t_{i+1}) + \psi(\cdot,t_i)}{2} - \psi(\cdot,t_{i+1/2})\right\|_0 \leqslant c(\Delta t)^2\|\psi_{tt}(\cdot,t_{i+\theta_2})\|_0,$$

和

$$\left\|\frac{3}{2}\psi(\cdot,t_{i+1})-\frac{1}{2}\psi(\cdot,t_i)-\psi(\cdot,t_{i+3/2})\right\|_0\leqslant c(\Delta t)^2\|\psi_{tt}(\cdot,t_{i+\theta_3})\|_0.$$

引理 3(Gronwall 不等式)　设 $\{a_n\},\{b_n\},\{c_n\},\{d_n\}$ 是非负实数列,$k,B\geqslant 0$,对任意的整数 $N\geqslant 1$, 如果

$$a_N+k\sum_{n=0}^{N}b_n\leqslant k\sum_{n=0}^{N-1}d_na_n+k\sum_{n=0}^{N}c_n+B,$$

则

$$a_N+k\sum_{n=0}^{N}b_n\leqslant \exp\left(k\sum_{n=0}^{N-1}d_n\right)\left(k\sum_{n=0}^{N}c_n+B\right).$$

2　全离散的非协调有限元方法

假设 $h>0,\boldsymbol{\tau}^h=\{K_j\}$ 和 $\boldsymbol{\tau}_U^h=\{K_{Uj}\}$ 分别是区域 Ω 和 Ω_U 的正则三角形剖分,h_K 和 h_{UK} 分别表示单元 K_j 和 K_{Uj} 的直径,且

$$h=\max_{K_j\in\tau^h}h_K,\ h_U=\max_{K_{Uj}\in\tau_U^h}h_{UK},$$

$$\bar\Omega=\cup\bar K_j,\ \Gamma_j=\partial\Omega\cap\partial K_j,\ \Gamma_{jk}=\partial K_j\cap\partial K_k,$$

ξ_k 和 ξ_{jk} 分别为 Γ_k 和 Γ_{jk} 的中点;$[\boldsymbol{y}]=\boldsymbol{y}|_{\Gamma_{jk}}-\boldsymbol{y}|_{\Gamma_{kj}}$ 表示跳量.

速度和压力的非协调有限元空间定义为

$$Y_h=\{\boldsymbol{y}\in(L^2(\Omega))^2:\boldsymbol{y}|_{K_j}\in(P_1(K_j))^2,\ \boldsymbol{y}(\xi_{jk})=\boldsymbol{y}(\xi_{kj}),\ \boldsymbol{y}(\xi_k)=0,\ \forall K_j,\ K_k\in\boldsymbol{\tau}^h\},$$

$$R_h=P_1=\{r\in H^1(\Omega)\cap R:\ r|_{K_j}\in P_1(K_j),\ \forall\ K_j\in\boldsymbol{\tau}^h\},$$

而

$$U_h=\{\boldsymbol{u}_h\in U\cap(H^1(\Omega_U))^2:\ \boldsymbol{u}_h|_{K_{Uj}}\in P_1(K_{Uj}),\ \forall K_{Uj}\in\boldsymbol{\tau}_U^h\},\ V_h=U_h\cap V.$$

这里 Y_h 是非协调 NCP_1 元, $P_n(K_j)$ 和 $P_n(K_{Uj})$ 分别表示单元 K_j 和 K_{Uj} 上所有次数小于等于 n 的多项式集合.

非协调空间 Y_h 满足下列相容性条件

$$\int_{\Gamma_{jk}}[\boldsymbol{y}]\mathrm{d}s=0,\ \int_{\Gamma_j}\boldsymbol{y}\mathrm{d}s=0$$

和如下逼近性质:对于任意的 $(\boldsymbol{y},r)\in Y\times R$,存在$(\boldsymbol{\Pi}(\boldsymbol{y}),\Pi(r))\in Y_h\times R_h$, 使得

$$\|\boldsymbol{y}-\boldsymbol{\Pi}(\boldsymbol{y})\|_{1,h}+\|r-\Pi(r)\|_0\leqslant Ch(\|\boldsymbol{y}\|_1+\|r\|_0),$$

对于任意的 $(\boldsymbol{y},r)\in((H^2(\Omega))^2\cap Y)\times(H^1(\Omega)\cap R)$,存在$(\boldsymbol{\Pi}(\boldsymbol{y}),\Pi(r))\in Y_h\times R_h$, 使得

$$\|\boldsymbol{y}-\boldsymbol{\Pi}(\boldsymbol{y})\|_0+h(\|\boldsymbol{y}-\boldsymbol{\Pi}(\boldsymbol{y})\|_{1,h}+\|r-\Pi(r)\|_0)\leqslant Ch^2(\|\boldsymbol{y}\|_2+\|r\|_1),$$

其中 $\|\cdot\|_{1,h}=(\sum_j|\cdot|_{1,K_j}^2)^{1/2}$ 为能量范数.

由于非协调元 NCP_1-P_1 不满足离散 inf-sup 稳定性条件,因此引入稳定项 $G_h(\cdot,\cdot)$:

$$G_h(p,q)=\sum_{K_j\in\tau^h}\left\{\int_{K_j,2}pq\mathrm{d}x-\int_{K_j,1}pq\mathrm{d}x\right\},\qquad\forall p,q\in L^2(\Omega),$$

其中 $\int_{K_j,i}g(x)\mathrm{d}x$ 表示单元 K_j 上的 Guass 积分, 且当多项式次数 $i\leqslant 2$ 时, 准确成立,$g(x)=p(x)q(x)$ 的次数小于等于 2.进而,定义 $\boldsymbol{\pi}_h:L^2(\Omega)\to W_0$($W_0$ 表示单元 K_j 上的分片常数集合)

为标准的 L^2 投影,满足如下性质[15-16]:

$$\begin{cases}(\pi_h(r),s)=(r,s), & \forall s\in R,\\ \|\pi_h(r)\|_0\leqslant C\|r\|_0, & \forall r\in R,\\ \|(I-\pi_h)(r)\|_0\leqslant Ch\|r\|_1, & \forall r\in H^1(\Omega)\cap R.\end{cases}\tag{7}$$

记

$$a_h(\boldsymbol{y},\boldsymbol{w})=\lambda\sum_j(\nabla\boldsymbol{y},\nabla\boldsymbol{w})_{K_j},\ d_h(\boldsymbol{y},r)=\sum_j(\nabla\cdot\boldsymbol{y},r)_{K_j},$$

$$G_h(r,s)=\sum_j((I-\pi_h)(r),(I-\pi_h)(s))_{K_j},$$

$$b_h(\boldsymbol{y},\boldsymbol{v},\boldsymbol{w})=\frac{1}{2}\sum_j(((\boldsymbol{y}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})_{K_j}-((\boldsymbol{y}\cdot\nabla)\boldsymbol{w},\boldsymbol{v})_{K_j}),$$

$$A_h(\boldsymbol{y},r;\boldsymbol{w},\phi)=a_h(\boldsymbol{y},\boldsymbol{w})+d_h(\boldsymbol{y},r)-d_h(\boldsymbol{w},\phi)+G_h(r,\phi).$$

