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梯度材料平板弯拉耦合力学的精确化支配方程*

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摘要: 基于非均匀材料弹性力学理论,采用算子谱分解和算子代数方法,对梯度材料平板结构弯曲与拉伸问题进行了研究.首次给出了指数梯度材料平板弯曲与拉伸的力学方程.研究结果表明:与各向同性平板结构弯曲和拉伸问题不同,在功能梯度平板中描述弯曲应力状态与描述拉伸应力状态的广义位移函数以及剪切函数都是耦合的.没有采用工程假设,推导得到的梯度材料平板结构力学方程是精确化的.通过分析可以认识和理解,分别对应于弯曲状态与拉伸状态的应力场耦合机理以及力学响应的构成等.给出的方程及其分析过程可望能够用于类平板形式热防护材料结构的应力分析与强度设计,推进热防护材料结构的轻型化.

关 键 词: 梯度板控制方程; 算子谱分解; Vieta 定理; 弯曲与拉伸耦合; 耦合机理与响应构成

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引 言

功能梯度材料的力学和热物理参数等沿壁厚方向是连续渐变的,从而满足工程结构不同部位对材料力/热性能的设计要求.与传统的各向同性材料相比,功能梯度材料结构的分析与设计中存在着更多的挑战性课题.众所周知,控制方程是实现结构力学分析计算的基础,它直接关系到材料结构力学响应的分析与计算结果[1-5].同时,由于功能梯度材料性能的非均匀性,难以直接采用那些适用于各向同性材料的结构力学模型化方法, 迫切要求探索更为有效的分析与求解方法.

功能梯度材料结构力学控制方程是目前人们研究的热点问题.Benachour 等[6]采用 Hamilton 原理给出了 4 变量平板精化理论.Woodward 和 Kashtalyan[7]提出了简支横观各向同性功能梯度板承受横向荷载作用下的三维弹性理论解,此时弹性模量和剪切模量沿厚度方向以指数函数形式变化,而 Poisson(泊松)比为常数.Thai 等[8]研究了功能梯度弹性地基板自由振动的

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精化剪切变形理论.可以看到,现存的对功能梯度平板的模型化方法主要还是基于一些经典假设进行的,其获得的控制方程具有较强的局限性.Zenkour 基于简单的剪切和法向变形理论[9],采用平板厚度和表面剪切应力自由的条件给出了 FGM 平板弯曲响应理论.

文献[10]曾基于弹性力学与算子谱分解方法,给出了各向同性平板弯曲和拉伸振动的精确化方程.本文将基于非均匀材料弹性力学理论,采用算子谱分解方法,对梯度材料平板弯拉耦合问题进行研究.给出指数梯度材料平板弯拉耦合力学方程,并对平板弯曲拉伸耦合机理与响应构成作分析.

1　梯度材料平板弯曲与拉压控制方程

根据非均匀材料三维弹性力学理论,无体力时固体材料内位移场的支配方程为[11]

$$\begin{cases}\mu(z)\,\nabla_0^2 u+\dfrac{\mu(z)}{1-2\nu}\dfrac{\partial}{\partial x}\theta+\mu'(z)\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right)=0,\\ \mu(z)\,\nabla_0^2 v+\dfrac{\mu(z)}{1-2\nu}\dfrac{\partial}{\partial y}\theta+\mu'(z)\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right)=0,\\ \mu(z)\,\nabla_0^2 w+\dfrac{\mu(z)}{1-2\nu}\dfrac{\partial}{\partial z}\theta+\dfrac{2\nu}{1-2\nu}\mu'(z)\theta+2\mu'(z)\dfrac{\partial w}{\partial z}=0,\end{cases}\tag{1}$$

μ,ν,ρ 分别为材料的 Lamé 常数、Poisson 比和材料密度,它们都是坐标 z 的函数;t 为时间;$\nabla_0=\boldsymbol{e}_1\partial/\partial x+\boldsymbol{e}_2\partial/\partial y+\boldsymbol{e}_3\partial/\partial z$ 为三维空间 Hamilton 算子,$\boldsymbol{e}_k(k=1,2,3)$ 是直角坐标系的单位矢量,相应的 Laplace 算子为 $\nabla_0^2=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2=\nabla^2+\partial^2/\partial z^2$;$\theta$ 是材料体积应变,$\theta=\partial u/\partial x+\partial v/\partial y+\partial w/\partial z$.