引理 4[15-16] 存在与 h, λ, Δt 无关的正常数 β,对任意的 $(\boldsymbol{y}_h, r_h)$, $(\boldsymbol{w}_h, \phi_h)\in Y_h\times R_h$, 使得

$$|A_h(\boldsymbol{y}_h,r_h;\boldsymbol{w}_h,\phi_h)|\leqslant C(\|\boldsymbol{y}_h\|_{1,h}+\|r_h\|_0)(\|\boldsymbol{w}_h\|_{1,h}+\|\phi_h\|_0),$$

$$\beta(\|\boldsymbol{y}_h\|_{1,h}+\|r_h\|_0)\leqslant\sup_{(\boldsymbol{w}_h,\phi_h)\in Y_h\times R_h}\frac{A_h(\boldsymbol{y}_h,r_h;\boldsymbol{w}_h,\phi_h)}{\|\boldsymbol{w}_h\|_{1,h}+\|\phi_h\|_0},$$

$$|G_h(r_h,\phi_h)|\leqslant C\|(I-\pi_h)(r_h)\|_0\|(I-\pi_h)(\phi_h)\|_0.$$

记 $\boldsymbol{y}_h^{n+1}$, $\boldsymbol{z}_h^{n+1}$, r_h^{n+1} 和 q_h^{n+1} 分别是 $\boldsymbol{y}^{n+1}$, $\boldsymbol{z}^{n+1}$, r^{n+1} 和 q^{n+1} 的空间逼近,且

$$g_h^{n+1/2}:=\frac{g_h^n+g_h^{n+1}}{2},\qquad g=\boldsymbol{y},\boldsymbol{z},r,q,\boldsymbol{f},\boldsymbol{u};\ n=0,1,\cdots,N-1.\tag{8}$$

基于以上记号,将局部投影 π_h 应用到该最优控制问题可以得到式(5)的一个全离散的非协调稳定格式为:求 $(\boldsymbol{y}_h^{n+1},r_h^{n+1};\boldsymbol{z}_h^{n+1},q_h^{n+1};\boldsymbol{u}_h^{n+1})\in Y_h\times R_h\times Y_h\times R_h\times V_h$,对任意 $(\boldsymbol{w}_h,\phi_h,\tilde{\boldsymbol{u}}_h)\in Y_h\times R_h\times V_h$, 满足

$$\begin{cases}\displaystyle\sum_j\left(\frac{\boldsymbol{y}_h^{n+1}-\boldsymbol{y}_h^n}{\Delta t},\boldsymbol{w}_h\right)_{K_j}+A_h(\boldsymbol{y}_h^{n+1/2},r_h^{n+1/2};\boldsymbol{w}_h,\phi_h)+b_h(E[\boldsymbol{y}_h^n,\boldsymbol{y}_h^{n-1}];\boldsymbol{y}_h^{n+1/2},\boldsymbol{w}_h)=\\ \qquad(\boldsymbol{f}(\boldsymbol{x},t_{n+1/2})+B\boldsymbol{u}_h^{n+1/2},\boldsymbol{w}_h),\\ \displaystyle-\sum_j\left(\frac{\boldsymbol{z}_h^{n+1}-\boldsymbol{z}_h^n}{\Delta t},\boldsymbol{w}_h\right)_{K_j}+A_h(\boldsymbol{z}_h^{n+1/2},q_h^{n+1/2};\boldsymbol{w}_h,\phi_h)+b_h(\boldsymbol{z}_h^{n+1/2};\boldsymbol{y}_h^{n+1/2},\boldsymbol{w}_h)+\\ \qquad b_h(\boldsymbol{y}_h^{n+1/2};\boldsymbol{z}_h^{n+1/2},\boldsymbol{w}_h)=(\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_{\mathrm{d}}^{n+1/2},\boldsymbol{w}_h),\\ (\alpha\boldsymbol{u}_h^{n+1/2}+B^*\boldsymbol{z}_h^{n+1/2},\tilde{\boldsymbol{u}}_h^{n+1/2}-\boldsymbol{u}_h^{n+1/2})_U\geqslant0,\\ (\boldsymbol{y}_h^0,\boldsymbol{w}_h)=(\boldsymbol{y}_0,\boldsymbol{w}_h),\ (\boldsymbol{z}_h^N,\boldsymbol{w}_h)=0,\end{cases}\tag{9}$$

其中

$$t_{n+1/2}=\frac{t_n+t_{n+1}}{2},\ E[\boldsymbol{y}_h^n,\boldsymbol{y}_h^{n-1}]=3\boldsymbol{y}_h^n/2-\boldsymbol{y}_h^{n-1}/2,\qquad(n=1,2,\cdots,N-1)$$

为外推公式.特别地,当 $n=0$ 时,令 $\boldsymbol{y}_h^{-1}=\boldsymbol{0}$.

注记 1 格式(9)中,时间上采用的 Crank-Nicolson 格式,由于外推公式的引入,将通常非线性项 $b_h(\boldsymbol{y}_h^{n+1/2};\boldsymbol{y}_h^{n+1/2},\boldsymbol{w}_h)$ 改进为线性格式 $b_h(E[\boldsymbol{y}_h^n,\boldsymbol{y}_h^{n-1}];\boldsymbol{y}_h^{n+1/2},\boldsymbol{w}_h)$, 进而减少了计算量,也避免了由于非线性引发的非物理震荡.

3 误差分析

3.1 稳定性

定理 1 格式(9)是稳定的,即对任意的 $h>0,\Delta t>0,n\geqslant 0$, 满足

$$\max_{0\leqslant i\leqslant N-1}\|\boldsymbol{y}_h^{n+1}\|_0^2+\Delta t\sum_{i=0}^{N-1}(\lambda\|\boldsymbol{y}_h^{i+1/2}\|_{1,h}^2+2\|(I-\boldsymbol{\pi}_h)(r_h^{i+1/2})\|_0^2)\leqslant$$
$$\|\boldsymbol{y}_h(0)\|_{0,K_j}^2+\frac{C}{\lambda}\sum_{i=0}^{N-1}(\|\boldsymbol{f}(t_{i+1/2})\|_{-1}^2+\|\boldsymbol{u}_h^{i+1/2}\|_{-1}^2),\tag{10}$$

和

$$\max_{0\leqslant i\leqslant N-1}\|\boldsymbol{z}_h^{n+1}\|_0^2+\Delta t\sum_{i=0}^{N-1}(\lambda\|\boldsymbol{z}_h^{i+1/2}\|_{1,h}^2+2\|(I-\boldsymbol{\pi}_h)(q_h^{i+1/2})\|_0^2)\leqslant$$
$$\frac{C}{\lambda}\sum_{i=0}^{N-1}\|\boldsymbol{y}_h^{i+1/2}-\boldsymbol{y}_{\mathrm{d}}^{i+1/2}\|_{-1}^2.\tag{11}$$

证明 在式(9)的第一个式子中,取 $(\boldsymbol{w}_h,\boldsymbol{\phi}_h)=(\boldsymbol{y}_h^{n+1/2},r_h^{n+1/2})$,由 $b_h(\cdot;\boldsymbol{y},\boldsymbol{y})=0$ 可得

$$\frac{1}{2\Delta t}(\|\boldsymbol{y}_h^{n+1}\|_0^2-\|\boldsymbol{y}_h^{n}\|_0^2)+\lambda\|\boldsymbol{y}_h^{n+1/2}\|_{1,h}^2+\|(I-\boldsymbol{\pi}_h)(r_h^{n+1/2})\|_0^2\leqslant$$
$$\frac{C}{\lambda}(\|\boldsymbol{f}(t_{i+1/2})\|_{-1}^2+\|\boldsymbol{u}_h^{n+1/2}\|_{-1}^2)+\frac{\lambda}{2}\|\boldsymbol{y}_h^{n+1/2}\|_{1,h}^2,$$

将上式两边同时积分,可得式(10).

同理,在式(9)的第二个式子中,取 $(\boldsymbol{w}_h,\boldsymbol{\phi}_h)=(\boldsymbol{z}_h^{n+1/2},q_h^{n+1/2})$,由 $b_h(\cdot;\boldsymbol{z}_h,\boldsymbol{z}_h)=0$ 可得

$$-\frac{1}{2\Delta t}(\|\boldsymbol{z}_h^{n+1}\|_0^2-\|\boldsymbol{z}_h^{n}\|_0^2)+\lambda\|\boldsymbol{z}_h^{n+1/2}\|_{1,h}^2+\|(I-\boldsymbol{\pi}_h)(q_h^{n+1/2})\|_0^2\leqslant$$
$$\frac{C}{\lambda}\|\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_{\mathrm{d}}^{n+1/2}\|_{-1}^2+\frac{\lambda}{4}\|\boldsymbol{z}_h^{n+1/2}\|_{1,h}^2+|b_h(\boldsymbol{z}_h^{n+1/2};\boldsymbol{y}_h^{n+1/2},\boldsymbol{z}_h^{n+1/2})|,\tag{12}$$

用 Young 不等式可得

$$|b_h(\boldsymbol{z}_h^{n+1/2};\boldsymbol{y}_h^{n+1/2},\boldsymbol{z}_h^{n+1/2})|\leqslant$$
$$\|\boldsymbol{z}_h^{n+1/2}\|_0^{1/2}\|\boldsymbol{z}_h^{n+1/2}\|_{1,h}^{3/2}\|\boldsymbol{y}_h^{n+1/2}\|_{1,h}^{1/2}\|\boldsymbol{y}_h^{n+1/2}\|_0^{1/2}\leqslant$$
$$\frac{C}{\lambda}\|\boldsymbol{z}_h^{n+1/2}\|_0^2\|\boldsymbol{y}_h^{n+1/2}\|_{1,h}^2\|\boldsymbol{y}_h^{n+1/2}\|_0^2+\frac{\lambda}{4}\|\boldsymbol{z}_h^{n+1/2}\|_{1,h}^2.\tag{13}$$

结合式(12)和式(13),对其两边同时积分,并根据 $\boldsymbol{z}_h(T)=\boldsymbol{0}$, 可得

$$\max_{0\leqslant i\leqslant N-1}\|\boldsymbol{z}_h^{i+1}\|_0^2+\Delta t\sum_{i=0}^{N-1}(\lambda\|\boldsymbol{z}_h^{i+1/2}\|_{1,h}^2+\|(I-\boldsymbol{\pi}_h)(q_h^{i+1/2})\|_0^2)\leqslant$$
$$\frac{C}{\lambda}\sum_{i=0}^{N-1}(\|\boldsymbol{y}^{i+1/2}-\boldsymbol{y}_{\mathrm{d}}^{i+1/2}\|_{-1}^2+\|\boldsymbol{z}_h^{i+1/2}\|_0^2\|\boldsymbol{y}_h^{i+1/2}\|_{1,h}^2\|\boldsymbol{y}_h^{i+1/2}\|_0^2).\tag{14}$$

根据式(10),可知

$$\frac{C}{\lambda}\sum_{i=0}^{N-1}\|\boldsymbol{y}_h^{i+1/2}\|_{1,h}^2\|\boldsymbol{y}_h^{i+1/2}\|_0^2<\infty,$$

因此结合引理 3,可得式(11).证毕.

3.2 误差估计

为了给出数值解的误差分析,需要引入辅助问题:

$$\begin{cases} \sum_j \left(\dfrac{\boldsymbol{y}_h^{n+1}(\boldsymbol{u}) - \boldsymbol{y}_h^n(\boldsymbol{u})}{\Delta t}, \boldsymbol{w}_h \right)_{K_j} + A_h(\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}), r_h^{n+1/2}(\boldsymbol{u}); \boldsymbol{w}_h, \phi_h) + \\ \quad b_h(E[\boldsymbol{y}_h^n(\boldsymbol{u}), \boldsymbol{y}_h^{n-1}(\boldsymbol{u})]; \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}), \boldsymbol{w}_h) = (\boldsymbol{f}(\boldsymbol{x}, t_{n+1/2}) + B\boldsymbol{u}^{n+1/2}, \boldsymbol{w}_h), \\ -\sum_j \left(\dfrac{\boldsymbol{z}_h^{n+1}(\boldsymbol{u}) - \boldsymbol{z}_h^n(\boldsymbol{u})}{\Delta t}, \boldsymbol{w}_h \right)_{K_j} + A_h(\boldsymbol{z}_h^{n+1/2}(\boldsymbol{u}), q_h^{n+1/2}(\boldsymbol{u}); \boldsymbol{w}_h, \phi_h) + \\ \quad b_h(\boldsymbol{z}_h^{n+1/2}(\boldsymbol{u}); \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}), \boldsymbol{w}_h) + b_h(\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}); \boldsymbol{z}_h^{n+1/2}(\boldsymbol{u}), \boldsymbol{w}_h) = \\ \quad (\boldsymbol{y}_h^{n+1/2} - \boldsymbol{y}_\mathrm{d}^{n+1/2}, \boldsymbol{w}_h), \end{cases} \tag{15}$$

这里,当 $n=0$ 时,$\boldsymbol{y}_h^{-1}(\boldsymbol{u}) = \boldsymbol{0}$.