研究物理参数典型分布情况,设材料参数分布为 $\mu(z)=\mu_0\mathrm{e}^{\beta z}$,$\rho(z)=\rho_0\mathrm{e}^{\beta z}$,$\nu=\mathrm{const}$.于是,方程(1)可改写为

$$\begin{cases}\left(\nabla^2+\dfrac{\partial^2}{\partial z^2}\right)u+\dfrac{1}{1-2\nu}\dfrac{\partial}{\partial x}\theta+\beta\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right)=0,\\ \left(\nabla^2+\dfrac{\partial^2}{\partial z^2}\right)v+\dfrac{1}{1-2\nu}\dfrac{\partial}{\partial y}\theta+\beta\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right)=0,\\ \left(\nabla^2+\dfrac{\partial^2}{\partial z^2}\right)w+\dfrac{1}{1-2\nu}\dfrac{\partial}{\partial z}\theta+\dfrac{2\nu}{1-2\nu}\beta\theta+2\beta\dfrac{\partial w}{\partial z}=0.\end{cases}\tag{2}$$

对于均匀材料而言,相对于几何中面的反对称变形只对应于内力矩,而对称变形对应于内力.对于功能梯度非均匀材料,采用物理中面的概念,将平板结构内应力-应变分解为相对于物理中面的弯曲应力状态和沿横截面的拉伸应力状态[12].平板受力时物理中面的定义为面内无弯曲变形.

在平板几何中面上建立直角坐标系,采用数字 $k=1,2,3$ 分别表示坐标 x,y,z.设功能梯度板的物理中面位于 $z=z_0$ 处,根据 Taylor 级数展开式,弹性平板内任意一点的位移可描述为

$$u_k(x,y,z,t)=\exp\left((z-z_0)\frac{\partial}{\partial z}\right)u_k(x,y,z_0,t)\qquad(k=1,2,3).\tag{3}$$

对于薄壁平板结构其物理中面位置坐标可写为

$$z_0=\frac{\displaystyle\int_{-h/2}^{h/2}zE_{\mathrm{M}}(z)\,\mathrm{d}z}{\displaystyle\int_{-h/2}^{h/2}E_{\mathrm{M}}(z)\,\mathrm{d}z}=\frac{1}{2}h\left[\coth\left(\frac{\beta h}{2}\right)-\frac{1}{\beta h}\right],\tag{4}$$

式中，$E_{\mathrm{M}}(z)$ 是平板材料的弹性模量；h 是平板的厚度.研究指数梯度材料情况，当功能梯度材料的物理参数为 $E_{\mathrm{M}}(z)=E_0\exp(\beta z)$，中性面的坐标是

$$z_0=\frac{1}{2}h\left[\coth\left(\frac{1}{2}\beta h\right)-\frac{2}{\beta h}\right]=\frac{1}{2}\gamma h .$$

当材料梯度指数很小时，即有 $\beta h/2\to 0$ 时，物理中面位置为 $z_0\to 0$.E_0，β 分别为材料在 $z=0$ 处的弹性模量和材料的梯度指数.与基于结构几何中面的功能梯度平板理论相比，物理中面内本构关系中没有弯曲与拉伸变形的耦合影响.

在弹性结构内，由式(1)和式(2)可得如下式子[10-11]：

$$\boldsymbol{\Phi}(x,y,z)=\exp\left((z-z_0)\frac{\partial}{\partial z}\right)\boldsymbol{\Phi}(x,y,z_0)=$$
$$\sum_{j=1}^{4}\exp((z-z_0)D_{1j})\varphi^j(x,y)=2\mathrm{Re}\left[\sum_{j=1}^{2}\exp((z-z_0)D_{1j})\varphi^j(x,y)\right], \tag{5a}$$

$$\boldsymbol{\Psi}(x,y,z)=\exp\left((z-z_0)\frac{\partial}{\partial z}\right)\boldsymbol{\Psi}(x,y,z_0)=$$
$$\sum_{j=1}^{2}\exp((z-z_0)D_{2j})\psi^j(x,y)=2\mathrm{Re}[\exp((z-z_0)D_{21})\psi^1(x,y)], \tag{5b}$$

式中

$$\boldsymbol{\Phi}=\sum_{j=1}^{4}\boldsymbol{\Phi}^j;\ \boldsymbol{\Psi}=\sum_{j=1}^{2}\boldsymbol{\Psi}^j;$$
$$\left(\frac{\partial}{\partial z}-D_{1j}\right)\boldsymbol{\Phi}^j=0,\ \left(\frac{\partial}{\partial z}-D_{2k}\right)\boldsymbol{\Psi}^k=0\qquad(j=1,2,3,4;k=1,2);$$

$D_i(i=1,2)$ 是微分算子 $\partial/\partial z$ 的谱，分别满足如下算子代数方程：

$$D_1^4\varphi-2\beta D_1^3\varphi+(2\nabla^2+\beta^2)D_1^2\varphi-2\beta\nabla^2D_1\varphi+\left(\nabla^2-\frac{\nu}{1-\nu}\beta^2\right)\nabla^2\varphi=0, \tag{6a}$$
$$(D_2^2+\beta D_2+\nabla^2)\psi=0. \tag{6b}$$

根据高等代数中 Vieta 定理[13]，方程(6a)的算子谱有如下关系：

$$D_{11}+D_{12}+D_{13}+D_{14}=2\beta,$$
$$D_{11}D_{12}+D_{11}D_{13}+D_{11}D_{14}+D_{12}D_{13}+D_{12}D_{14}+D_{13}D_{14}=(2\nabla^2+\beta^2),$$
$$D_{11}D_{12}D_{13}+D_{11}D_{12}D_{14}+D_{11}D_{13}D_{14}+D_{12}D_{13}D_{14}=2\beta\nabla^2,$$
$$D_{11}D_{12}D_{13}D_{14}=\left(\nabla^2-\frac{\nu}{1-\nu}\beta^2\right)\nabla^2,$$
$$D_{21}+D_{22}=2\mathrm{Re}(D_{21})=-\beta,\ |D_{21}|^2=\nabla^2.$$

梯度材料平板结构内位移的表达式为[11]

$$u_x(x,y,z)=\frac{1}{2\mu_{\mathrm{M}}(z)}\left(\nu\nabla_0^2-\frac{\partial^2}{\partial z^2}\right)\frac{\partial}{\partial x}\boldsymbol{\Phi}+\frac{\partial}{\partial y}\boldsymbol{\Psi}=$$
$$\frac{1}{\mu_{\mathrm{M}}(z)}\frac{\partial}{\partial x}\mathrm{Re}\left\{\sum_{j=1}^{2}[\nu\nabla^2-(1-\nu)D_{1j}^2]\exp((z-z_0)D_{1j})\varphi^j\right\}+$$
$$2\frac{\partial}{\partial y}\mathrm{Re}\{\exp((z-z_0)D_{21})\psi^1\}, \tag{7a}$$

$$u_y(x,y,z)=\frac{1}{2\mu_{\mathrm{M}}(z)}\left(\nu\nabla_0^2-\frac{\partial^2}{\partial z^2}\right)\frac{\partial}{\partial y}\boldsymbol{\Phi}-\frac{\partial}{\partial x}\boldsymbol{\Psi}=$$

$$\frac{1}{\mu_{\mathrm{M}}(z)}\frac{\partial}{\partial y}\mathrm{Re}\left\{\sum_{j=1}^{2}[\nu\nabla^2-(1-\nu)D_{1j}^2]\exp((z-z_0)D_{1j})\varphi^j\right\}-$$

$$2\frac{\partial}{\partial x}\mathrm{Re}\{\exp((z-z_0)D_{2j})\psi^1\}, \tag{7b}$$

$$u_z(x,y,z)=\frac{1}{\mu_{\mathrm{M}}(z)}\left(\nabla_0^2-\frac{\partial^2}{\partial z^2}\right)\frac{\partial}{\partial z}\Phi-\frac{\partial}{\partial z}\left[\frac{1}{2\mu_{\mathrm{M}}(z)}\left(\nu\nabla_0^2-\frac{\partial^2}{\partial z^2}\right)\Phi\right]=$$

$$\frac{1}{\mu_{\mathrm{M}}(z)}\mathrm{Re}\left\{\sum_{j=1}^{2}[(2-\nu)\nabla^2+(1-\nu)D_{1j}^2]D_{1j}\exp((z-z_0)D_{1j})\varphi^j\right\}+$$

$$\frac{\beta}{\mu_{\mathrm{M}}(z)}\mathrm{Re}\left\{\sum_{j=1}^{2}[\nu\nabla^2-(1-\nu)D_{1j}^2]\exp((z-z_0)D_{1j})\varphi^j\right\}, \tag{7c}$$

其中，$\nabla_0^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\nabla^2+\frac{\partial^2}{\partial z^2}$ 为 Laplace 算子；μ_{M} 为材料的 Lamé 常数，$\mu_{\mathrm{M}}=E_{\mathrm{M}}/(2(1+\nu))$；$\nu$ 为材料的 Poisson 比，假设为常数．

梯度板结构内物理中面的位移和法线转角以及横向正应变的表达式为[10]

$$U_1=u_1|_{z=z_0}=\frac{1}{\mu_{\mathrm{M}}(z_0)}\frac{\partial}{\partial x}\mathrm{Re}\left\{\sum_{j=1}^{2}[\nu\nabla^2-(1-\nu)D_{1j}^2]\varphi^j\right\}+2\frac{\partial}{\partial y}\mathrm{Re}(\psi^1), \tag{8a}$$

$$U_2=u_2|_{z=z_0}=\frac{1}{\mu_{\mathrm{M}}(z_0)}\frac{\partial}{\partial y}\mathrm{Re}\left\{\sum_{j=1}^{2}[\nu\nabla^2-(1-\nu)D_{1j}^2]\varphi^j\right\}-2\frac{\partial}{\partial x}\mathrm{Re}(\psi^1), \tag{8b}$$

$$W=u_z|_{z=z_0}=\frac{1}{\mu_{\mathrm{M}}(z_0)}\mathrm{Re}\left\{\sum_{j=1}^{2}[(2-\nu)\nabla^2+(1-\nu)D_{1j}^2]D_{1j}\varphi^j\right\}+$$