再引入投影算子 (P_h, Q_h):

定义 1 对任意 $(\boldsymbol{w}, \phi) \in Y \times R$, $(\boldsymbol{w}_h, \phi_h) \in Y_h \times R_h$,令投影算子 $(P_h, Q_h): Y \times R \to Y_h \times R_h$ 满足如下关系:

$$A_h(P_h(\boldsymbol{w}, \phi), Q_h(\boldsymbol{w}, \phi); \boldsymbol{w}_h, \phi_h) = \tilde{A}(\boldsymbol{w}, \phi; \boldsymbol{w}_h, \phi_h),$$

这里 $\tilde{A}(\boldsymbol{w}, \phi; \boldsymbol{w}_h, \phi_h) = A_h(\boldsymbol{w}, \phi; \boldsymbol{w}_h, \phi_h) - G_h(\phi, \phi_h)$.

引理 5[15-16] 投影算子 (P_h, Q_h) 满足如下性质:对任意的 $\boldsymbol{w}, \phi \in Y \times R$, 则

$$\| \boldsymbol{w} - P_h(\boldsymbol{w}, \phi) \|_{1,h} + \| \phi - Q_h(\boldsymbol{w}, \phi) \|_0 \leqslant c(\| \boldsymbol{w} \|_1 + \| \phi \|_0),$$

对任意的 $\boldsymbol{w}, \phi \in ((H^2(\Omega))^2 \cap Y) \times (H^1(\Omega) \cap R)$, 则

$$\| \boldsymbol{w} - P_h(\boldsymbol{w}, \phi) \|_0 + h(\| \boldsymbol{w} - P_h(\boldsymbol{w}, \phi) \|_{1,h} + \| \phi - Q_h(\boldsymbol{w}, \phi) \|_0) \leqslant ch^2(\| \boldsymbol{w} \|_2 + \| \phi \|_1).$$

假设 1 假设 $(\boldsymbol{y}, r; \boldsymbol{z}, q; \boldsymbol{u}) \in Y \times R \times Y \times R \times V$ 是问题(5)的解,且满足如下正则性条件:

$$\begin{aligned} \max(& \| \boldsymbol{y} \|_{L^\infty(0,T,W^{1,\infty}(\Omega))}, \| \boldsymbol{y} \|_{L^\infty(0,T,H^2(\Omega))}, \| \boldsymbol{y}_t \|_{L^2(0,T,H^1(\Omega))}, \| \boldsymbol{y}_{tt} \|_{L^2(0,T,H^1(\Omega))}, \\ & \| \boldsymbol{y}_{ttt} \|_{L^2(0,T,L^2(\Omega))}, \| \boldsymbol{z} \|_{L^\infty(0,T,W^{1,\infty}(\Omega))}, \| \boldsymbol{z} \|_{L^\infty(0,T,H^2(\Omega))}, \| \boldsymbol{z}_t \|_{L^2(0,T,H^1(\Omega))}, \\ & \| \boldsymbol{z}_{tt} \|_{L^2(0,T,H^1(\Omega))}, \| \boldsymbol{z}_{ttt} \|_{L^2(0,T,L^2(\Omega))}, \| r \|_{L^\infty(0,T,H^1(\Omega))}, \| q \|_{L^\infty(0,T,H^1(\Omega))}, \\ & \| \boldsymbol{u} \|_{L^\infty(0,T,H^1(\Omega_U))}) \leqslant C. \end{aligned} \tag{16}$$

引理 6 若 $(\boldsymbol{y}_h(\boldsymbol{u}), r_h(\boldsymbol{u}); \boldsymbol{z}_h(\boldsymbol{u}), q_h(\boldsymbol{u})) \in Y_h \times R_h \times Y_h \times R_h$ 是问题(15)的解,且假设 1 成立,则有估计:

$$\max_{1 \leqslant i \leqslant N} \| \boldsymbol{y}_h^i(\boldsymbol{u}) - \boldsymbol{y}^i \|_0^2 + \Delta t \sum_{i=0}^{N-1} (\| \boldsymbol{y}_h^{i+1/2}(\boldsymbol{u}) - \boldsymbol{y}^{i+1/2} \|_{1,h}^2 + 2 \| (I - \boldsymbol{\pi}_h)(r_h^{i+1/2}(\boldsymbol{u}) - r^{i+1/2}) \|_0^2) \leqslant \frac{C}{\lambda}[h^2 + (\Delta t)^4], \tag{17}$$

和

$$\max_{1 \leqslant i \leqslant N} \| \boldsymbol{z}_h^i(\boldsymbol{u}) - \boldsymbol{z}^i \|_0^2 + \Delta t \sum_{i=0}^{N-1} (\| \boldsymbol{z}_h^{i+1/2}(\boldsymbol{u}) - \boldsymbol{z}^{i+1/2} \|_{1,h}^2 + 2 \| (I - \boldsymbol{\pi}_h)(q_h^{i+1/2}(\boldsymbol{u}) - q^{i+1/2}) \|_0^2) \leqslant \frac{C}{\lambda}[h^2 + (\Delta t)^4]. \tag{18}$$

证明 在最优控制问题(5)中,取 $t = t_{n+1/2}$, 利用 Green 公式,可得对任意的 $(\boldsymbol{w}_h, \phi_h) \in Y_h \times R_h$, 有

$$\begin{aligned} & (\boldsymbol{y}_t(t_{n+1/2}), \boldsymbol{w}_h) + A_h(\boldsymbol{y}(t_{n+1/2}), r(t_{n+1/2}); \boldsymbol{w}_h, \phi_h) + b_h(\boldsymbol{y}(t_{n+1/2}), \boldsymbol{y}(t_{n+1/2}), \boldsymbol{w}_h) = \\ & \quad - r_n^1(\boldsymbol{y}(t_{n+1/2}), \boldsymbol{w}_h) - r_n^2(r(t_{n+1/2}), \boldsymbol{w}_h) + G_h(r(t_{n+1/2}), \phi_h) + \\ & \quad (f(\boldsymbol{x}, t_{n+1/2}) + B\boldsymbol{u}(\boldsymbol{x}, t_{n+1/2}), \boldsymbol{w}_h), \end{aligned} \tag{19}$$

其中

$$r_n^1(\boldsymbol{y}(t_{n+1/2}),\boldsymbol{w}_h)=\lambda\sum_j\left\langle\frac{\partial\boldsymbol{y}(t_{n+1/2})}{\partial n_j},\boldsymbol{w}_h\right\rangle_{\partial K_j},$$

$$r_n^2(r(t_{n+1/2}),\boldsymbol{w}_h)=\sum_j\langle r(t_{n+1/2}),\boldsymbol{w}_h\cdot n_j\rangle_{\partial K_j}.$$