$$\frac{\beta}{\mu_{\mathrm{M}}(z_0)}\mathrm{Re}\left\{\sum_{j=1}^{2}[\nu\nabla^2-(1-\nu)D_{1j}^2]\varphi^j\right\}, \tag{8c}$$

$$\phi_1=-\left.\frac{\partial u_1}{\partial z}\right|_{z=z_0}=-\frac{1}{\mu_{\mathrm{M}}(z_0)}\frac{\partial}{\partial x}\mathrm{Re}\left\{\sum_{j=1}^{2}[\nu\nabla^2-(1-\nu)D_{1j}^2]D_{1j}\varphi^j\right\}+$$

$$\frac{\beta}{\mu_{\mathrm{M}}(z_0)}\frac{\partial}{\partial x}\mathrm{Re}\left\{\sum_{j=1}^{2}[\nu\nabla^2-(1-\nu)D_{1j}^2]\varphi^j\right\}-2\frac{\partial}{\partial y}\mathrm{Re}(D_{21}\psi^1), \tag{8d}$$

$$\phi_2=-\left.\frac{\partial u_2}{\partial z}\right|_{z=z_0}=-\frac{1}{\mu_{\mathrm{M}}(z_0)}\frac{\partial}{\partial y}\mathrm{Re}\left\{\sum_{j=1}^{2}[\nu\nabla^2-(1-\nu)D_{1j}^2]D_{1j}\varphi^j\right\}+$$

$$\frac{\beta}{\mu_{\mathrm{M}}(z_0)}\frac{\partial}{\partial y}\mathrm{Re}\left\{\sum_{j=1}^{2}[\nu\nabla^2-(1-\nu)D_{1j}^2]\varphi^j\right\}+2\frac{\partial}{\partial x}\mathrm{Re}(D_{21}\psi^1), \tag{8e}$$

$$E=\left.\frac{\partial u_z}{\partial z}\right|_{z=z_0}=\frac{1}{\mu_{\mathrm{M}}(z_0)}\mathrm{Re}\left\{\sum_{j=1}^{2}[(2-\nu)\nabla^2+(1-\nu)D_{1j}^2]D_{1j}^2\varphi^j\right\}-$$

$$\frac{2(1-\nu)\beta}{\mu_{\mathrm{M}}(z_0)}\mathrm{Re}\left[\sum_{j=1}^{2}(\nabla^2+D_{1j}^2)D_{1j}\varphi^j\right]-\frac{\beta^2}{\mu_{\mathrm{M}}(z_0)}\mathrm{Re}\left\{\sum_{j=1}^{2}[\nu\nabla^2-(1-\nu)D_{1j}^2]\varphi^j\right\}. \tag{8f}$$

将弹性结构内物理中面转角和位移函数等作如下形式分解[10,14]：

$$\phi_1=\frac{\partial}{\partial x}F^{(1)}+\frac{\partial}{\partial y}f^{(1)},\ \phi_2=\frac{\partial}{\partial y}F^{(1)}-\frac{\partial}{\partial x}f^{(1)}, \tag{9}$$

$$U_1=\frac{\partial}{\partial x}F^{(2)}+\frac{\partial}{\partial y}f^{(2)},\ U_2=\frac{\partial}{\partial y}F^{(2)}-\frac{\partial}{\partial x}f^{(2)}. \tag{10}$$

通过演算推导,可得梯度材料平板结构内未知函数的表达式:

$$\mathrm{Re}(\psi^1) = \frac{1}{2}f^{(2)},\ \mathrm{Im}(\psi^1) = \frac{1}{2}[\mathrm{Im}(D_{21})]^{-1}(f^{(1)} + \mathrm{Re}(D_{21})f^{(2)}), \tag{11}$$

$$\varphi^1 = \frac{2\mu_M(z_0)}{\Delta}(D_{12} - D_{13})(D_{12} - D_{14})(D_{13} - D_{14})(s_{11}W + s_{12}F^{(1)} + s_{13}E + s_{14}F^{(2)}) = (S_{11}W + S_{12}F^{(1)} + S_{13}E + S_{14}F^{(2)}), \tag{12a}$$

$$\varphi^2 = -\frac{2\mu_M(z_0)}{\Delta}(D_{11} - D_{13})(D_{11} - D_{14})(D_{13} - D_{14})(s_{21}W + s_{22}F^{(1)} + s_{23}E + s_{24}F^{(2)}) = (S_{21}W + S_{22}F^{(1)} + S_{23}E + S_{24}F^{(2)}), \tag{12b}$$

$$\varphi^3 = \frac{2\mu_M(z_0)}{\Delta}(D_{11} - D_{12})(D_{11} - D_{14})(D_{12} - D_{14})(s_{31}W + s_{32}F^{(1)} + s_{33}E + s_{34}F^{(2)}) = (S_{31}W + S_{32}F^{(1)} + S_{33}E + S_{34}F^{(2)}) = (\bar{S}_{11}W + \bar{S}_{12}F^{(1)} + \bar{S}_{13}E + \bar{S}_{14}F^{(2)}), \tag{12c}$$

$$\varphi^4 = -\frac{2\mu_M(z_0)}{\Delta}(D_{11} - D_{12})(D_{11} - D_{13})(D_{12} - D_{13})(s_{41}W + s_{42}F^{(1)} + s_{43}E + s_{44}F^{(2)}) = (S_{41}W + S_{42}F^{(1)} + S_{43}E + S_{44}F^{(2)}) = (\bar{S}_{21}W + \bar{S}_{22}F^{(1)} + \bar{S}_{23}E + \bar{S}_{24}F^{(2)}), \tag{12d}$$

式中,算子的负指数表示算子的逆;

$$\Delta = -2(1-\nu)(1-2\nu)\nabla^2\nabla^2\nabla^2 \prod_{i=11,\ j=12;\ i<j}^{i=13,\ j=14}(D_i - D_j),$$

$$D_{13} = \bar{D}_{11},\ D_{14} = \bar{D}_{12};$$

以及

$$s_{j1} = -\frac{1}{D_{1j}^2}\{(1-2\nu)\nabla^2[\nu\beta^2 - (1-\nu)\nabla^2] + 2(1-3\nu+2\nu^2)\beta\nabla^2 D_{1j} + \nu[(1-\nu)^2\beta^2 + (1-3\nu+\nu^2)\nabla^2]D_{1j}^2\}\nabla^2\nabla^2 \quad (j=1,2),$$

$$s_{j2} = \frac{1}{D_{1j}^2}\{(1-2\nu)\nabla^2[\nu\beta^2 - (1-\nu)\nabla^2] + 2(1-3\nu+2\nu^2)\beta\nabla^2 D_{1j} + [\nu(1-\nu)^2\beta^2 - (2-5\nu+3\nu^2-\nu^3)\nabla^2]D_{1j}^2\}\nabla^2\nabla^2 \quad (j=1,2),$$

$$s_{j3} = \frac{1}{D_{1j}}\{2(1-\nu)[\nu\beta^2 - 2\nu\beta D_{1j} + \nu D_{1j}^2 - (1-\nu)\nabla^2]\}\nabla^2\nabla^2 \quad (j=1,2),$$

$$s_{j4} = \frac{1}{D_{1j}}\{2[\nu^2\beta^2 - (1-2\nu+2\nu^2)\beta D_{1j} + (1-\nu)^2 D_{1j}^2 - \nu(1-\nu)\nabla^2]\}\nabla^2\nabla^2\nabla^2 \quad (j=1,2).$$

这样,梯度材料平板弯曲与拉伸的应力分量表达式为[14]

$$\tau_{zx} = -\nabla^2\frac{\partial^2\Phi}{\partial x\partial z} + \mu(z)\frac{\partial^2\Psi}{\partial y\partial z} = -2\nabla^2\frac{\partial}{\partial x}\mathrm{Re}\left\{\sum_{j=1}^{2}D_{1j}\exp((z-z_0)D_{1j})\varphi^j\right\} + 2\mu(z)\frac{\partial}{\partial y}\mathrm{Re}\{D_{21}\exp((z-z_0)D_{21})\psi^1\}, \tag{13a}$$

$$\tau_{zy} = -\nabla^2\frac{\partial^2\Phi}{\partial y\partial z} - \mu(z)\frac{\partial^2\Psi}{\partial x\partial z} = -2\nabla^2\frac{\partial}{\partial y}\mathrm{Re}\left\{\sum_{j=1}^{2}D_{1j}\exp((z-z_0)D_{1j})\varphi^j\right\} - 2\mu(z)\frac{\partial}{\partial x}\mathrm{Re}\{D_{21}\exp((z-z_0)D_{21})\psi^1\}, \tag{13b}$$

$$\sigma_z = \nabla^2\nabla^2\Phi = 2\nabla^2\nabla^2\mathrm{Re}\left\{\sum_{j=1}^{2}\exp((z-z_0)D_{1j})\varphi^j\right\}. \tag{13c}$$