用式(19)减去辅助问题(15)的第一式,可得对任意的 $(\boldsymbol{w}_h,\phi_h)\in Y_h\times R_h$, 有

$$\begin{aligned}&\sum_j\left(\boldsymbol{y}_t(t_{n+1/2})-\frac{\boldsymbol{y}_h^{n+1}(\boldsymbol{u})-\boldsymbol{y}_h^n(\boldsymbol{u})}{\Delta t},\boldsymbol{w}_h\right)_{K_j}+\\&A_h(\boldsymbol{y}(t_{n+1/2})-\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}),r(t_{n+1/2})-r_h^{n+1/2}(\boldsymbol{u});\boldsymbol{w}_h,\phi_h)+\\&b_h(\boldsymbol{y}(t_{n+1/2}),\boldsymbol{y}(t_{n+1/2}),\boldsymbol{w}_h)-b_h(E[\boldsymbol{y}_h^n(\boldsymbol{u}),\boldsymbol{y}_h^{n-1}(\boldsymbol{u})];\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}),\boldsymbol{w}_h)=\\&-r_n^1(\boldsymbol{y}(t_{n+1/2}),\boldsymbol{w}_h)-r_n^2(r(t_{n+1/2}),\boldsymbol{w}_h)+G_h(r(t_{n+1/2}),\phi_h)+\\&(B(\boldsymbol{u}(t_{n+1/2})-\boldsymbol{u}^{n+1/2}),\boldsymbol{w}_h).\end{aligned}\tag{20}$$

在式(20)左右两边同时加入

$$\begin{aligned}&\sum_j\left(\frac{\boldsymbol{y}^{n+1}-\boldsymbol{y}^n}{\Delta t},\boldsymbol{w}_h\right)_{K_j}+a_h(\boldsymbol{y}^{n+1/2},\boldsymbol{w}_h)+d_h(\boldsymbol{y}^{n+1/2},\phi_h)-d_h(\boldsymbol{w}_h,r^{n+1/2})+\\&b_h(\boldsymbol{y}(t_{n+1/2}),E[\boldsymbol{y}^n,\boldsymbol{y}^{n-1}]+E[\boldsymbol{y}_h^n(\boldsymbol{u}),\boldsymbol{y}_h^{n-1}(\boldsymbol{u})];\boldsymbol{y}^{n+1/2},\boldsymbol{w}_h),\end{aligned}$$

可得

$$\begin{aligned}&\sum_j\left(\frac{\boldsymbol{e}_{\boldsymbol{y}}^{n+1}-\boldsymbol{e}_{\boldsymbol{y}}^n}{\Delta t},\boldsymbol{w}_h\right)_{K_j}+a_h(\boldsymbol{e}_{\boldsymbol{y}}^{n+1/2},\boldsymbol{w}_h)+d_h(\boldsymbol{e}_{\boldsymbol{y}}^{n+1/2},\phi_h)-d_h(\boldsymbol{w}_h,e_r^{n+1/2})-\\&G_h(r_h^{n+1/2}(\boldsymbol{u}),\phi_h)=(D_1^{n+1/2},\boldsymbol{w}_h)-b_h(E[\boldsymbol{y}_h^n(\boldsymbol{u}),\boldsymbol{y}_h^{n-1}(\boldsymbol{u})],\boldsymbol{e}_{\boldsymbol{y}}^{n+1/2},\boldsymbol{w}_h)-\\&b_h(E[\boldsymbol{e}_{\boldsymbol{y}}^n,\boldsymbol{e}_{\boldsymbol{y}}^{n-1}],\boldsymbol{y}^{n+1/2},\boldsymbol{w}_h),\end{aligned}\tag{21}$$

其中

$$\begin{aligned}(D_1^{n+1/2},\boldsymbol{w}_h)=&\sum_j\left(\frac{\boldsymbol{y}^{n+1}-\boldsymbol{y}^n}{\Delta t}-\boldsymbol{y}_t(t_{n+1/2}),\boldsymbol{w}_h\right)_{K_j}+a_h(\boldsymbol{y}(t_{n+1/2})-\boldsymbol{y}^{n+1/2},\boldsymbol{w}_h)-\\&d_h(\boldsymbol{w}_h,r(t_{n+1/2})-r^{n+1/2})-r_n^1(\boldsymbol{y}(t_{n+1/2}),\boldsymbol{w}_h)-r_n^2(r(t_{n+1/2}),\boldsymbol{w}_h)+\\&(B(\boldsymbol{u}(t_{n+1/2})-\boldsymbol{u}^{n+1/2}),\boldsymbol{w}_h)+b_h(\boldsymbol{y}(t_{n+1/2});\boldsymbol{y}^{n+1/2}-\boldsymbol{y}(t_{n+1/2}),\boldsymbol{w}_h)+\\&b_h(E[\boldsymbol{y}^n,\boldsymbol{y}^{n-1}]-\boldsymbol{y}(t_{n+1/2});\boldsymbol{y}^{n+1/2},\boldsymbol{w}_h).\end{aligned}$$

引入记号:

$$\boldsymbol{e}_{\boldsymbol{y}}^n:=\boldsymbol{y}^n-\boldsymbol{y}_h^n(\boldsymbol{u})=(\boldsymbol{y}^n-P_h(\boldsymbol{y}^n,r^n))+(P_h(\boldsymbol{y}^n,r^n)-\boldsymbol{y}_h^n(\boldsymbol{u}))=\boldsymbol{\eta}^n+\boldsymbol{\xi}_h^n,$$

$$e_r^n:=r^n-r_h^n(\boldsymbol{u})=(r^n-Q_h(\boldsymbol{y}^n,r^n))+(Q_h(\boldsymbol{y}^n,r^n)-r_h^n(\boldsymbol{u}))=\theta^n+\gamma_h^n,$$

$$\boldsymbol{e}_{\boldsymbol{y}}^{n+1/2}:=\frac{\boldsymbol{e}_{\boldsymbol{y}}^n+\boldsymbol{e}_{\boldsymbol{y}}^{n+1}}{2}=\boldsymbol{\xi}_h^{n+1/2}+\boldsymbol{\eta}^{n+1/2},$$

$$e_r^{n+1/2}:=\frac{e_r^n+e_r^{n+1}}{2}=\gamma_h^{n+1/2}+\theta^{n+1/2}.$$

利用定义 1,容易知道 $a_h(\boldsymbol{\eta}^{n+1/2},\boldsymbol{w}_h)+d_h(\boldsymbol{\eta}^{n+1/2},\phi_h)-d_h(\boldsymbol{w}_h,\theta^{n+1/2})=G_h(Q_h(\boldsymbol{w}_h,\phi),\phi_h)$, 于是结合以上记号有

$$\begin{aligned}&\sum_j\left(\frac{\boldsymbol{\xi}_h^{n+1}-\boldsymbol{\xi}^n}{\Delta t},\boldsymbol{w}_h\right)_{K_j}+A_h(\boldsymbol{\xi}_h^{n+1/2},\gamma_h^{n+1/2};\boldsymbol{w}_h,\phi_h)=\\&\quad-\sum_j\left(\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n}{\Delta t},\boldsymbol{w}_h\right)_{K_j}-b_h(E[\boldsymbol{y}_h^n(\boldsymbol{u}),\boldsymbol{y}_h^{n-1}(\boldsymbol{u})],\boldsymbol{e}_{\boldsymbol{y}}^{n+1/2},\boldsymbol{w}_h)-\end{aligned}$$

$$b_h(E[\boldsymbol{e}_y^n,\boldsymbol{e}_y^{n-1}],\boldsymbol{y}^{n+1/2},\boldsymbol{w}_h)+(D_1^{n+1/2},\boldsymbol{w}_h). \tag{22}$$

在式(22)中取 $(\boldsymbol{w}_h,\phi_h)=(\boldsymbol{\xi}_h^{n+1/2},\gamma_h^{n+1/2})$，可得

$$\frac{1}{2\Delta t}(\|\boldsymbol{\xi}_h^{n+1}\|_0^2-\|\boldsymbol{\xi}_h^n\|_0^2)+\lambda\|\boldsymbol{\xi}_h^{n+1/2}\|_{1,h}^2+\|(I-\pi_h)(\gamma_h^{n+1/2})\|_0^2\leqslant$$
$$\left|-\sum_j\left(\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n}{\Delta t},\boldsymbol{\xi}_h^{n+1/2}\right)_{K_j}\right|+|-b_h(E[\boldsymbol{y}_h^n(\boldsymbol{u}),\boldsymbol{y}_h^{n-1}(\boldsymbol{u})],\boldsymbol{e}_y^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})-$$
$$b_h(E[\boldsymbol{e}_y^n,\boldsymbol{e}_y^{n-1}],\boldsymbol{y}^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|+|(D_1^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|=$$
$$I_1+I_2+I_3. \tag{23}$$

用 Cauchy-Schwarz 不等式和 Young 不等式可得

$$I_1\leqslant\frac{C}{\lambda}\left\|\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n}{\Delta t}\right\|_{-1}^2+\frac{\lambda}{\epsilon_1}\|\boldsymbol{\xi}_h^{n+1/2}\|_{1,h}^2. \tag{24}$$

对于 I_2，由三角不等式，Young 不等式，逆不等式和 $b_h(\cdot;\boldsymbol{\xi}_h^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})=0$ 可得

$$I_2\leqslant|b_h(E[\boldsymbol{y}_h^n(\boldsymbol{u}),\boldsymbol{y}_h^{n-1}(\boldsymbol{u})],\boldsymbol{\eta}^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|+$$
$$|b_h(E[\boldsymbol{\xi}_h^n,\boldsymbol{\xi}_h^{n-1}],\boldsymbol{y}^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|+|b_h(E[\boldsymbol{\eta}^n,\boldsymbol{\eta}^{n-1}],\boldsymbol{y}^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|\leqslant$$
$$|b_h(E[\boldsymbol{y}^n,\boldsymbol{y}^{n-1}],\boldsymbol{\eta}^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|+|b_h(E[\boldsymbol{\eta}^n,\boldsymbol{\eta}^{n-1}],\boldsymbol{\eta}^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|+$$
$$|b_h(E[\boldsymbol{\xi}_h^n,\boldsymbol{\xi}_h^{n-1}],\boldsymbol{\eta}^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|+|b_h(E[\boldsymbol{\xi}_h^n,\boldsymbol{\xi}_h^{n-1}],\boldsymbol{y}^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|+$$
$$|b_h(E[\boldsymbol{\eta}^n,\boldsymbol{\eta}^{n-1}],\boldsymbol{y}^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|\leqslant$$
$$\frac{5\lambda}{\epsilon_1}\|\boldsymbol{\xi}_h^{n+1/2}\|_{1,h}^2+\frac{C}{\lambda}\|\boldsymbol{\eta}^{n+1/2}\|_{1,h}^2[1+\|\boldsymbol{\eta}^n\|_{1,h}^2+\|\boldsymbol{\eta}^{n-1}\|_{1,h}^2+$$
$$h^{-1}(\|\boldsymbol{\xi}_n^h\|_0^2+\|\boldsymbol{\xi}_{n-1}^h\|_0^2)]+$$
$$\frac{C}{\lambda}(\|\boldsymbol{\eta}^n\|_{1,h}^2+\|\boldsymbol{\eta}^{n-1}\|_{1,h}^2+\|\boldsymbol{\xi}_n^h\|_0^2+\|\boldsymbol{\xi}_{n-1}^h\|_0^2). \tag{25}$$

现在考虑 $I_3=|(D_1^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|$，由 Young 不等式，引理 2 和假设 1，可得

$$\begin{cases}\left|\left(\dfrac{\boldsymbol{y}^{n+1}-\boldsymbol{y}^n}{\Delta t}-\boldsymbol{y}_t(t_{n+1/2}),\boldsymbol{\xi}_h^{n+1/2}\right)\right|\leqslant\\ \qquad\dfrac{\lambda}{\epsilon_1}\|\boldsymbol{\xi}_h^{n+1/2}\|_{1,h}^2+\dfrac{C(\Delta t)^4}{\lambda}\|\boldsymbol{y}_{ttt}(t_{n+\theta_1})\|_0^2,\\ |a_h(\boldsymbol{y}(t_{n+1/2})-\boldsymbol{y}^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})-d_h(\boldsymbol{\xi}_h^{n+1/2},r(t_{n+1/2})-r^{n+1/2})|\leqslant\\ \qquad\dfrac{\lambda}{\epsilon_1}\|\boldsymbol{\xi}_h^{n+1/2}\|_{1,h}^2+C(\Delta t)^4(\lambda\|\boldsymbol{y}_{tt}(t_{n+\theta_2})\|_{1,h}^2+\lambda^{-1}\|r_{tt}(t_{n+\theta_3})\|_0^2),\\ |-r_n^1(\boldsymbol{y}(t_{n+1/2}),\boldsymbol{\xi}_h^{n+1/2})-r_n^2(r(t_{n+1/2}),\boldsymbol{\xi}_h^{n+1/2})|\leqslant\\ \qquad\dfrac{2\lambda}{\epsilon_1}\|\boldsymbol{\xi}_h^{n+1/2}\|_{1,h}^2+\dfrac{Ch^2}{\lambda}(\|\boldsymbol{y}(t_{n+1/2})\|_2^2+\|r(t_{n+1/2})\|_1^2),\\ |(B(\boldsymbol{u}(t_{n+1/2})-\boldsymbol{u}^{n+1/2}),\boldsymbol{\xi}_h^{n+1/2})|\leqslant\\ \qquad\|\boldsymbol{\xi}_h^{n+1/2}\|_0^2+C(\Delta t)^4\|\boldsymbol{u}_{tt}(t_{n+1/2})\|_{0,U}^2,\end{cases} \tag{26}$$

和

$$|b_h(\boldsymbol{y}(t_{n+1/2});\boldsymbol{y}^{n+1/2}-\boldsymbol{y}(t_{n+1/2}),\boldsymbol{\xi}_h^{n+1/2})+$$
$$b_h(E[\boldsymbol{y}^n,\boldsymbol{y}^{n-1}]-\boldsymbol{y}(t_{n+1/2});\boldsymbol{y}^{n+1/2},\boldsymbol{\xi}_h^{n+1/2})|\leqslant$$

$$\frac{2\lambda}{\epsilon_1}\|\boldsymbol{\xi}_h^{n+1/2}\|_{1,h}^2 + C\lambda(\Delta t)^4(\|\boldsymbol{y}_{tt}(t_{n+\theta_4})\|_{1,h}^2 + \|\boldsymbol{y}_{tt}(t_{n-1+\theta_5})\|_{1,h}^2), \tag{27}$$