把平板上下表面剪力与正应力为0的边界条件引入到方程(13)中,可得如下等式:

$$\frac{\partial}{\partial x}\left\{\nabla^2\mathrm{Re}\left[\sum_{j=1}^{2}D_{1j}\exp\left(\left(\pm\frac{h}{2}-z_0\right)D_{1j}\right)\varphi^j\right]\right\} - \mu\left(\pm\frac{h}{2}\right)\frac{\partial}{\partial y}\mathrm{Re}\left[D_{21}\exp\left(\left(\pm\frac{h}{2}-z_0\right)D_{21}\right)\psi^1\right] = 0, \tag{14}$$

$$\frac{\partial}{\partial y}\left\{\nabla^2\mathrm{Re}\left[\sum_{j=1}^{2}D_{1j}\exp\left(\left(\pm\frac{h}{2}-z_0\right)D_{1j}\right)\varphi^j\right]\right\} + \mu\left(\pm\frac{h}{2}\right)\frac{\partial}{\partial x}\mathrm{Re}\left[D_{21}\exp\left(\left(\pm\frac{h}{2}-z_0\right)D_{21}\right)\psi^1\right] = 0, \tag{15}$$

$$\nabla^2\nabla^2\mathrm{Re}\left[\sum_{j=1}^{2}\exp\left(\left(\pm\frac{h}{2}-z_0\right)D_{1j}\right)\varphi^j\right] = 0, \tag{16}$$

式中,h 是平板结构的厚度.

根据复变函数理论,方程(14)和(15)可看成是一个解析函数的实部和虚部所满足的Cauchy-Riemann[10]条件,于是可有

$$\nabla^2\mathrm{Re}\left[\sum_{j=1}^{2}D_{1j}\exp\left(\left(\pm\frac{h}{2}-z_0\right)D_{1j}\right)\varphi^j\right] = 0, \tag{17}$$

$$\mathrm{Re}\left[D_{21}\exp\left(\left(\pm\frac{h}{2}-z_0\right)D_{21}\right)\psi^1\right] = 0. \tag{18}$$

将方程(11)代入到式(18)中,可得如下方程:

$$\nabla^2\frac{\sin(\mathrm{Im}(D_{21})h)}{\mathrm{Im}(D_{21})}f = 0, \tag{19}$$

式中,$D_{21} = a + \mathrm{i}b$.

于是,可以得到分别描述弯曲和拉伸应力状态的剪切变形模式函数为[15]

$$f^{(1)} = C_{11}f_1 + C_{21}f_2, \tag{20a}$$

$$f^{(2)} = C_{12}f_1 + C_{22}f_2, \tag{20b}$$

其中,函数 $f = f_1 + f_2$,函数 $f_i(i=1,2)$ 满足如下微分方程:

$$\left(\nabla^2 - \frac{\pi^2}{h^2} - \frac{1}{4}\beta^2\right)f_1 = 0, \tag{21a}$$

$$\nabla^2 f_2 = 0, \tag{21b}$$

以及 $C_{ij}(i=1,2;\ j=1,2)$ 是由方程(18)得到的伴随矩阵的元素,表达式分别为

$$C_{11} = -b^{-1}\nabla^2\sin\left((1+\gamma)\frac{bh}{2}\right),\ C_{12} = -b^{-1}\nabla^2\sin\left((1-\gamma)\frac{bh}{2}\right),$$

$$C_{21} = ab^{-1}\sin\left((1+\gamma)\frac{bh}{2}\right) - \cos\left((1+\gamma)\frac{bh}{2}\right),$$

$$C_{22} = ab^{-1}\sin\left((1-\gamma)\frac{bh}{2}\right) + \cos\left((1-\gamma)\frac{bh}{2}\right).$$

将方程(17)和(16)联立后,可得如下方程:

$$\nabla^2\mathrm{Re}\left\{\sum_{j=1}^{2}D_{1j}\exp\left(\left(\pm\frac{h}{2}-z_0\right)D_{1j}\right)\varphi^j\right\} = 0, \tag{22a}$$

$$\nabla^2\nabla^2\mathrm{Re}\left\{\sum_{j=1}^{2}\exp\left(\left(\pm\frac{h}{2}-z_0\right)D_{1j}\right)\varphi^j\right\}=0. \tag{22b}$$

将式(12)代入到式(22)中,可得如下式子:

$$\boldsymbol{DS}[\boldsymbol{W}\quad \boldsymbol{F}^{(1)}\quad \boldsymbol{E}\quad \boldsymbol{F}^{(2)}]^{\mathrm{T}}=\boldsymbol{A}[\boldsymbol{W}\quad \boldsymbol{F}^{(1)}\quad \boldsymbol{E}\quad \boldsymbol{F}^{(2)}]^{\mathrm{T}}=\boldsymbol{0}, \tag{23}$$