其中

$$\theta_1,\theta_2,\theta_3,\theta_4,\theta_5 \in (0,1).$$

综合式(19)~(27),取 $\epsilon_1 = 24$,从 1 到 n 相加,并乘以 $2\Delta t$, 再由假设 1,可得

$$\|\boldsymbol{\xi}_h^{n+1}\|_0^2 + \lambda\|\boldsymbol{\xi}_h^{n+1/2}\|_{1,h}^2 + 2\|(I-\boldsymbol{\pi}_h)(\gamma_h^{n+1/2})\|_0^2 \leqslant$$
$$\|\boldsymbol{\xi}_h^1\|_0^2 + \frac{C}{\lambda}[h^2 + (\Delta t)^4] + C\left[1 + \frac{(1+h)\Delta t}{\lambda}\right]\sum_{i=1}^{n}\|\boldsymbol{\xi}_h^i\|_0^2. \tag{28}$$

当 $n=0$ 时,取 $\boldsymbol{y}_h^{-1}(\boldsymbol{u})=\boldsymbol{0}$, 上述式(19)~(27)的证明中只有三线性项的估计有所不同.事实上,注意到只需要将式(25)的最后一行估计中取 $\boldsymbol{\eta}^{-1} = \boldsymbol{e}_y^{-1} = \boldsymbol{0}$, 即可.因此有

$$\|\boldsymbol{\xi}_h^1\|_0^2 + \lambda\|\boldsymbol{\xi}_h^{1/2}\|_{1,h}^2 + 2\|(I-\boldsymbol{\pi}_h)(\gamma_h^{1/2})\|_0^2 \leqslant$$
$$\frac{C}{\lambda}[h^2 + (\Delta t)^4] + C\left[1 + \frac{(1+h)\Delta t}{\lambda}\right]\|\boldsymbol{\xi}_h^1\|_0^2. \tag{29}$$

最后综合式(28)和(29),用三角不等式和引理 3 可得式(17).同理可以证明得到式(18).证毕.

引理 7　若 $(\boldsymbol{y}_h(\boldsymbol{u}), r_h(\boldsymbol{u}); \boldsymbol{z}_h(\boldsymbol{u}), q_h(\boldsymbol{u})) \in Y_h \times R_h \times Y_h \times R_h$ 是问题(15)的解,而$(\boldsymbol{y}_h, r_h; \boldsymbol{z}_h, q_h; \boldsymbol{u}_h) \in Y_h \times R_h \times Y_h \times R_h \times V_h$ 是问题(9)的解,可以得到如下估计:

$$\max_{1\leqslant i\leqslant N}\|\boldsymbol{y}_h^i - \boldsymbol{y}_h^i(\boldsymbol{u})\|_0^2 + \Delta t\sum_{i=0}^{N-1}(\|\boldsymbol{y}_h^{i+1/2} - \boldsymbol{y}_h^{i+1/2}(\boldsymbol{u})\|_{1,h}^2 +$$
$$2\|(I-\boldsymbol{\pi}_h)(r_h^{i+1/2} - r_h^{i+1/2}(\boldsymbol{u}))\|_0^2) \leqslant \frac{C}{\lambda}\Delta t\sum_{i=0}^{N-1}\|\boldsymbol{u}_h^{i+1/2} - \boldsymbol{u}^{i+1/2}\|_{0,U}^2, \tag{30}$$

和

$$\max_{1\leqslant i\leqslant N}\|\boldsymbol{z}_h^i - \boldsymbol{z}_h^i(\boldsymbol{u})\|_0^2 + \Delta t\sum_{i=0}^{N-1}(\|\boldsymbol{z}_h^{i+1/2} - \boldsymbol{z}_h^{i+1/2}(\boldsymbol{u})\|_{1,h}^2 +$$
$$2\|(I-\boldsymbol{\pi}_h)(q_h^{i+1/2} - q_h^{i+1/2}(\boldsymbol{u}))\|_0^2) \leqslant \frac{C}{\lambda}\Delta t\sum_{i=0}^{N-1}\|\boldsymbol{u}_h^{i+1/2} - \boldsymbol{u}^{i+1/2}\|_{0,U}^2. \tag{31}$$

证明　用式(9)中第一式减去式(15)中第一式,可得,

$$\sum_j\left(\frac{(\boldsymbol{y}_h^{n+1} - \boldsymbol{y}_h^{n+1}(\boldsymbol{u})) - (\boldsymbol{y}_h^n - \boldsymbol{y}_h^n(\boldsymbol{u}))}{\Delta t}, \boldsymbol{w}_h\right)_{K_j} +$$
$$A_h(\boldsymbol{y}_h^{n+1/2} - \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}), r_h^{n+1/2} - r_h^{n+1/2}(\boldsymbol{u}); \boldsymbol{w}_h, \phi_h) =$$
$$-b_h(\boldsymbol{y}_h^{n+1/2}; \boldsymbol{y}_h^{n+1/2}, \boldsymbol{w}_h) + b_h(\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}); \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}), \boldsymbol{w}_h) -$$
$$(B(\boldsymbol{u}^{n+1/2} - \boldsymbol{u}_h^{n+1/2}), \boldsymbol{w}_h). \tag{32}$$

在式(32)中,取 $(\boldsymbol{w}_h, \phi_h) = (\boldsymbol{y}_h^{n+1/2} - \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}), r_h^{n+1/2} - r_h^{n+1/2}(\boldsymbol{u}))$,根据 $b_h(\boldsymbol{w}; \boldsymbol{v}, \boldsymbol{v}) = 0$, 可得

$$\frac{1}{2\Delta t}(\|\boldsymbol{y}_h^{n+1} - \boldsymbol{y}_h^{n+1}(\boldsymbol{u})\|_0^2 - \|\boldsymbol{y}_h^n - \boldsymbol{y}_h^n(\boldsymbol{u})\|_0^2) + \lambda\|\boldsymbol{y}_h^{n+1/2} - \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u})\|_{1,h}^2 +$$
$$\|(I-\boldsymbol{\pi}_h)(r_h^{n+1/2} - r_h^{n+1/2}(\boldsymbol{u}))\|_0^2 \leqslant$$
$$|b_h(\boldsymbol{y}_h^{n+1/2}; \boldsymbol{y}_h^{n+1/2}, \boldsymbol{y}_h^{n+1/2} - \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u})) -$$
$$b_h(\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}); \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}), \boldsymbol{y}_h^{n+1/2} - \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}))| +$$
$$|(B(\boldsymbol{u}^{n+1/2} - \boldsymbol{u}_h^{n+1/2}), \boldsymbol{y}_h^{n+1/2} - \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}))| \leqslant$$
$$|b_h(\boldsymbol{y}_h^{n+1/2} - \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}); \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}), \boldsymbol{y}_h^{n+1/2} - \boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}))| +$$

$$|(B(\boldsymbol{u}^{n+1/2}-\boldsymbol{u}_h^{n+1/2}),\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}))|. \tag{33}$$