其中

$$\boldsymbol{D}=\begin{bmatrix} D_{11}\mathrm{e}^{(h/2-z_0)D_{11}} & D_{12}\mathrm{e}^{(h/2-z_0)D_{12}} & D_{13}\mathrm{e}^{(h/2-z_0)D_{13}} & D_{14}\mathrm{e}^{(h/2-z_0)D_{14}} \\ D_{11}\mathrm{e}^{-(h/2+z_0)D_{11}} & D_{12}\mathrm{e}^{-(h/2+z_0)D_{12}} & D_{13}\mathrm{e}^{-(h/2+z_0)D_{13}} & D_{14}\mathrm{e}^{-(h/2+z_0)D_{14}} \\ \mathrm{e}^{(h/2-z_0)D_{11}} & \mathrm{e}^{(h/2-z_0)D_{12}} & \mathrm{e}^{(h/2-z_0)D_{13}} & \mathrm{e}^{(h/2-z_0)D_{14}} \\ \mathrm{e}^{-(h/2+z_0)D_{11}} & \mathrm{e}^{-(h/2+z_0)D_{12}} & \mathrm{e}^{-(h/2+z_0)D_{13}} & \mathrm{e}^{-(h/2+z_0)D_{14}} \end{bmatrix},$$

$$\boldsymbol{S}=\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}.$$

矩阵(23)所对应的行列式为

$$|\boldsymbol{DS}|=|\boldsymbol{D}||\boldsymbol{S}|=\frac{2\mathrm{e}^{-\beta(h+2z_0)}h^4[\mu_{\mathrm{M}}(z_0)]^4[2(\nabla^2-4\beta^2)h^2-15\beta h-15]}{9(1-\nu)(1-2\nu)}\nabla^6. \tag{24}$$

根据式(24)的算子矩阵行列式,可得梯度材料平板弯曲-拉伸广义位移函数的支配方程:

$$\nabla^2\nabla^2\nabla^2\left(\nabla^2-\frac{1}{4}\beta^2-\frac{15}{4}\beta^2\gamma^2+\frac{15}{2h}\beta\gamma-\frac{15}{2h^2}\right)H=0, \tag{25}$$

对应的广义位移函数 $W,F^{(1)}$ 和 $E,F^{(2)}$ 的表达式为[15]

$$\begin{cases} W=B_{11}H_1+B_{21}H_2+B_{31}H_3+B_{41}H_4, \\ F^{(1)}=B_{12}H_1+B_{22}H_2+B_{32}H_3+B_{42}H_4, \\ E=B_{13}H_1+B_{23}H_2+B_{33}H_3+B_{43}H_4, \\ F^{(2)}=B_{14}H_1+B_{24}H_2+B_{34}H_3+B_{44}H_4, \end{cases} \tag{26}$$

其中, $B_{ij}(i=1,2,3,4;\ j=1,2,3,4)$ 是矩阵 $\boldsymbol{A}$ 的伴随矩阵的元素:

$$\begin{aligned} B_{11}=h^2\{&-24(1-2\nu)+4(1-2\nu)(1+3\gamma)\beta h- \\ &\{\beta^2h^2[\nu^2(3\gamma^2+2\gamma-1)-\nu(3\gamma+1)^2+3\gamma^2+2\gamma+1]- \\ &\frac{1}{(1-\nu)}h^2\nabla^2[\nu^3(3\gamma^2+2\gamma-1)+\nu^2(3\gamma^2+2\gamma+7)- \\ &\nu(15\gamma^2+10\gamma+7)+6\gamma^2+4\gamma+2]\}\}, \end{aligned}$$

$$\begin{aligned} B_{12}=h^2\{&-24(1-2\nu)-4(1-2\nu)(1-3\gamma)\beta h- \\ &\{\beta^2h^2[\nu^2(3\gamma^2-2\gamma-1)-\nu(1-3\gamma)^2+3\gamma^2-2\gamma+1]- \\ &\frac{1}{(1-\nu)}h^2\nabla^2[\nu^3(3\gamma^2-2\gamma-1)+\nu^2(3\gamma^2-2\gamma+7)+ \\ &\nu(-15\gamma^2+10\gamma-7)+6\gamma^2-4\gamma+2]\}\}, \end{aligned}$$

$$\begin{aligned} B_{13}=h\{&48(1-2\nu)-24(1-2\nu)\gamma\beta h+ \\ &2\beta^2h^2[3\nu^2(\gamma^2-1)-9\nu\gamma^2+\nu+3\gamma^2+1]- \\ &\frac{2}{(1-\nu)}\nabla^2h^2[3\nu^3(\gamma^2-1)+\nu^2(3\gamma^2+13)-3\nu(5\gamma^2+3)+6\gamma^2+2]\}, \end{aligned}$$

$$B_{14}=-B_{13},\ B_{21}=B_{11},\ B_{22}=B_{12},\ B_{23}=B_{13},\ B_{24}=-B_{13},$$

$$B_{31}=\frac{2\nu(1-2\nu)}{(1-\nu)}h^3\nabla^2[2(1+3\gamma)+(1+2\gamma+3\gamma^2)\beta h],$$

$$B_{32}=\frac{2\nu(1-2\nu)}{(1-\nu)}h^3\nabla^2[2(1+3\gamma)+(1-2\gamma+3\gamma^2)\beta h],$$

$$B_{33}=\frac{4\nu(1-2\nu)}{(1-\nu)}h^2\nabla^2[6\gamma-\beta h(1+3\gamma^2)],$$