利用引理 1,Young 不等式和 $\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u})=\boldsymbol{y}^{n+1/2}-\boldsymbol{e}_y^{n+1/2}$, 可得

$$\begin{aligned}&|b_h(\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u});\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}),\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u}))|\leqslant\\&C\|\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u})\|_{1,h}^{3/2}\|\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u})\|_0^{1/2}(\|\boldsymbol{y}^{n+1/2}\|_{1,h}^{1/2}\|\boldsymbol{y}^{n+1/2}\|_0^{1/2}+\\&\|\boldsymbol{e}_y^{n+1/2}\|_{1,h}^{1/2}\|\boldsymbol{e}_y^{n+1/2}\|_0^{1/2})\leqslant\\&\frac{\lambda}{2}\|\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u})\|_{1,h}^2+\frac{C}{\lambda}(\|\boldsymbol{y}^{n+1/2}\|_{1,h}^{1/2}\|\boldsymbol{y}^{n+1/2}\|_0^{1/2}+\\&\|\boldsymbol{e}_y^{n+1/2}\|_{1,h}^{1/2}\|\boldsymbol{e}_y^{n+1/2}\|_0^{1/2})\|\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u})\|_0^2.\end{aligned} \tag{34}$$

结合式(33)和式(34)可得

$$\begin{aligned}&\frac{1}{2\Delta t}(\|\boldsymbol{y}_h^{n+1}-\boldsymbol{y}_h^{n+1}(\boldsymbol{u})\|_0^2-\|\boldsymbol{y}_h^{n}-\boldsymbol{y}_h^{n}(\boldsymbol{u})\|_0^2)+\lambda\|\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u})\|_{1,h}^2+\\&\|(I-\pi_h)(r_h^{n+1/2}-r_h^{n+1/2}(\boldsymbol{u}))\|_0^2\leqslant\\&\frac{\lambda}{2}\|\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u})\|_{1,h}^2+\|\boldsymbol{u}^{n+1/2}-\boldsymbol{u}_h^{n+1/2}\|_{0,U}^2+\\&\frac{C}{\lambda}(1+\|\boldsymbol{y}^{n+1/2}\|_{1,h}^{1/2}\|\boldsymbol{y}^{n+1/2}\|_0^{1/2}+\\&\|\boldsymbol{e}_y^{n+1/2}\|_{1,h}^{1/2}\|\boldsymbol{e}_y^{n+1/2}\|_0^{1/2})\|\boldsymbol{y}_h^{n+1/2}-\boldsymbol{y}_h^{n+1/2}(\boldsymbol{u})\|_0^2.\end{aligned} \tag{35}$$

根据引理 6 可知

$$\frac{1}{\lambda}\sum_{i=1}^{n}(1+\|\boldsymbol{y}^{n+1/2}\|_{1,h}^{1/2}\|\boldsymbol{y}^{n+1/2}\|_0^{1/2}+\|\boldsymbol{e}_y^{n+1/2}\|_{1,h}^{1/2}\|\boldsymbol{e}_y^{n+1/2}\|_0^{1/2})$$

有界,因此将式(35)两边同时求和,并用引理 3,可得式(30).同理可得式(31).证毕.

引理 8[24] 设 $(\boldsymbol{u},\boldsymbol{y})$ 和 $(\boldsymbol{u}_h,\boldsymbol{y}_h)$ 分别是问题(5)和问题(9)的解,则有

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,U}\leqslant\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|_{0,U}^2+\frac{1}{\alpha}\|z(\boldsymbol{u})-z_h(\boldsymbol{u})\|_{0,U}^2,$$

其中 $\hat{\boldsymbol{u}}$ 是 $\boldsymbol{u}$ 在空间 U 的最优插值.

定理 2 若 $(\boldsymbol{y}_h,r_h;z_h,q_h;\boldsymbol{u}_h)\in Y_h\times R_h\times Y_h\times R_h\times V_h$ 是问题(9)的解,且假设 1 成立,则有估计:

$$\begin{aligned}&\max_{0\leqslant n\leqslant N-1}\|\boldsymbol{y}^{n+1}-\boldsymbol{y}_h^{n+1}\|_0^2+\Delta t\sum_{n=0}^{N-1}(\lambda\|\boldsymbol{y}^{n+1/2}-\boldsymbol{y}_h^{n+1/2}\|_{1,h}^2+\\&2\|(I-\pi_h)(r^{n+1/2}-r_h^{n+1/2})\|_0^2)\leqslant\frac{C}{\lambda}[h^2+h_U{}^2+(\Delta t)^4],\end{aligned} \tag{36}$$

和

$$\begin{aligned}&\max_{0\leqslant n\leqslant N-1}\|z^{n+1}-z_h^{n+1}\|_0^2+\Delta t\sum_{n=0}^{N-1}(\lambda\|z^{n+1/2}-z_h^{n+1/2}\|_{1,h}^2+\\&2\|(I-\pi_h)(q^{n+1/2}-q_h^{n+1/2})\|_0^2)\leqslant\frac{C}{\lambda}[h^2+h_U{}^2+(\Delta t)^4].\end{aligned} \tag{37}$$

证明 该定理的证明由引理 6、引理 7、引理 8 容易得到.证毕.

4 结论

针对非定常 N-S 方程最优控制问题,研究了非协调有限元稳定化方法.为了绕开 inf-sup 条件对离散解的束缚,本文基于局部 Guass 积分,研究了一种新的局部投影稳定化方法.为了回

避非线性项的计算,通过外推公式,建立了一种线性的全离散格式.通过详细的推导,给出了稳定性分析和误差估计.误差结果表明,该线性格式具有二阶精度.

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A Local Stabilized Nonconforming Finite Element Method for the Optimal Control of Navier-Stokes Equations

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Abstract: For the optimal control of Navier-Stokes equations, a new local stabilized nonconforming finite element method was proposed. The time-dependent problem was fully discretized with lowest-equal-order nonconforming finite element NCP_1-P_1 in the velocity and pressure spaces and the reduced Crank-Nicolson scheme in the time domain. The scheme was stable for the equal-order combination of discrete velocity and pressure spaces through the addition of a local L^2 projection term. Specially, based on an extrapolation formula, the method requires only the solution of one linear system per time step. Stability of the method was proved. For the state, adjoint state and control variables, the a priori error estimates were obtained. The error estimation results show that the method has 2nd-order accuracy.

Key words: Navier-Stokes equation; optimal control; stabilized method; extrapolation formula

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