$$B_{34}=-B_{33},$$

$$B_{41}=2(1-2\nu)h^3[2(1+3\gamma)-(1+2\gamma+3\gamma^2)\beta h],$$

$$B_{42}=2(1-2\nu)h^3[-2(1+3\gamma)-(1-2\gamma+3\gamma^2)\beta h],$$

$$B_{43}=4(1-2\nu)h^2[-6\gamma+(1+3\gamma^2)\beta h],$$

$$B_{44}=-B_{43}.$$

广义位移函数

$$H(x,y)=H_1(x,y)+H_2(x,y)+H_3(x,y)+H_4(x,y),\tag{27}$$

其中,函数 $H_i(i=1,2,3,4)$ 分别满足

$$\begin{cases}\left(\nabla^2-\dfrac{1}{4}\beta^2-\dfrac{15}{4}\beta^2\gamma^2+\dfrac{15}{2h}\beta\gamma-\dfrac{15}{2h^2}\right)H_1=0,\\ \nabla^2H_2=H_3,\ \nabla^2H_3=H_4,\ \nabla^2H_4=0.\end{cases}\tag{28}$$

将式(27)代入到式(26)中,可得广义位移函数 $W,F^{(1)}$ 和 $E,F^{(2)}$ 的具体表达式

$$W=h^2\left[-24(1-2\nu)-\frac{1}{(1-\nu)}h^2\nabla^2(\nu^3-7\nu^2+7\nu-2)\right](H_1+H_2)+\frac{4\nu(1-2\nu)}{(1-\nu)}h^3\nabla^2H_3+4(1-2\nu)h^3H_4,\tag{29a}$$

$$F^{(1)}=h^2\left[-24(1-2\nu)-\frac{1}{(1-\nu)}h^2\nabla^2(\nu^3-7\nu^2+7\nu-2)\right](H_1+H_2)+\frac{4\nu(1-2\nu)}{(1-\nu)}h^3\nabla^2H_3-4(1-2\nu)h^3H_4,\tag{29b}$$

$$E=h\left[48(1-2\nu)+\frac{2}{(1-\nu)}h^2\nabla^2(3\nu^3-13\nu^2+9\nu-2)\right](H_1+H_2),\tag{29c}$$

$$F^{(2)}=-h\left[48(1-2\nu)+\frac{2}{(1-\nu)}h^2\nabla^2(3\nu^3-13\nu^2+9\nu-2)\right](H_1+H_2).\tag{29d}$$

2 结　　论

与假设位移分布函数方法不同,本文基于非均匀介质三维弹性力学和物理中面的概念,采用了 Poisson 比等于常数的工程假设,直接采用算子谱分解和 Vieta 定理、解析函数 Cauchy-Riemann 条件等推导得到了梯度平板弯曲与拉伸耦合力学的精确化方程.

本论文的主要创新之处在于: 1) 基于物理中面的概念,首次给出了指数梯度材料平板弯曲与拉伸力学的精确化支配方程; 2) 没有采用直法线假设,以及力矩平衡的方法,而是直接基于弹性力学方程,采用算子谱和算子代数的方法,得到了梯度板弯曲与拉伸耦合力学的支配方程; 3) 与各向同性平板弯曲和拉伸问题不同,此时弯曲应力与拉伸应力状态是不耦合的,

而在功能梯度平板中描述弯曲应力状态的广义位移函数与描述拉伸应力状态的广义位移函数是耦合的，并且分别对应于弯曲和拉伸应力状态的剪切函数也是耦合的。

当前，功能梯度材料在工程中广泛应用，其结构力学建模十分重要，直接关系到力学和热物理行为的分析计算与预测。梯度材料结构的力学与热物理分析与工程设计中存在着许多挑战性课题，迫切要求我们探索更为有效的模型化方法及其分析计算。

本文提出的梯度平板结构弯曲-拉伸耦合力学方程，可望能够在空天飞行器热防护材料与结构的热防护分析与设计中得到应用。

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Refined Equations for Functionally Graded Material Plates Under Bending-Tension Coupling

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Abstract: Based on the theory of elasticity for inhomogeneous media, the spectral compositions of operators and the Vieta's theorem of algebra were applied, and the bending-tension coupling problem of plates of functionally graded material (FGM) was investigated. The refined equations for FGM plates under bending-tension coupling were given. It is shown that, unlike those for the isotropic plate under bending and tension, both the generalized displacement function and the shear function describing the bending stress state and the tension stress state for FGM plates are coupled. Since the derivation of the governing equations was conducted without prior assumptions, the proposed equations for FGM plates can be regarded as exact ones. The work also found out the coupling mechanism and the response structure. The proposed governing equations can be used to analyze the stress of the plate-like FGM structures for thermal protection, and to advance the lightweight design.

Key words: refined theory for FGM plates; operator spectral decomposition; Vieta's theorem; bending-tension coupling; coupling mechanism and response structure

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